

**QCD thermodynamics with $N_f = 2$ and $N_f = 2 + 1 + 1$ flavors
at maximal twist :**

Where do we stand ? Where shall we go ?

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BLTP, JINR Dubna**

**Mini-Workshop on Lattice and Functional Techniques
for the Exploration of Phase Structure and Transport Properties of QCD**

Dubna, July 11 and 12, 2017

The tmfT Collaboration

(“twisted mass at finite temperature”) :

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Outline:

1. Brief History of Lattice QCD in Dubna
2. Motivating the Current Project : Twisted Mass Approach at $T \neq 0$
3. Simulation Setup to Find the Crossover Temperatures for $N_f = 2$
4. Towards the Chiral Limit of T_χ for $N_f = 2$
5. Simulation Setup for $N_f = 2 + 1 + 1$ flavors
6. The Equation of State for $N_f = 2$ and $N_f = 2 + 1 + 1$
7. Relating Lattice and Continuum QCD : Propagators at $T \neq 0$
8. Perspectives of the Project: new Questions/new Partners for Dubna
9. Conclusions and Outlook

1. Brief History of Lattice QCD in Dubna

Lattice simulations in Russia have begun 1981/82 simultaneously in

- ITEP (led by M. I. Polikarpov (1952-2013))
- JINR (BLTP and LIT)

The JINR task force group consisted of :

- V. P. Gerdt (JINR LIT)
- E.-M. Ilgenfritz (JINR BLTP, from Leipzig)
- V. K. Mitrjushkin (JINR BLTP)
- M. Müller-Preussker (1946-2015) (JINR BLTP, from Berlin)
- A. M. Zadorozhny (JINR LIT)

D. V. Shirkov (BLTP) and M. G. Meshcheryakov (LIT) have immediately understood the potential importance of this development and have become active supporters.

Collaborators that have joined us (temporarily, except the last two):

- A. S. Ilchev (JINR BLTP, then from Sofia, Bulgaria)
- N. V. Makhaldiani (JINR LIT)
- S. Yu. Shmakov (JINR LIT)
- I. K. Sobolev (JINR BLTP)
- I. L. Bogolubsky (JINR LIT, still active in lattice QCD)
- V. G. Bornyakov (IHEP Protvino, now also at ITEP Moscow and FEFU Vladivostok)

Collaboration of the German-Russian group with other institutes :

- DESY Hamburg/Zeuthen (G. Schierholz)
(G. Schierholz gave constant support to LGT in Russia)
- Niels Bohr Institute Copenhagen (M. L. Laursen)
- University of Amsterdam (A. J. van der Sijs)
- Universität Bielefeld (F. Karsch, J. Engels et al.)
- Universität Wuppertal (K. Schilling)
- Università di Pisa (A. DiGiacomo et al.)
- ITEP Moscow (M. I. Polikarpov, A. I. Veselov et al.)
- University of Kanazawa (T. Suzuki)
- University of Leiden (P. van Baal)

Topics under investigation before 2000 :

- Vacuum structure : instantons, monopoles, gluon condensate
- Thermal phase transition and topological aspects
- Mean field and Complex Langevin for finite baryonic density
- Effective algorithms for fermions
- Improved gauge fixing : Maximal Abelian gauge and Landau gauge

The biggest international event ever in Dubna in Lattice Field Theory :

“Lattice fermions and structure of the vacuum”,

NATO Advanced Research Workshop, Dubna, Russia, October 5-9, 1999

organised by

V. K. Mitrjushkin (Dubna, JINR) and G. Schierholz (DESY, Zeuthen)

Joint Institute for Nuclear Research
Bogoliubov Laboratory of Theoretical Physics

NATO Advanced Research Workshop

Lattice Fermions and Structure of the Vacuum

October 5-9, 1999

Dubna, Russia

Sponsored by:
NATO, UNESCO and
Heisenberg-Landau
Program

TOPICS:

- Chiral fermions on the lattice
- Issues of chiral extrapolation
- Topology on the lattice
- Confinement mechanisms

ORGANIZING COMMITTEE:

M. Creutz (BNL, USA)
U. Heller (SCRI, FSU, USA)
V. Mitrjushkin (BLTP, JINR)
M. Mueller-Preussker (Humboldt Uni., Germany)
G. Schierholz (DESY, Zeuthen, Germany) - Co-director
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Organisation of the Lattice Summer Schools in Dubna in 2010's

supported by “Helmholtz Association” of German Research Centers

- Dubna International Advanced School of Theoretical Physics
Helmholtz International School “Lattice QCD, Hadron Structure and Hadronic Matter”, September 5 - 17, 2011
Organized by R. Sommer (DESY Zeuthen) with A. Sorin and M. Müller-Preussker
- Dubna International Advanced School of Theoretical Physics
Helmholtz International School “Lattice QCD, Hadron Structure and Hadronic Matter”, August 25 to September 6, 2014
Organized by E.-M. Ilgenfritz (JINR) with O. Teryaev and O. Philipsen

- Dubna International Advanced School of Theoretical Physics
Helmholtz International School “Hadron Structure, Hadronic Matter and Lattice QCD”, August 20 to September 2, 2017
Organized by O. Kaczmarek (Univ. Bielefeld), O. Teryaev and E.-M. Ilgenfritz (JINR Dubna)

Topics of the Dubna-Humboldt-ITEP-IHEP Lattice Collaboration
in the years 2000 – 2013 after the Advanced Research Workshop in Dubna

- Study of ghost and gluon propagators of Yang-Mills Theory and full QCD (IR asymptotics at $T = 0$ and behavior at $T \approx T_c$)
- Fighting the Gribov Ambiguity in Landau gauge fixing (clarifying the IR asymptotics at $T = 0$, decoupling vs. scaling solution)
- Thermal monopoles near the phase transition (resp. crossover) of Yang-Mills Theory and full QCD
- Effect of the thermal phase transitions on gluon and ghost propagators (and discovering the transition from them)
- Topological structure near the phase transition (resp. crossover) of Yang-Mills Theory and full QCD (searching calorons and dyons)

Part of this activity was awarded by the JINR Prize 2015
for a cycle of 20 theoretical papers

“Lattice studies of Landau gauge gluon and ghost propagators in
Quantum Chromodynamics”,

authored by the Russian-German group

I. Bogolubsky, V. Bornyakov, E.-M. Ilgenfritz, V. Mitrjushkin,
M. Müller-Preussker and A. Sternbeck,

their most-cited paper is :

I.L. Bogolubsky et al., *Phys.Lett. B676 (2009) 69* → cited by 340 records

After that “chapter was history”, it was time for a re-orientation to
optimize our contribution to the physics of the Quark-Gluon-Plasma :
by more systematically doing Lattice Thermodynamics, combined with
links to propagator, vertex etc. studies in order to keep contact with
Continuum QCD approaches (DSE and FRG, effective potential).

2. Motivating the Current Project : Twisted Mass Approach at $T \neq 0$

What changes are going on in Lattice QCD ?

Which development BLTP should join ? What comes to mind ?

- **Nuclear Physics** gets increasing contribution from lattice. Connecting QCD with nuclear physics via the lattice – a viable option for BLTP ?
- **Spectroscopy** for low energy high precision physics (FAIR \rightarrow PANDA) hadron spectroscopy, exotica, quarkonia, glueballs, hadron structure
- **QGP** : a long way to go towards Lattice QCD at baryonic chemical potential $\mu_B \neq 0$; progress has been recently achieved (complexification, complex Langevin simulation, Lefschetz thimbles, dualization)
- **QGP** : fruitful cooperation with Continuum QCD (SDE and FRG) (helps extending quenched \rightarrow unquenched, extending $\mu = 0 \rightarrow \mu \neq 0$ etc., gives assessments of approximations being made there [SDE])

- **QGP** : new observables became accessible to describe the QGP (jet quenching, transport coefficients, hadronization, real-time extensions (non-equilibrium QCD and kinetics), spectral functions (gluons and quarks), their extraction is non-trivial !)

Thus, taking these numerous “hot cross-links” with **QGP** into account, it seemed that a more explorative direction like

- conventional hadron gas vs. quark-gluon plasma thermodynamics (**justified by an unconventional fermion discretization**) remains an adequate research option, even if restricted to zero baryonic density.

For this reason,

twisted mass lattice QCD has been selected as the topic of first choice for collaboration between Dubna and Humboldt Universität Berlin !

A suitable topic also for training of **new lattice people in and for Dubna !**
(Also students or PhD students from JINR member states !)

One student in **Dubna University** (from Kazakhstan) and one at **Taras Shevchenko University in Kiev** (Ukraine) have been supervised with this aim to prepare them for work in Lattice QCD :

- **successful master thesis** (Orinay Baidlaeva),
- **successful bachelor degree** (Oleksii Grinyuk),

A series of **lectures has been read in BLTP of National Academy of Sciences in Kiev** during 2013 with the aim to attract more students.

One Ukrainian colleague (Maksym Teslyk) became member of my sector No. 17 in BLTP (then he left Dubna again due to the Ukrainian-Russian crisis). I had introduced him into **real-time lattice techniques**.

One successful newcomer in Dubna : Anton Trunin (PhD 2014)

Two publications resulted from his contributions to the
“Strangeness in Quark Matter” Conference, Dubna 2015 :

1. **Topological susceptibility** from $N_f = 2 + 1 + 1$ lattice QCD at nonzero temperature,
A. Trunin, F. Burger, E.-M. Ilgenfritz, M. P. Lombardo,
M. Müller-Preussker,
J. Phys. Conf. Ser. 668 (2016) no.1, 012123 (arXiv:1510.02265)
2. Towards the **quark–gluon plasma Equation of State** with dynamical strange and charm quarks,
F. Burger, E.-M. Ilgenfritz, M. P. Lombardo, M. Müller-Preussker,
A. Trunin,
J. Phys. Conf. Ser. 668 (2016) no.1, 012092 (arXiv:1510.02262)

Another publication resulted from another of his various activities :

3. Finite temperature **gluon spectral functions** from $N_f=2+1+1$ lattice QCD

E.-M. Ilgenfritz, J. M. Pawłowski, A. Rothkopf, A. Trunin

e-Print: arXiv:1701.08610 [hep-lat]

Finally, the topology project became unexpectedly popular :

4. **Topology (and axion's properties)** from lattice QCD with a dynamical charm

F. Burger, E.-M. Ilgenfritz, M. P. Lombardo, M. Müller-Preussker,
A. Trunin

Contribution to Quark Matter 2017

e-Print: arXiv:1705.01847 [hep-lat]

Unfortunately, A. Trunin has left BLTP in May 2017 for personal reasons.

After closing of the lattice group of M. Müller-Preussker in HU Berlin, we are still facing a dramatic situation. New options are opening :

- **Huge lattice ensembles remaining from simulations**, which so far are only partly evaluated (have been moved from HLRN to CINECA, Italy, to Giessen and Jena) and are still incomplete.
- New partners from **other German Universities are interested to use the data**, mainly from the point of view of Continuum QCD (Heidelberg and Giessen). (see the contribution of Lukas Holicki)
- New partners from **other German Universities are considering now to become more active** (or, active again !) “on lattice” with manpower, master students and hardware (Giessen and Jena [A. Sternbeck]).

A reasonable time to continue/start working in Lattice QCD in Dubna !

Physical motivation : Current situation in Lattice Thermodynamics :

- Thermodynamic simulations mostly done with **staggered fermions** :
Computationally least demanding ! Has remnant of chiral symmetry.
Many improvements made since the advent of staggered fermions :
p4, asqtad, stout → Highly Improved Staggered Quarks (HISQ).
- They had to fight with problems like **taste symmetry breaking**.
- Other problems remain untouched : **rooting ? locality of the action ?**
- Comparison/parallel studies with **other fermion discretization schemes obviously are an urgent “must”** :
 1. **Wilson fermions (have recently become relatively popular)**
 2. Domain wall fermions (still difficult, remain a bit “exclusive”)
 3. Overlap fermions (still difficult, expensive, but “perfect”)

Improvement in a necessary. Two methods exist for Wilson fermions :

1. **Clover improvement**, by adding a Pauli term w and w/o stout smearing (CP-PACS-Coll., WHOT-QCD-Coll., DESY-ITEP-Kanazawa-Coll.)
2. **Twisted mass improvement** (European Twisted Mass Coll., a collaboration concentrating on $T = 0$ hadron physics)

The latter improvement scheme has been applied to finite temperature only by few people, the tmfT Collaboration (proposed at LATTICE 2006).

The Wilson twisted mass fermion action is written **for a doublet** in a twisted basis $\bar{\psi}, \psi$ (τ^3 in flavor space):

$$S_F[\psi, \bar{\psi}, U] = a^4 \sum_x [\bar{\psi}(x) (1 + \kappa D_W[U] + i\kappa\mu_l \gamma_5 \tau^3) \psi(x)]$$

where $D_W[U] = \gamma_\mu \frac{1}{2a} (\nabla_\mu + \nabla_\mu^*) - \frac{ar}{2} \nabla_\mu \nabla_\mu^*$ is the Wilson-Dirac operator.

Describes a minimum of $N_f = 2$ light quarks : μ_l regulates the light sector.

Use of this **fermion action** is made in conjunction with a **tree-level Symanzik improved gauge action** : public code available, see <https://github.com/etmc/tmLQCD/>

$$S_G[U] = \beta \left[c_0 \sum_P \left(1 - \frac{1}{3} \mathbf{Re} (\text{tr} [U(P)]) \right) + c_1 \sum_R \left(1 - \frac{1}{3} \mathbf{Re} (\text{tr} [U(R)]) \right) \right]$$

where $\beta = 6/g_0^2$ and $U(P)$, $U(R)$ are plaquette and rectangle loops.

With weight coefficients $c_0 + 8c_1 = 1$, and $c_1 = -1/12$

This choice corresponds to the **conventions adopted by European Twisted Mass Collaboration (ETMC) for the hadron sector at $T = 0$ with $N_f = 2$ (referred to for calibration of our $T \neq 0$ lattices).**

Main advantages of the twisted mass approach :

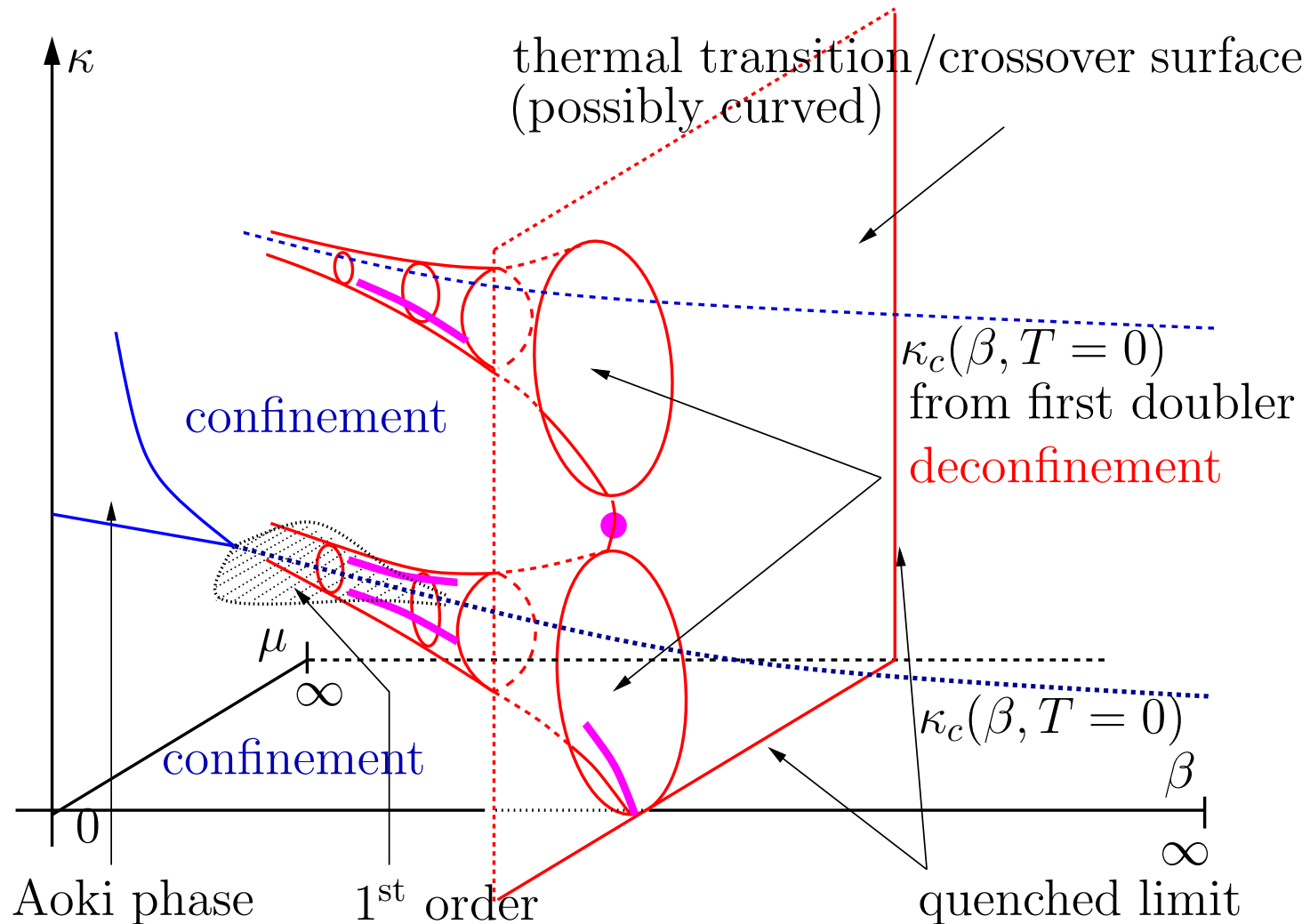
- prevents **exceptional configurations** which are spoiling, for example, clover improved Wilson fermion simulations
- twisted mass provides a **natural infrared cutoff** (the twisted mass μ_l)
- at **maximal twist**, with κ set to $\kappa_c(\beta)$, the twisted mass term μ_l takes over the role of the mass term, while **automatic $O(a)$ improvement** is guaranteed.

This **attractive feature of automatic improvement** had been observed by R. Frezzotti and G. C. Rossi, JHEP 0408 (2004) 007

A slight disadvantage :

- **explicit flavor symmetry breaking**, which is a small remnant of the “Aoki phase” : scale setting is necessary with m_{π^\pm} instead of m_{π^0} .

Observations for $N_f = 2$ collected in a suggestive **3-dimensional phase diagram** (illustrating the “**cone conjecture**” by M. Creutz)



What is the physically relevant branch ?

- Only the **lower cone** is connected with the quenched limit !
Varying the quark mass $\rightarrow \infty$, a **first a critical endpoint is passed, at which the crossover goes over into a first order transition.**
The first order line finally ends in the quenched endpoint at $\kappa = 0$.
- How does a **line of constant physics** (LCP) intersect Creutz' cone ?
 - LCP **has to run** at maximal twist ! **Tuning $\kappa = \kappa_c(\beta)$ is required.**
 - LCP **must not run** at $\mu_l = \text{const}$! **Tuning $\mu_l = \mu_0(\beta)$ is required.**

The analysis rests on **calibration simulations made at $T = 0$,**

- done by the **ETM Collaboration**
- additional $T = 0$ simulations by the **tmfT Collaboration**

Chiral limit ? Simulations made at unphysically massive light quarks !

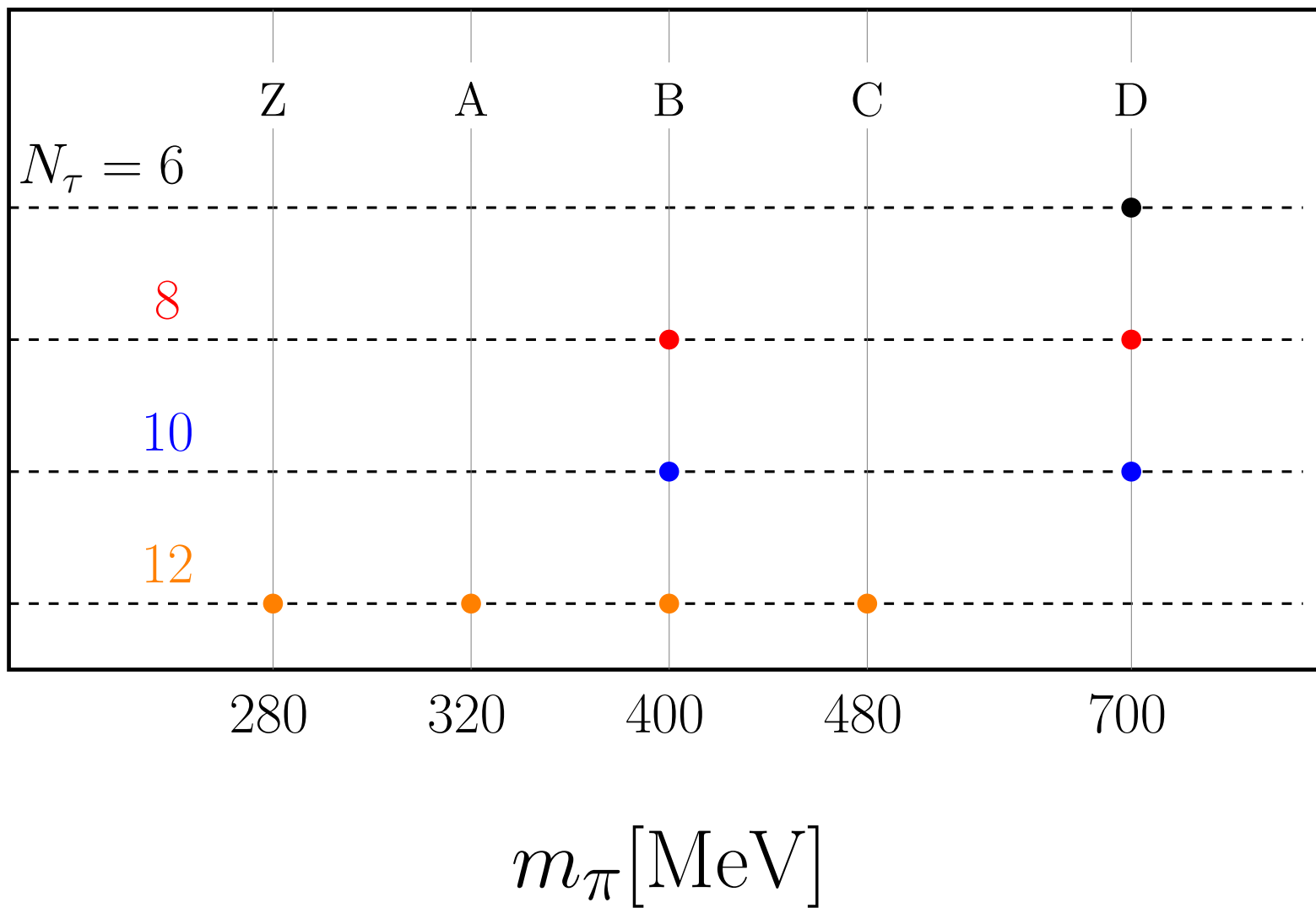
- For natural quark masses : **crossover instead of a phase transition.**
- Nature of the phase transition with $N_f = 2$ flavors in the chiral limit is of **high theoretical interest** (not settled up to now), it is depending how the $U_A(1)$ symmetry is restored (B. B. Brandt, H. Meyer et al., arXiv:1608.06882).
- **Does “our QCD” belong to the $Z(2)$ or $O(4)$ equivalence classes ?**
- Majority of previous results for $N_f = 2$ compatible with $O(4)$.

Next step ? How can we simulate more flavors (partly natural masses) ?

- Rotation in flavor space requires doublets : one degenerate for $N_f = 2$
- Two doublets (light degenerate, heavy non-degenerate) : $N_f = 2 + 1 + 1$

3. Simulation Setup to Find the Crossover Temperatures for $N_f = 2$

- Evaluated : four families of ensembles : A, B, C, Z
(capital referring to π^\pm mass)
- A12 etc. referring to N_τ (number of time slices, fine or coarse lattice)
- populate the **three-dimensional phase diagram** (β, κ, μ_0)
- a **β scan** should fix the position of the crossover line
- maximal twist: requires **tuning of** $\kappa = \kappa_c(T = 0, \beta)$ during β scan
- fixed m_{π^\pm} : requires **tuning of** $a\mu_l = a\mu_0(\beta) = C \exp(-\beta/(12\beta_0))$
(obtained from a one-loop fit or a two-loop fit)
- such fits for **various families of $T = 0$ simulations** are based on data of the ETM-Collaboration [published in JHEP 08 097 (2010)]



List of β -scans (here only for smallest a , $N_\tau = 12$)

- **A12:** $32^3 \times 12$, $3.84 \leq \beta \leq 3.99$,
 $m_\pi = 316(16)$ **MeV**, $r_0 m_\pi = 0.673(42)$
 $\beta_\chi = 3.89(3)$,
 $T_\chi = 202(7)$ **MeV**
- **B12:** $32^3 \times 12$, $3.86 \leq \beta \leq 4.35$,
 $m_\pi = 398(20)$ **MeV**, $r_0 m_\pi = 0.847(53)$
 $\beta_\chi = 3.93(2)$, $\beta_{\text{deconf}} = 4.027(14)$,
 $T_\chi = 217(5)$ **MeV**, $T_{\text{deconf}} = 249(5)$ **MeV**
- **C12:** $32^3 \times 12$, $3.90 \leq \beta \leq 4.07$,
 $m_\pi = 469(24)$ **MeV**, $r_0 m_\pi = 0.998(62)$
 $\beta_\chi = 3.97(3)$, $\beta_{\text{deconf}} = 4.050(15)$,
 $T_\chi = 229(5)$ **MeV**, $T_{\text{deconf}} = 258(5)$ **MeV**

One T_χ or two temperatures T_χ and T_{deconf} (chiral and deconfining) are localized by considering ($T_\chi < T_{\text{deconf}}$) :

- **chiral susceptibility** (looking for sort of a **step function**)

$$\chi_{\bar{\psi}\psi} = \frac{\partial \langle \bar{\psi}\psi \rangle}{\partial m_q}$$

- **disconnected part of it** (looking for a **Gaussian peak**)

$$\sigma_{\bar{\psi}\psi}^2 = \frac{V}{T} (\langle (\bar{\psi}\psi)^2 \rangle - \langle \bar{\psi}\psi \rangle^2)$$

with $\langle \bar{\psi}\psi \rangle$ evaluated as a stochastic estimator

- **renormalized (subtracted) chiral condensate**

$$R_{\langle \bar{\psi}\psi \rangle} = \frac{\langle \bar{\psi}\psi \rangle(T, \mu_l) - \langle \bar{\psi}\psi \rangle(0, \mu_l) + \langle \bar{\psi}\psi \rangle(0, 0)}{\langle \bar{\psi}\psi \rangle(0, 0)}$$

- **renormalized Polyakov loop** (searching for an **inflection point**)

$$\langle \text{Re}(L) \rangle_R = \langle \text{Re}(L) \rangle_{\text{bare}} \exp(V(r_0)/2T)$$

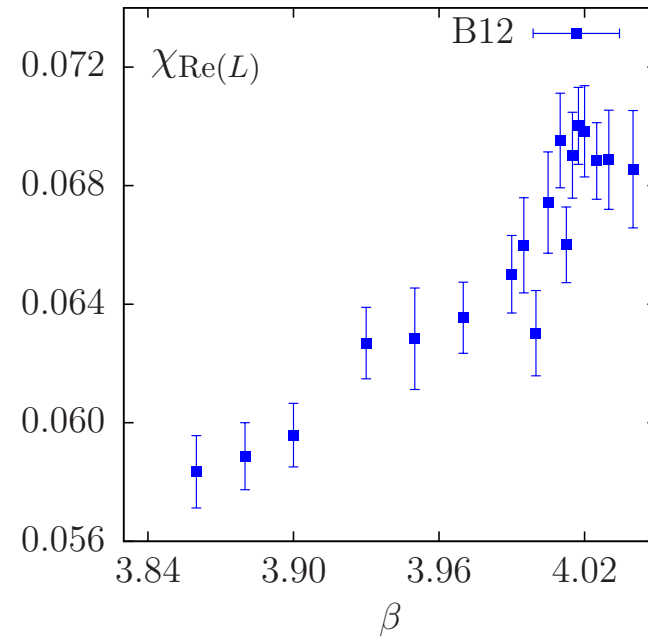
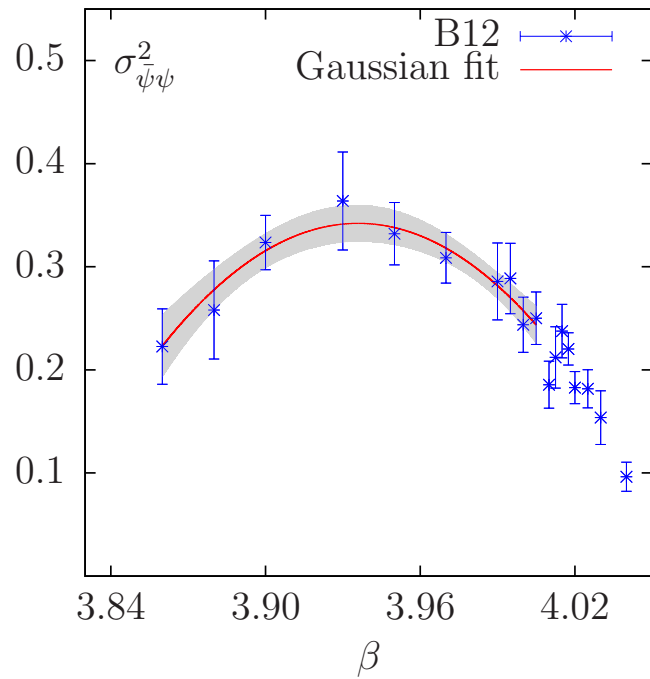
- **and its susceptibility** (searching for a **Gaussian peak**)

$$\chi_{\text{Re}(L)}$$

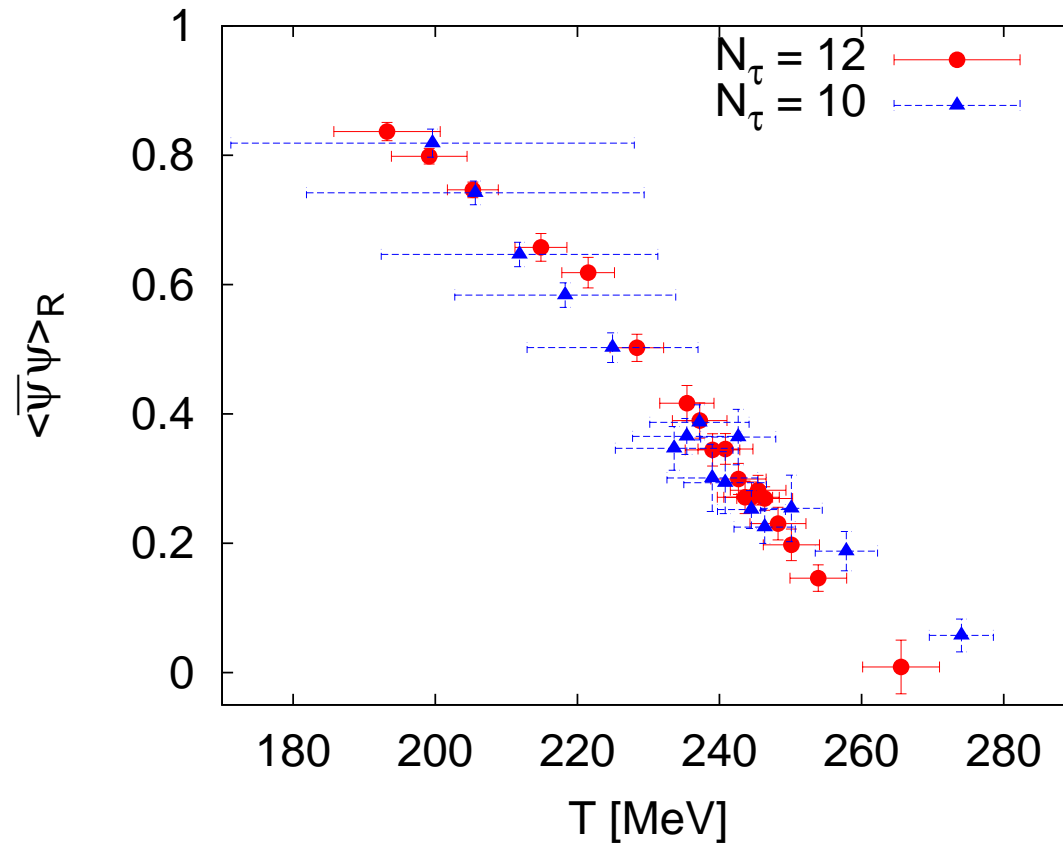
The m_π -dependence and chiral extrapolations are discussed in papers :

- arXiv:1102.4530v2 (finally Phys. Rev. D 87 (2013) 074508)
- arXiv:1212.0982 (F. Burger et al., at LATTICE 2012)

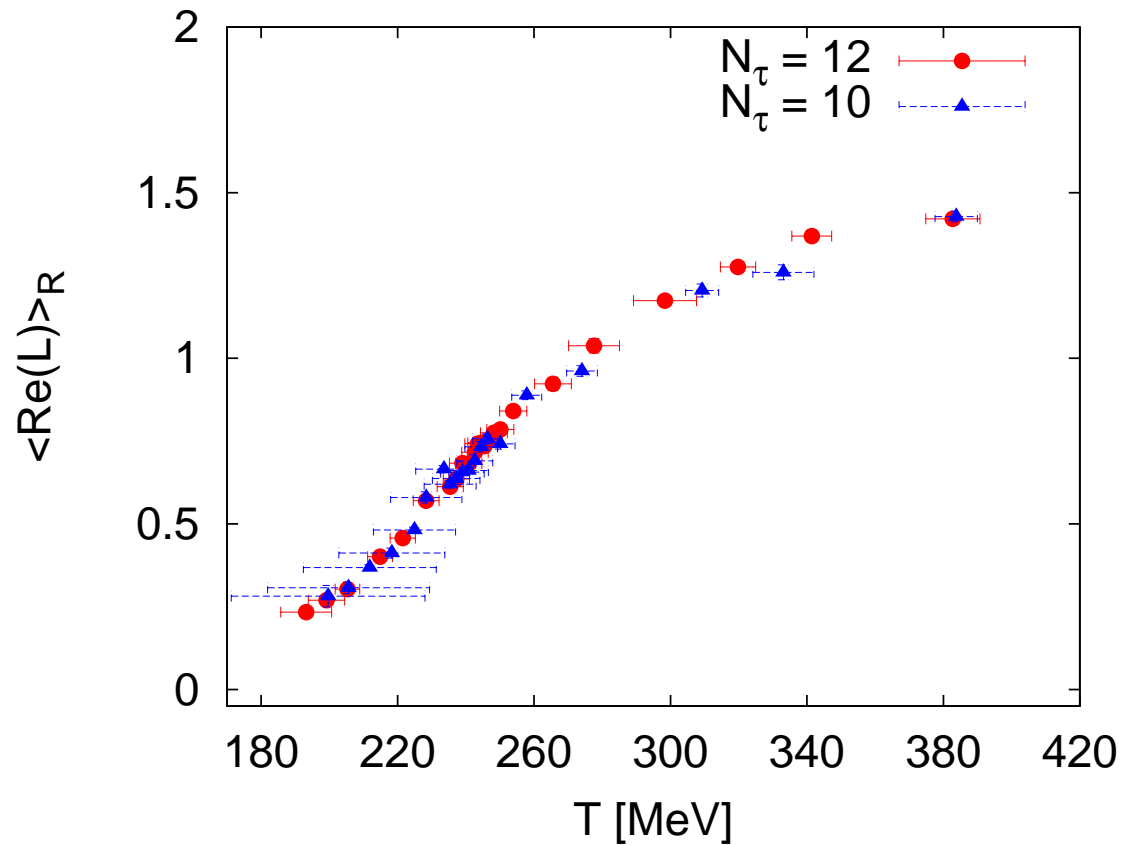
Chiral susceptibility and Polyakov loop susceptibility for B12



Renormalized chiral condensate $\langle \bar{\psi}\psi \rangle$ for B12 and B10
(with the N_τ dependence successfully removed)



Renormalized Polyakov loop $\langle \text{Re}(L) \rangle_R$ for B12 and B10
(with the N_τ dependence successfully removed)



4. Towards the Chiral Limit of T_χ for $N_f = 2$

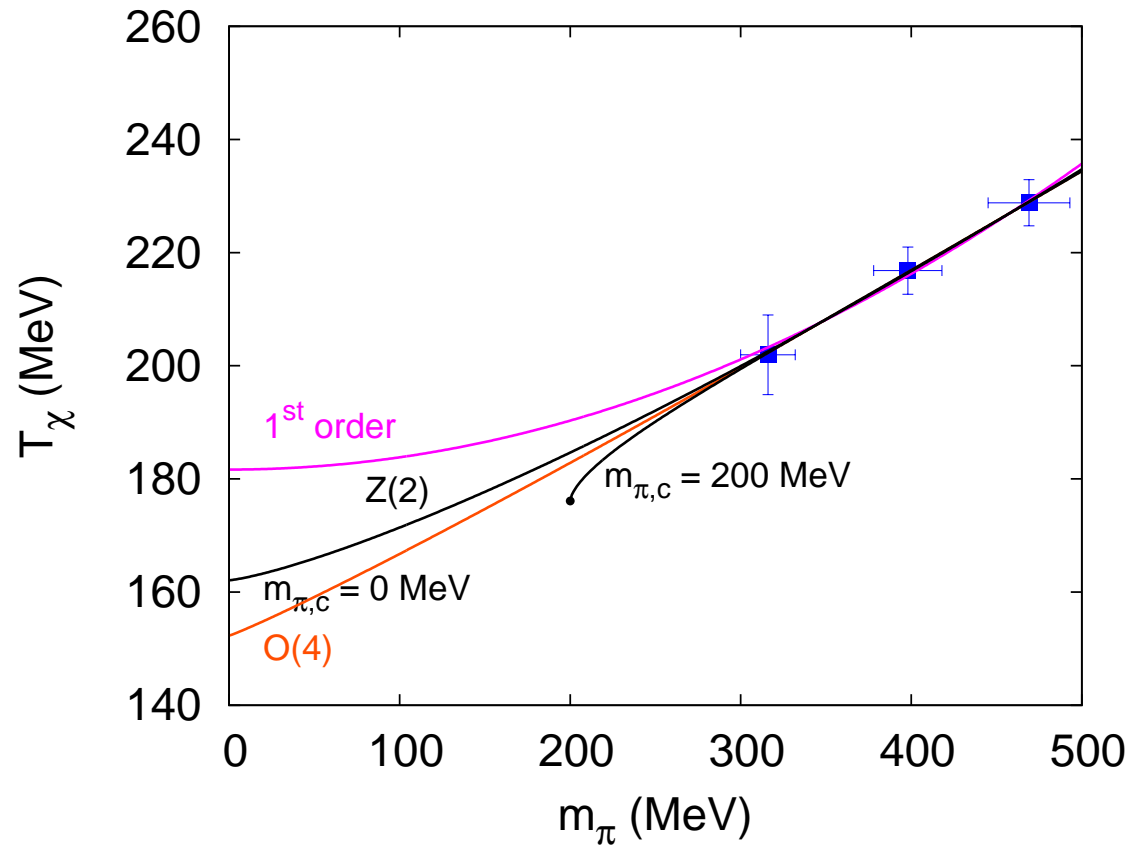
Chiral extrapolations for $T_\chi(m_\pi)$ for various scenarios (from χ PT)

$$T_\chi(m_\pi) = T_\chi(m_\pi = 0) + A m_\pi^{2/(\tilde{\beta}\delta)}$$

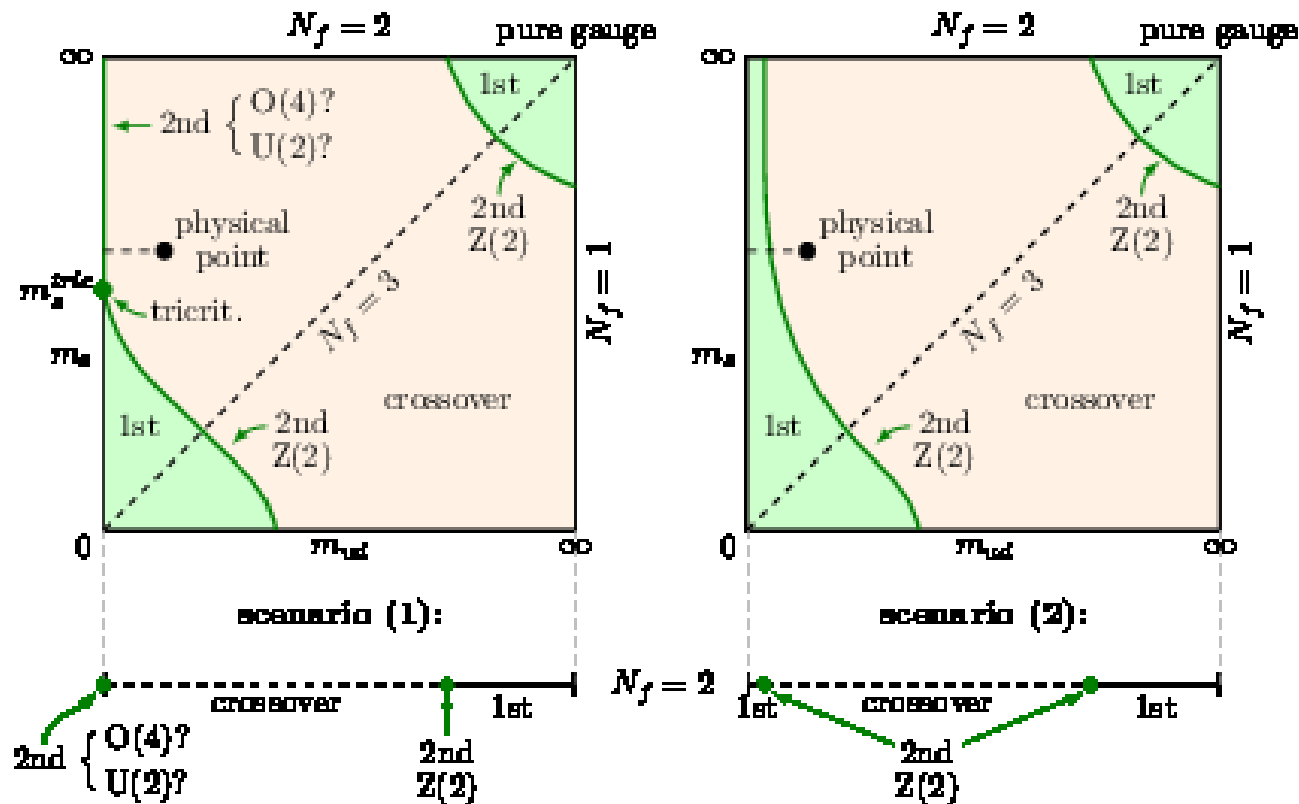
with critical indices $\tilde{\beta}$, δ corresponding to the respective equivalence classes of **three-dimensional spin models : $O(4)$ or $Z(2)$ (Ising)**

- $O(4)$: $2/(\tilde{\beta}\delta) = 1.08$ (close to linear fit) leads to $T_\chi(m_\pi = 0) = 152(26)$ MeV
- $Z(2)$: two cases $m_{\pi,c} = 0$ or $m_{\pi,c} \neq 0$;
these lead to $T_\chi(m_\pi \rightarrow 0)$ **between $O(4)$ and 1-st order scenario**
- **first order** : in literature one takes formally $2/(\tilde{\beta}\delta) = 2$; the quadratic fit leads to $T_\chi(m_\pi = 0) = 182(14)$ MeV
(applicability of these “critical indices” questionable in this context)

Chiral extrapolations for $T_\chi(m_\pi)$ for various scenarios



Quark mass dependence of the Nature of the thermal phase transition with three flavours (for $N_f = 2$, look at the upper rim)



The Columbia Plot

Summary : scenarios for the $N_f = 2$ chiral limit

1. first order transition:

fit gives $T_\chi(m_\pi = 0) = 182(14)$ MeV

2. $O(4)$ second order transition down to $m_\pi = 0$:

fit gives $T_\chi(m_\pi = 0) = 152(26)$ MeV

3. second order $Z(2)$ transition, fits give $T_\chi(m_\pi \rightarrow 0)$ in between.

- either with a **critical point** separating second order ($m_\pi > m_{\pi,c}$) from first order transition (for $m_\pi < m_{\pi,c}$) with $m_{\pi,c} \neq 0$
- or as second order transition down to $m_{\pi,c} = 0$: leading to $T_\chi(m_\pi = 0) = 162(16)$ MeV .

The Z12 results ($m_\pi \approx 280$ MeV) was hoped to exclude the first order scenario ! Could not be clarified before the $N_f = 2$ project was finished.

5. Simulation Setup for $N_f = 2 + 1 + 1$ flavors

Our aims :

- Exploring QCD thermodynamics with realistic strange and charm quark masses, with π^\pm masses from approx. 210 to 470 MeV
- Assessing the influence of heavy flavors on the crossover behavior
- Fixing the contribution of strange and charm quarks to the Equation of State within a T interval from 150 to 600 MeV
- For this purpose, making use of the twisted mass approach for two doublets (a light and a heavy) of Wilson fermions
- Instead of locating the transition by β -scans (as we did for $N_f = 2$), we were now making use of the fixed-scale approach (fixing β)

- The approach to the **continuum limit is taken under control** by simulating for a set of β -values (lattice spacings)
- This philosophy allows to make contact to **$T = 0$ results of the ETM Collaboration** ($T = 0$ vacuum expectation values, masses vs. bare parameters, β -functions and anomalous dimensions from a global fitting)

To describe $2 + 1 + 1$ flavors, one adds a second, non-degenerate doublet of heavy h to the light, degenerate doublet l :

(τ^3 and τ^1 in flavor space)

$$S_F[\psi, \bar{\psi}, U] = a^4 \sum_x [\bar{\psi}_l(x) (1 - \kappa D_W[U] + i2\kappa a \mu_l \gamma_5 \tau^3) \psi_l(x)] \\ + a^4 \sum_x [\bar{\psi}_h(x) (1 - \kappa D_W[U] + i2\kappa a \mu_\sigma \gamma_5 \tau^1 + 2\kappa a \mu_\delta \tau^3) \psi_h(x)]$$

where the renormalized s and c quark masses are related by

$$(m_{s,c})_R = \frac{1}{Z_P} \left(\mu_\sigma \mp \frac{Z_P}{Z_S} \mu_\delta \right)$$

The Iwasaki action is used now in this context for the gauge sector :

(with $c_0 = 3.648$ and $c_1 = -0.331$)

$$S_g[U] = \beta \left(c_0 \sum_P \left[1 - \frac{1}{3} \text{ReTr} (U_P) \right] + c_1 \sum_R \left[1 - \frac{1}{3} \text{ReTr} (U_R) \right] \right).$$

How is the simulation done ?

- Standard Hybrid Monte Carlo simulation is used for the two doublets instead of one (two fermionic forces in Hamiltonian dynamics). T. Chiarappa et al., Numerical simulation of QCD with u, d, s and c quarks in the twisted-mass Wilson formulation, Eur. Phys. J. C50 (2007) 373, hep-lat/0606011
- Modern HMC with all tricks. C. Urbach and K. Jansen, “tmLQCD: A Program suite to simulate Wilson Twisted mass Lattice QCD”, Comput. Phys. Commun. 180 (2009) 2717, arXiv:0905.3331
- The hopping parameter $\kappa_l = \kappa_c(\beta)$ must be tuned to maximal twist.
- The “light” twisted-mass parameter μ_l characterizes now the (eventually non-physical, however constant) π^\pm -mass.

- The **two “heavy” twisted-mass parameters** μ_σ and μ_δ are tuned to the physical point (originally via the K and D masses, finally the nucleon sector).

So far we have **generated finite temperature configurations** for **eight sets of parameters** corresponding to

- four values of the **charged pion mass** of about 470, 370, 260 and 210 MeV for which ...
- two, three, two and one values of the **lattice spacing** (of β) have been considered, respectively : the nomenclature relates capital A, B, D to lattice spacings :

$$a(A) \approx 0.09 \text{ fm} > a(B) \approx 0.08 \text{ fm} > a(D) \approx 0.06 \text{ fm}$$

- The corresponding lattice spacings have been taken from the paper arXiv: 1406.4310, C. Alexandrou et al. (2014), where they are **fixed by comparison with masses in the nucleon sector.**

our	ETMC	m_π^\pm [MeV]	a [fm]	$N_\tau \times N_\sigma^3$	statistics
D210	D15.48	213(9)	0.0646	$\{4, 6, 8, 10, 12, 14, 16, 18, 20, 24\} \times 48^3$	1k-7k
A260	A30.32	261(11)	0.0936	$\{4, 6, 8, 10, 11, 12, 14\} \times 32^3$ $\{16\} \times 40^3$ $\{20\} \times 48^3$	1k-5k 3k 4k
B260	B25.32	256(12)	0.0823	$\{4, 5, 6, 8, 10, 12, 14, 16, 18\} \times 40^3$	1k-8k
A370	A60.24	364(15)	0.0936	$\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \times 24^3$ $\{13, 14\} \times 32^3$	2k-9k 5k,27k
B370	B55.32	372(17)	0.0823	$\{3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16\} \times 32^3$	2k-10k
B370.24	B55.32	372(17)	0.0823	$\{4, 6, 8, 10, 11, 12\} \times 24^3$	3k-10k
D370	D45.32	369(15)	0.0646	$\{5, 6, 7, 8, 9, 10, 12, 14, 16\} \times 32^3$ $\{18\} \times 40^3$ $\{20\} \times 48^3$	1k-12k 10k 10k
A470	A100.24s	466(19)	0.0936	$\{4, 5, 6, 7, 8, 9, 10, 11, 12\} \times 24^3$ $\{14\} \times 32^3$	3k-8k 8k
B470	B85.24	465(21)	0.0823	$\{4, 5, 6, 7, 8, 9, 10, 11, 12\} \times 24^3$ $\{13, 14\} \times 32^3$	2k-4k 2.5k,7k

$T = 0$ base ensembles of ETMC with the respective charged pion masses and lattice spacings a , and a list of finite temperature (i.e. various N_τ) ensembles generated so far by tmfT with the given (typical) statistics. The temperature is assigned to the ensembles by $1/T = aN_\tau$.

A new way to renormalize the “light” chiral condensate :

A new possibility appears to define a **renormalized (subtracted) chiral condensate** when a third (heavier) flavor is included (s denotes “strange”).

$$\Delta_{l,s} = \frac{\langle \bar{\psi}\psi \rangle_l - \frac{\mu_l}{\mu_s} \langle \bar{\psi}\psi \rangle_s}{\langle \bar{\psi}\psi \rangle_l^{T=0} - \frac{\mu_l}{\mu_s} \langle \bar{\psi}\psi \rangle_s^{T=0}} .$$

This implies **subtraction of a divergent part** (proportional to the quark mass), here known from the ratio

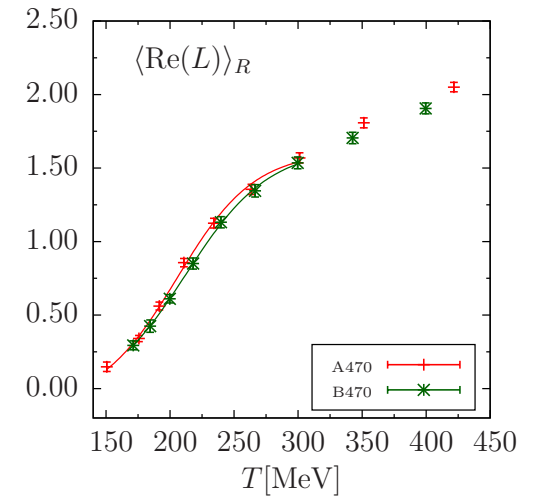
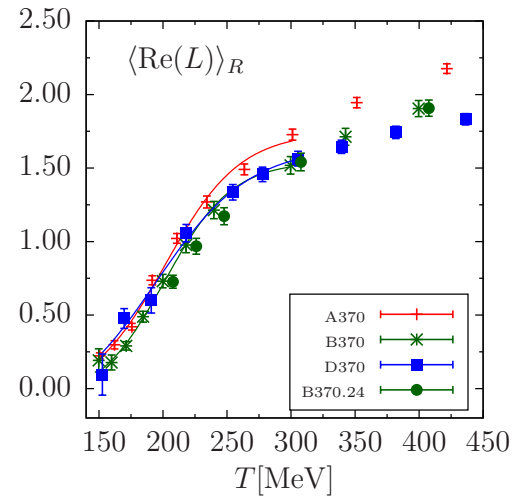
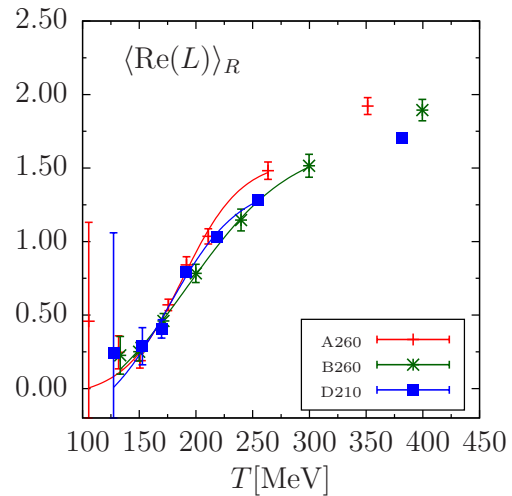
$$\frac{\text{strange quark condensate}}{\text{strange quark mass}}$$

Determination of the pseudo-critical temperature (confinement)

from the renormalized Polyakov loop $\langle \text{Re}(L) \rangle_R(T)$

This requires again a measurement of $V(r_0)$!

The static quark-antiquark potential $V(r)$ has been evaluated using APE smearing. For the Sommer scale $r_0 \approx 0.5$ fm we use the values determined by ETMC, for the respective parameter β .



The renormalized Polyakov loop.

Left: for $m_\pi = 210$ MeV (blue points) and for $m_\pi = 260$ MeV.

Middle: for $m_\pi = 370$ MeV. The data of the finite size test ensemble B370.24 is shown slightly shifted to ease reading the figure (no volume effect seen !). **Right:** for $m_\pi = 470$ MeV.

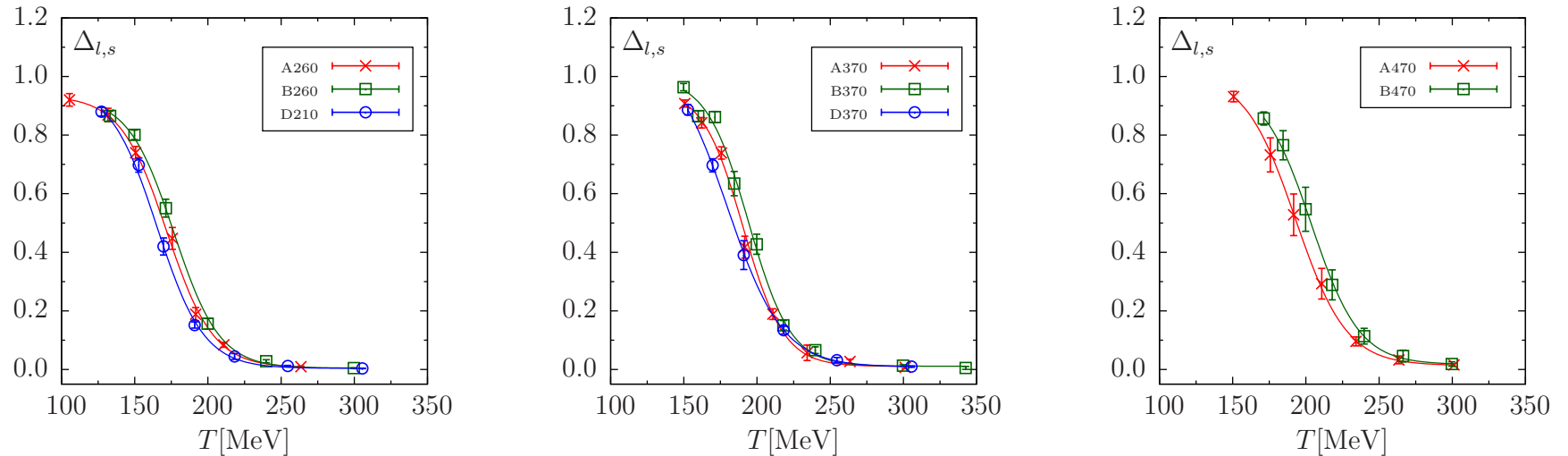
The deconfinement transition temperature T_{deconf} is read off as **inflection point** of a hyperbolic tangent function fit to the **renormalized Polyakov loop** data

$$\langle \text{Re}(L) \rangle_R = A_P + B_P \tanh(C_P(T - T_{\text{deconf}})) .$$

The data becomes more and more noisy for larger N_τ which mostly reduces the data quality for the small mass points.

Fits are restricted to $T < 310$ MeV.

Determination of the pseudo-critical temperature (chiral transition)
from the renormalized (subtracted) chiral condensate $\Delta_{l,s}(T)$



The renormalized (subtracted) chiral condensate $\Delta_{l,s}$.

Left: for $m_\pi = 210$ MeV (blue points) and for $m_\pi = 260$ MeV.

Middle: for $m_\pi = 370$ MeV. **Right:** for $m_\pi = 470$ MeV

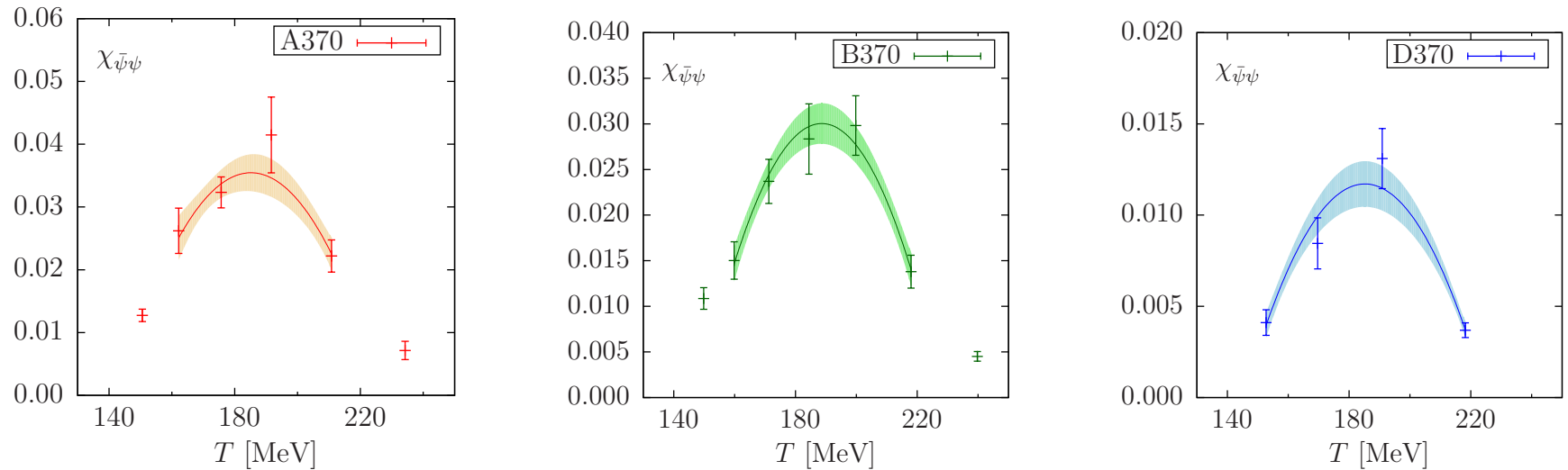
The chiral transition temperature T_Δ is read off as the **inflection point** of a hyperbolic tangent function fit to the **renormalized (subtracted) chiral condensate data**

$$\Delta_{ls}(T) = A_\Delta + B_\Delta \tanh(-C_\Delta(T - T_\Delta)) .$$

We always **used all available data at low temperatures** and used two upper limits for the fit ranges. **The main fits were obtained with data with $T < 350$ MeV.**

Another fit has been done with $T < 450$ MeV. Half the deviation of the latter from the main fit results was used to obtain the **systematic error.**

An alternative determination of the pseudo-critical (chiral) temperature
from the chiral susceptibility $\chi_{\bar{\psi}\psi}(T)$ (as practiced before for $N_f = 2$)



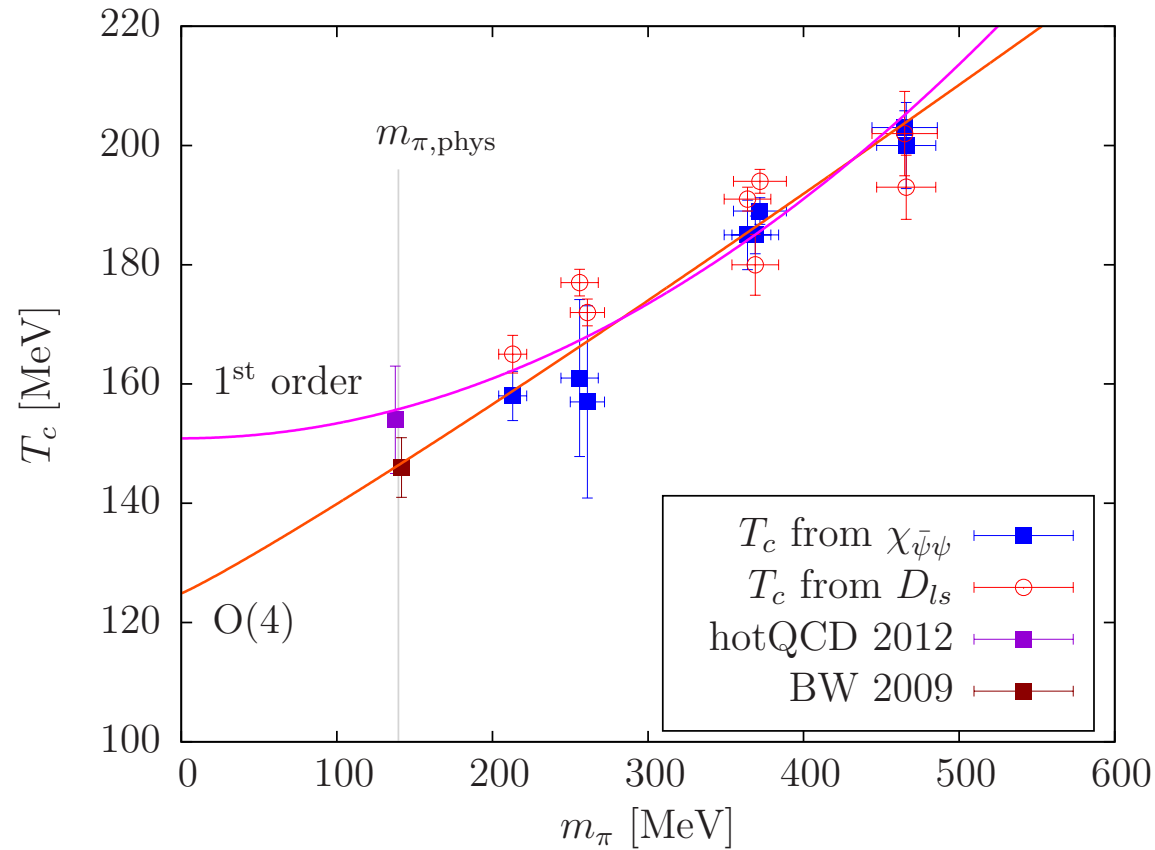
The bare (disconnected) chiral susceptibility $\chi_{\bar{\psi}\psi}$ and fits for different ensembles. **Left: A370, middle: B370, Right: D370** (data are shown here only for $m_\pi \approx 370$ MeV).

The chiral transition temperature T_χ is obtained from fits with the parabolic ansatz (not a model !) for $\chi_{\bar{\psi}\psi}(T)$

$$\chi_{\bar{\psi}\psi}(T) = A_\chi + B_\chi (T - T_\chi)^2 .$$

Ensemble	a [fm]	m_π [MeV]	T_χ [MeV]	T_Δ [MeV]	T_{deconf} [MeV]
D210	0.065	213	158(1)(4)	165(3)(1)	176(8)(8)
A260	0.094	261	157(8)(14)	172(2)(1)	188(6)(1)
B260	0.082	256	161(13)(2)	177(2)(1)	192(9)(2)
A370	0.094	364	185(5)(3)	191(2)(0)	202(3)(0)
B370	0.082	372	189(2)(1)	194(2)(0)	201(6)(0)
D370	0.065	369	185(1)(3)	180(5)(1)	193(13)(2)
A470	0.094	466	200(4)(6)	193(5)(2)	205(4)(2)
B470	0.082	465	203(2)(2)	202(7)(1)	212(6)(1)

Summary of fit-estimated pseudo-critical temperatures using fermionic and gluonic observables. (so far unpublished !)



Chiral extrapolation and extrapolation to the physical pion mass using the first order and second order chiral scenarios. Comparison with staggered results. Only data T_χ (obtained from $\chi_{\bar{\psi}\psi}$) has been used in the fits.

Conclusion for this part

Concerning the influence of the strange and charm quarks, the consensus is that the strange quark has a strong influence in the transition region, while the charm quark does not have any influence up to about 300 MeV.

There the charm quark starts contributing to the EoS.

The disconnected chiral susceptibility up to high T gives a hint concerning the T and mass dependence of the topol. susceptibility. (see the talk of M. P. Lombardo)

6. The Equation of State for $N_f = 2$ and $N_f = 2 + 1 + 1$

Using the integral method. Trace anomaly for $N_f = 2$

$$\begin{aligned}
 \frac{I}{T^4} &= \frac{\epsilon - 3p}{T^4} = -\frac{T}{VT^4} \left\langle \frac{d \ln Z}{d \ln a} \right\rangle_{\text{sub}} \\
 &= N_\tau^4 B_\beta \frac{1}{N_\sigma^3 N_\tau} \left\{ \frac{c_0}{3} \langle \text{ReTr} \sum_P U_P \rangle_{\text{sub}} \right. \\
 &\quad + \frac{c_1}{3} \langle \text{ReTr} \sum_R U_R \rangle_{\text{sub}} \\
 &\quad + B_\kappa \langle \bar{\chi} D_W[U] \chi \rangle_{\text{sub}} \\
 &\quad \left. - [2(a\mu)B_\kappa + 2\kappa_c(a\mu)B_\mu] \langle \bar{\chi} i\gamma_5 \tau^3 \chi \rangle_{\text{sub}} \right\}
 \end{aligned}$$

$\langle \dots \rangle_{\text{sub}} \equiv \langle \dots \rangle_{T>0} - \langle \dots \rangle_{T=0}$ denotes subtraction of vacuum contributions.

Details and results given here only for $N_f = 2$.

$$N_f = 2$$

F. Burger et al., Phys. Rev. D91 (2015) 074504, arXiv:1412.6748

$$N_f = 2 + 1 + 1$$

F. Burger et al., J. Phys. Conf. Ser. 668 (2016) no.1, 012092

arXiv:1510.02262

work still unfinished !

B_β , B_μ and B_κ are (related to) derivatives of the bare parameters with respect to the lattice spacing:

$$B_\beta = a \frac{d\beta}{da}, \quad B_\mu = \frac{1}{(a\mu)} \frac{\partial(a\mu)}{\partial\beta}, \quad B_\kappa = \frac{\partial\kappa_c}{\partial\beta}$$

Evaluation of the **pressure by integrating the identity**

$$\frac{I}{T^4} = T \frac{\partial}{\partial T} \left(\frac{p}{T^4} \right)$$

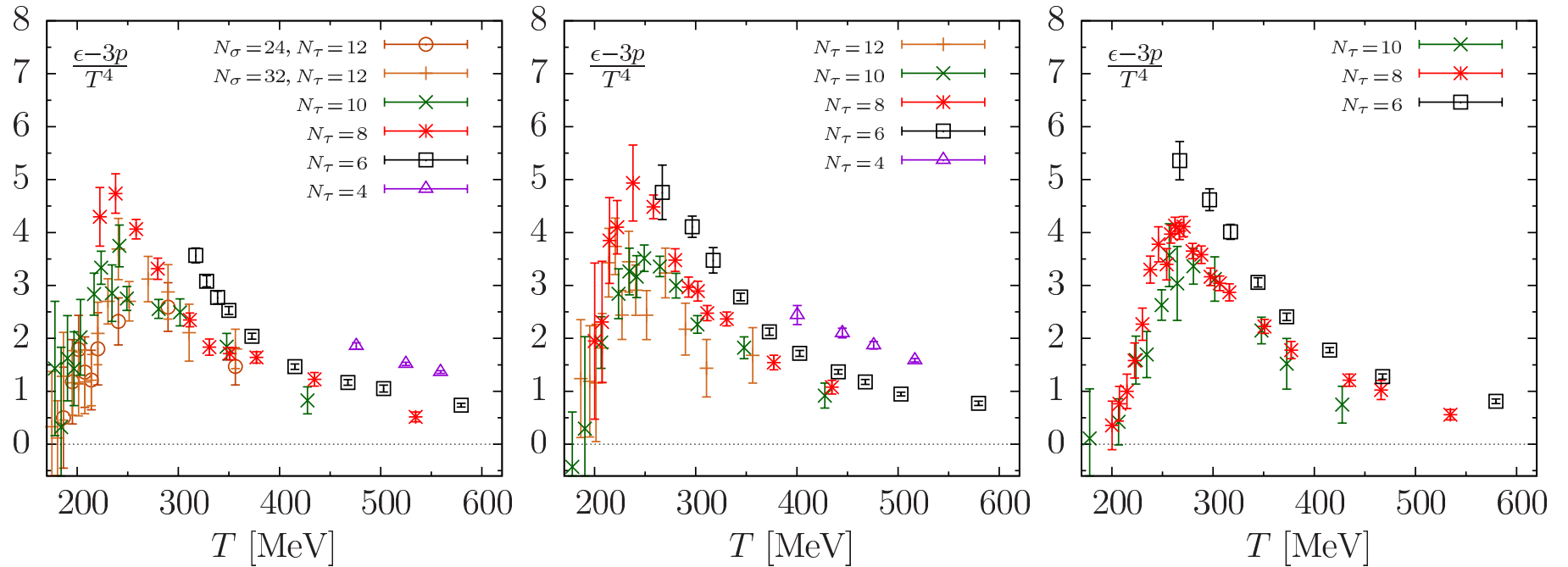
along the line of constant physics (LCP):

$$\frac{p}{T^4} - \frac{p_0}{T_0^4} = \int_{T_0}^T d\tau \frac{\epsilon - 3p}{\tau^5} \Big|_{\text{LCP}}$$

The available lattice data of $\frac{I}{T^4}$ have been fitted to the ansatz

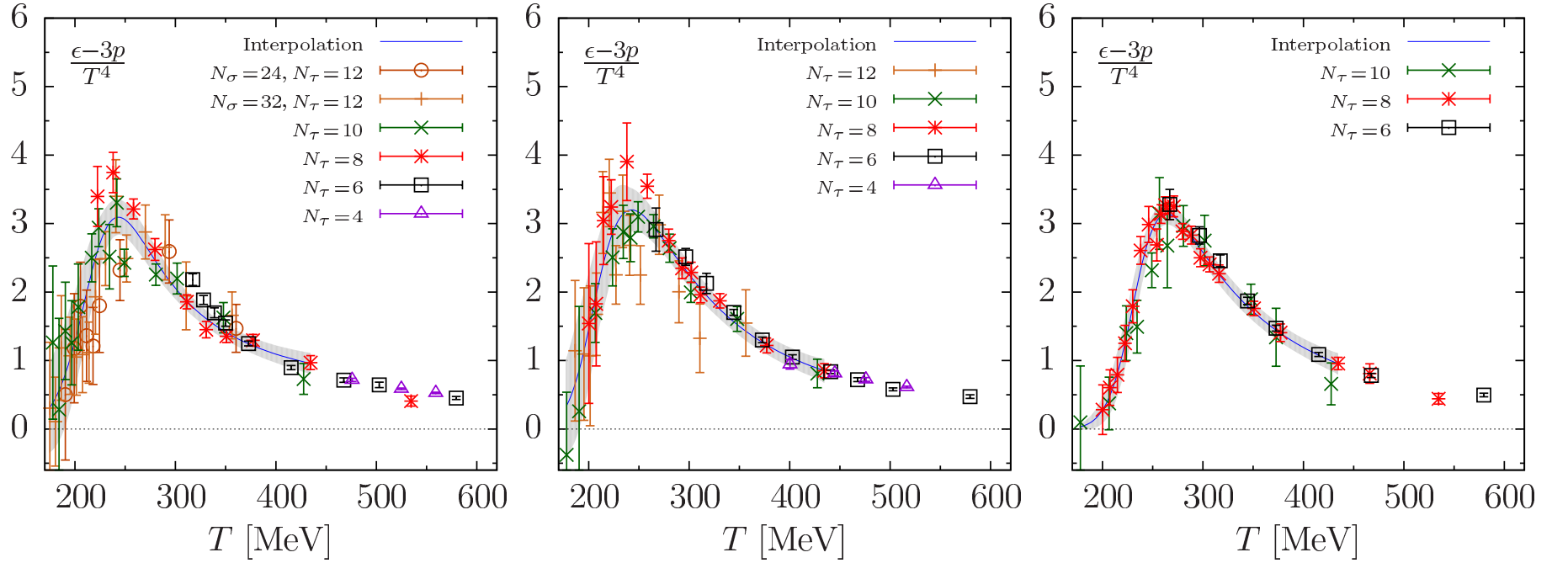
$$\frac{I}{T^4} = \exp(-h_1 \bar{t} - h_2 \bar{t}^2) \cdot \left(h_0 + \frac{f_0 \{ \tanh(f_1 \bar{t} + f_2) \}}{1 + g_1 \bar{t} + g_2 \bar{t}^2} \right)$$

where $\bar{t} = T/T_0$ and T_0 is a further free parameter in the fit.

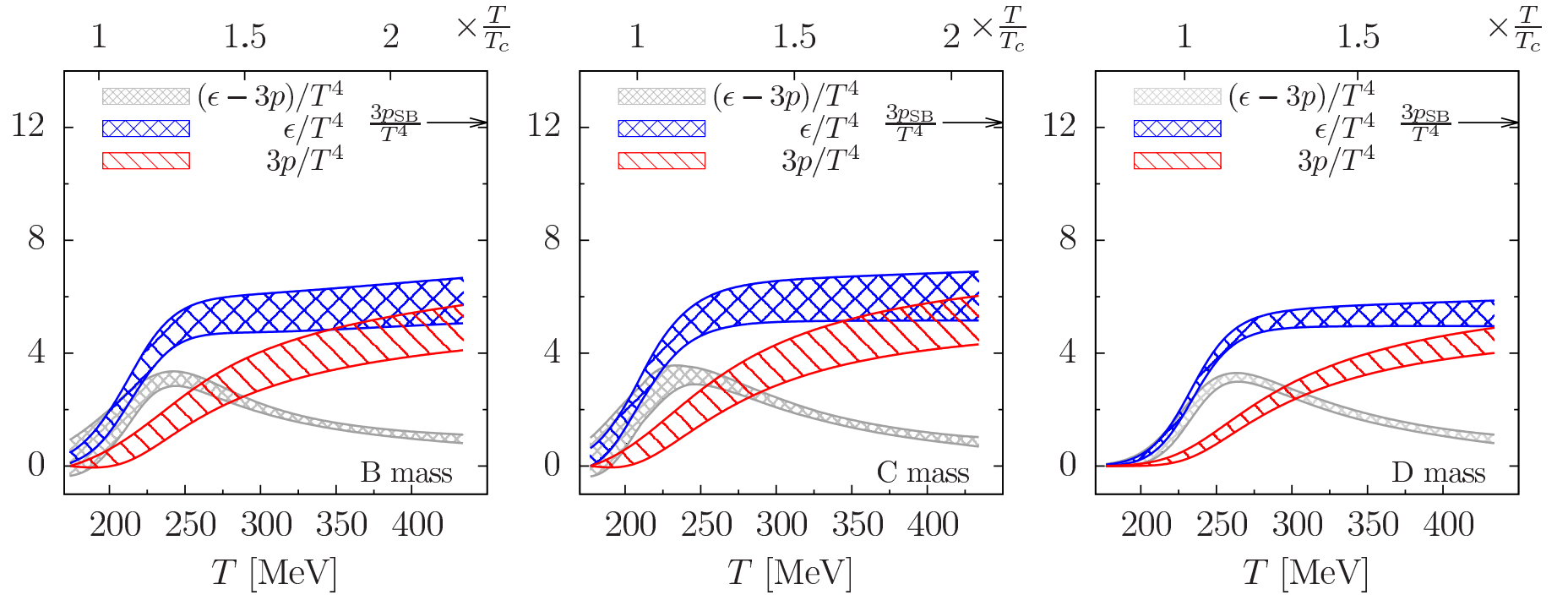


Left: Raw data for the trace anomaly for the B mass obtained for different values of the temporal extent N_τ . **Middle:** The same quantity for the C mass. **Right:** The same quantity for the D mass.

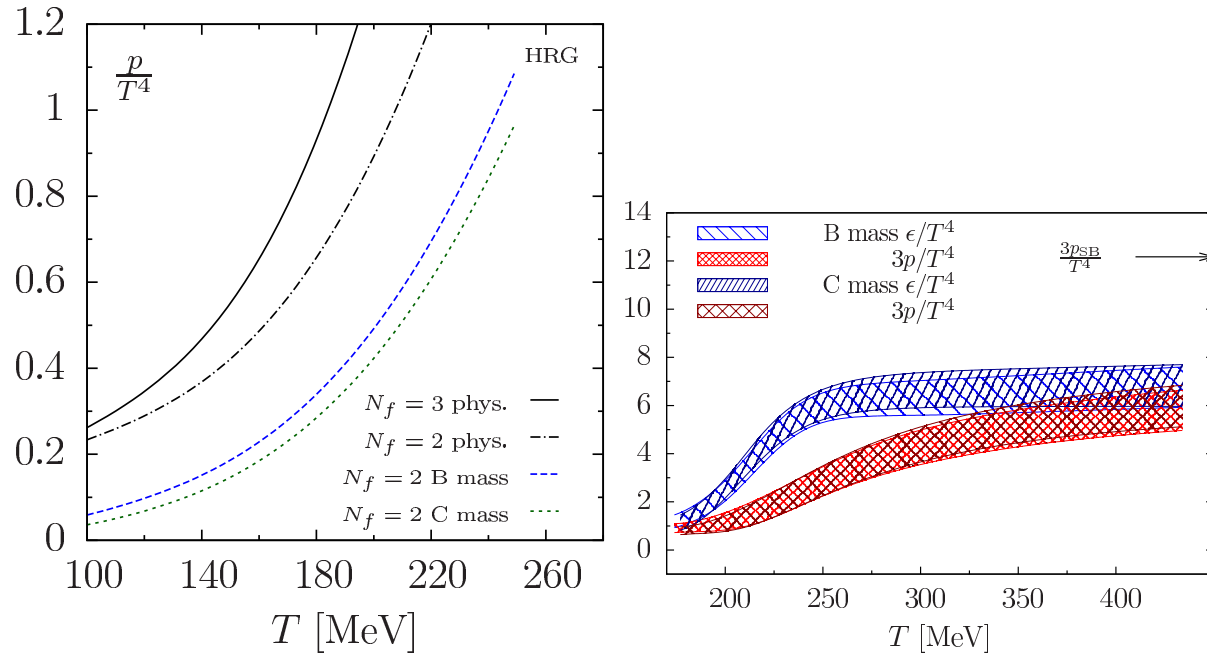
For the B mass the results obtained on the smaller spatial volume are superimposed slightly shifted for better visibility.



Left: The tree-level corrected trace anomaly for the B mass obtained for different values of the temporal extent N_τ ($T_\chi = 208$ MeV and $T_{\text{deconf}} = 225$ MeV). **Middle:** The same quantity for the C mass ($T_\chi = 208$ MeV and $T_{\text{deconf}} = 225$ MeV). **Right:** The same quantity for the D mass ($T_\chi = 229$ MeV and $T_{\text{deconf}} = 244$ MeV). Also shown is a fit to all N_τ .



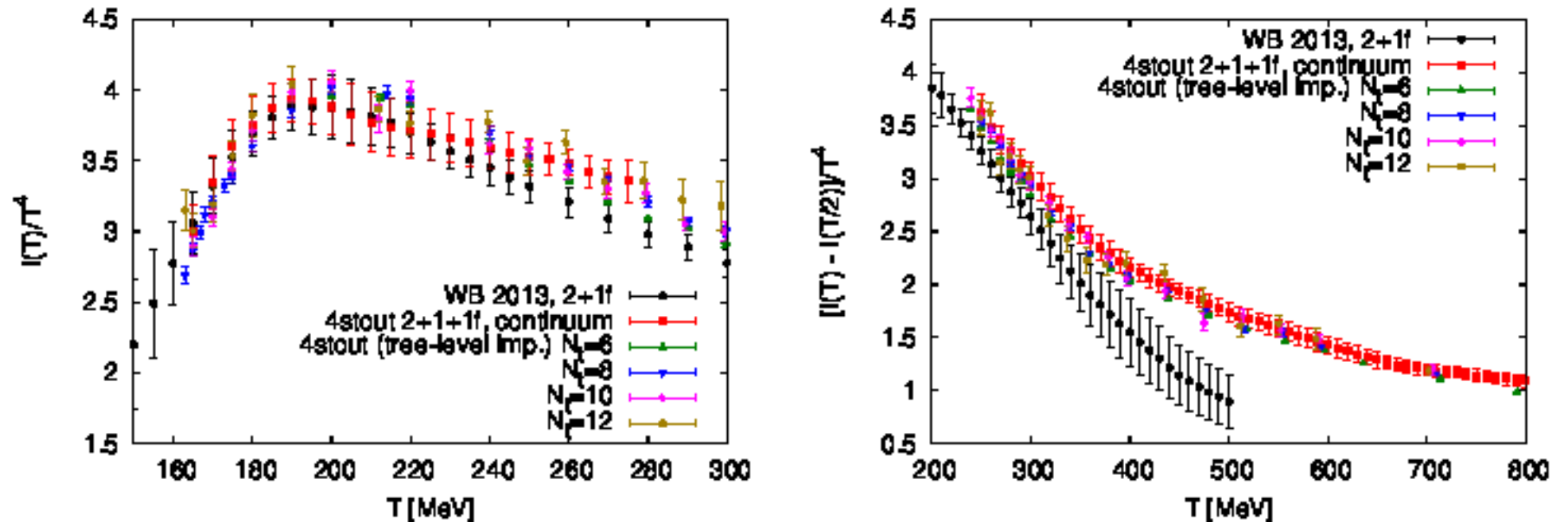
Left: Final result for the pressure p and the energy density ϵ in units of T^4 for the B mass ensemble. Also the interpolation of the trace anomaly used for integration to get the pressure is shown. The arrow shows the expected Stefan-Boltzmann limit for the pressure. **Middle:** The same for the C mass. **Right:** The same for the D mass.



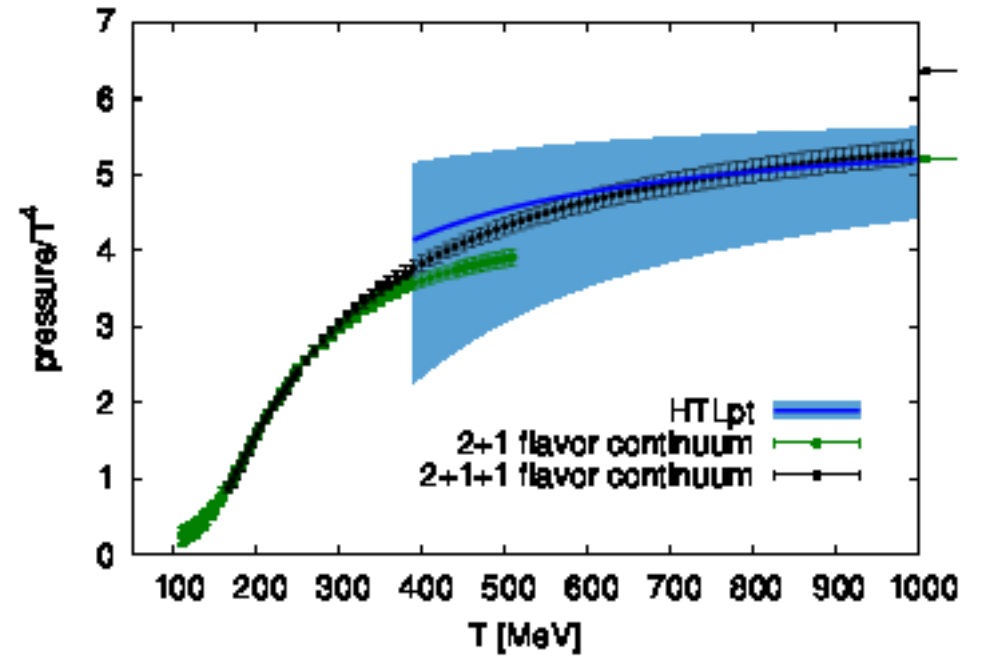
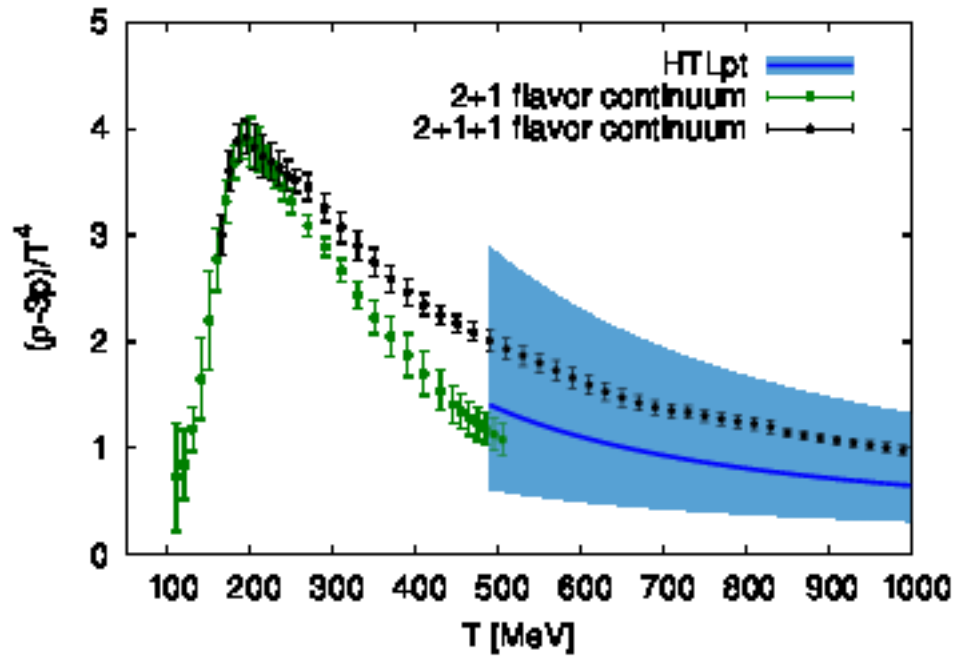
Left: The pressure as obtained from several HRG models is shown (different selections from the PDG table). **Right:** The pressure and energy density for the B mass ($m_\pi \sim 360$ MeV) and for the C mass ($m_\pi \sim 430$ MeV) as obtained **when the HRG model pressure is used to fix the lower integration constant p_0 .**

Other collaborations about heavy quark contributions to the EoS ?

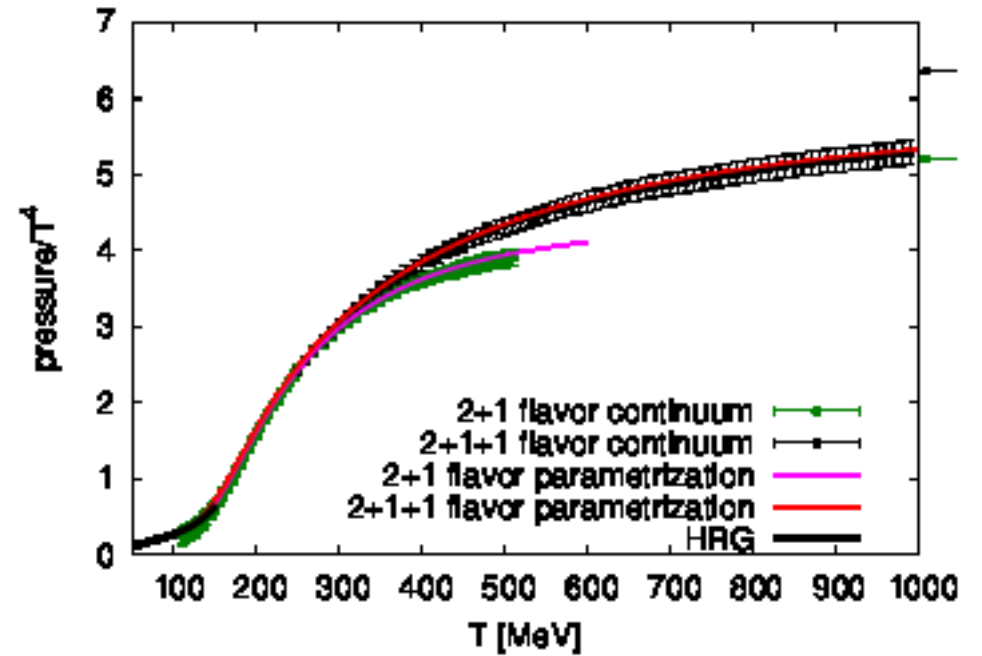
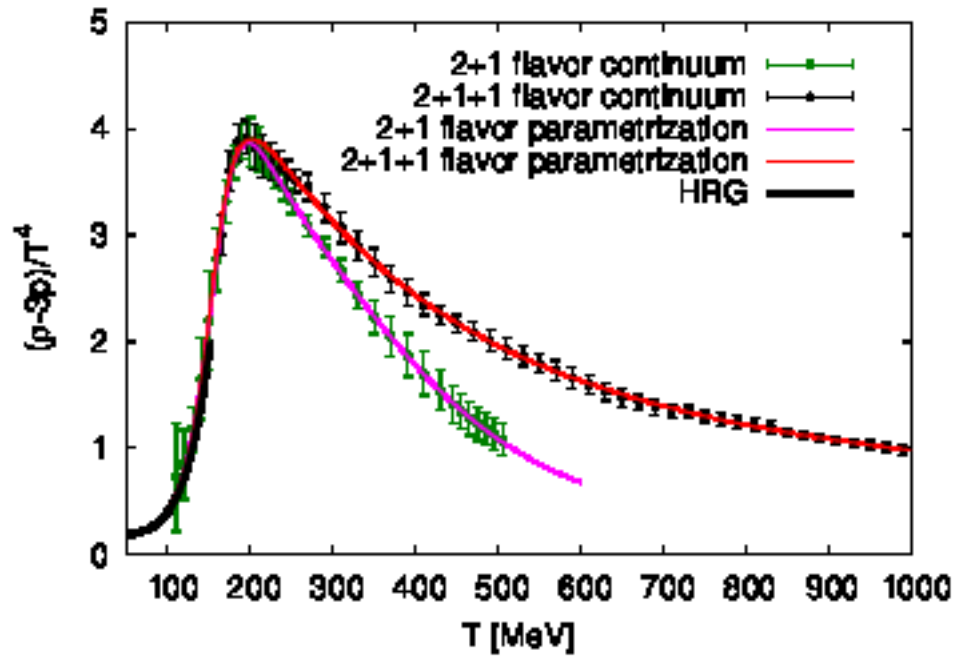
Sz. Borsanyi et al., Lattice QCD for Cosmology, arXiv:1606.07494



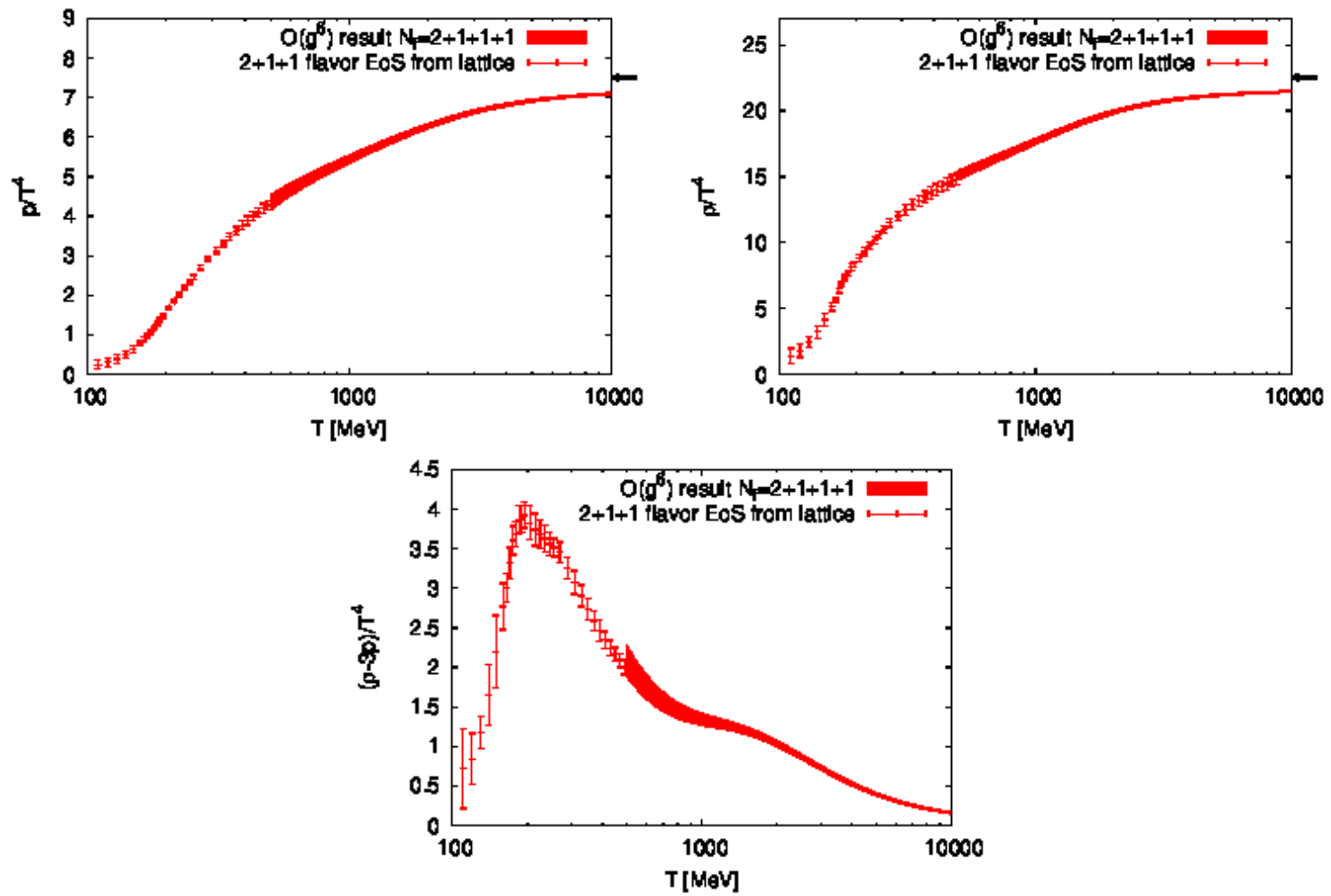
The trace anomaly renormalized with zero temperature simulations (left panel) and the subtracted trace anomaly (right panel) in the 2+1+1 and 2+1 flavor theories. For $T < 300$ MeV the two results agree within errors.



The QCD trace anomaly and pressure in the 2+1+1 and 2+1 flavor theories. Also the four flavor NNLO HTL result at high temperatures is given (Andersen:2010).



The QCD trace anomaly and pressure in the 2+1+1 and 2+1 flavor theories in the parametrization of arXiv:1606.07494. The Hadron Resonance Gas model prediction is shown for comparison.



QCD pressure, energy density and trace anomaly in the 2+1+1+1 theory.

7. Relating Lattice and Continuum QCD : Propagators at $T \neq 0$

Gauge fixing : Landau gauge

$$\nabla_\mu A_\mu(x) = \sum_\mu (A_\mu(x + \hat{\mu}/2) - A_\mu(x - \hat{\mu}/2)) = 0$$

$$A_\mu(x + \hat{\mu}/2) = \frac{1}{2ia g_0} (U_{x\mu} - U_{x\mu}^\dagger)_{\text{traceless}}$$

implemented by **maximization** of the “gauge functional”

$$F_U[g] = \frac{1}{3} \sum_{x,\mu} \text{Re tr} \left(g_x U_{x\mu} g_{x+\mu}^\dagger \right)$$

with respect to suitable gauge transformations g_x .

Remarks

- **Gauge dependent observables** (like propagators) and gauge fixing itself didn't belong to the **instruments of lattice QCD** for long time, **in contrast to QCD formulated in continuum** (Dyson-Schwinger equations, Functional RG method).
- Both is indispensable for any productive exchange with other, **non-lattice approaches to field theory and hadron physics**.
- Gauge fixing leads to a multitude of copies. How to select the **relevant copy** (or copies) ? This is still under debate.
- If the **global maximum** is searched, simulated annealing helps a lot (“maximal Landau gauge”).
- Finally, we went beyond the level of methodical studies !

- **Gauge fixing** is performed for relevant ensembles of Monte Carlo configurations by a special algorithm, irrespective of their origin.
- Ghosts are not explicit, studied only algebraically, by **inversion of the Faddeev-Popov operator**.
- The **quark propagator for the gluon configurations generated with twisted mass fermions by TMC** at $T = 0$ has been studied in :
F. Burger et al., Phys.Rev. D 87 (2013) 034514, arXiv:1210.0838
- We wanted to study the **effect of the crossover on the gluon (and ghost) propagator**, both **quenched and in the presence of quarks**.
- The **quark propagator** should also be studied near the crossover and **compared with Dyson-Schwinger equation results**.
This is left to the future.

7.1 First step: study of the quenched propagator at finite T

R. Aouane et al., Phys. Rev. D 85 (2012) 034501, arXiv:1108.1735
(with Wilson action, for various lattices)

Glueon propagator in momentum space as ensemble average :

$$D_{\mu\nu}^{ab}(q) = \left\langle \tilde{A}_\mu^a(k) \tilde{A}_\nu^b(-k) \right\rangle$$
$$q_\mu(k_\mu) = \frac{2}{a} \sin\left(\frac{\pi k_\mu}{N_\mu}\right) \quad \text{for Matsubara frequency } q_4 = 0$$

For non-zero temperature, Euclidean invariance is broken, then it is useful to split $D_{\mu\nu}^{ab}(q)$ into two components, ($N_g = N_c^2 - 1$ and $N_c = 3$)

- **transversal D_T (“chromomagnetic”)** propagator
- **longitudinal D_L (“chromoelectric”)** propagator

$$D_{\mu\nu}^{ab}(q) = \delta^{ab} \left(P_{\mu\nu}^T D_T(q_4^2, \vec{q}^2) + P_{\mu\nu}^L D_L(q_4^2, \vec{q}^2) \right)$$

Propagators $D_{T,L}$ (or their respective **dimensionless dressing functions** $Z_{T,L}(q) = q^2 D_{T,L}(q)$) obtained from the Fourier transforms

$$D_T(q) = \frac{1}{2N_g} \left\langle \sum_{i=1}^3 \tilde{A}_i^a(k) \tilde{A}_i^a(-k) - \frac{q_4^2}{\vec{q}^2} \tilde{A}_4^a(k) \tilde{A}_4^a(-k) \right\rangle$$

and

$$D_L(q) = \frac{1}{N_g} \left(1 + \frac{q_4^2}{\vec{q}^2} \right) \left\langle \tilde{A}_4^a(k) \tilde{A}_4^a(-k) \right\rangle$$

The **corresponding renormalized functions**, in momentum subtraction (MOM) schemes, can be obtained from

$$Z_{T,L}^{ren}(q, \mu) \equiv \tilde{Z}_{T,L}(\mu) Z_{T,L}(q),$$

with the \tilde{Z} -factors being defined such that $Z_{T,L}^{ren}(\mu, \mu) = 1$.

Fitting gluon dressing functions we used the **Gribov-Stingl** form

$$Z_{\text{fit}}(q) = q^2 \frac{c (1 + d q^{2n})}{(q^2 + r^2)^2 + b^2}$$

Our main emphasis : Finite-volume and discretization studies, providing **continuum parametrizations** for various temperatures, to be used as input (or benchmark) for finite- T continuum studies (performed by DS equations or Functional RG studies)

- Gribov copy and finite volume effects turned out to be of minor importance in the momentum range under study
- fits of the momentum dependence of the propagators
 $0.6 \text{ GeV} < q < 3.0 \text{ GeV}$
- in the temperature range
 $0.65 < T/T_{\text{deconf}} < 2.97$

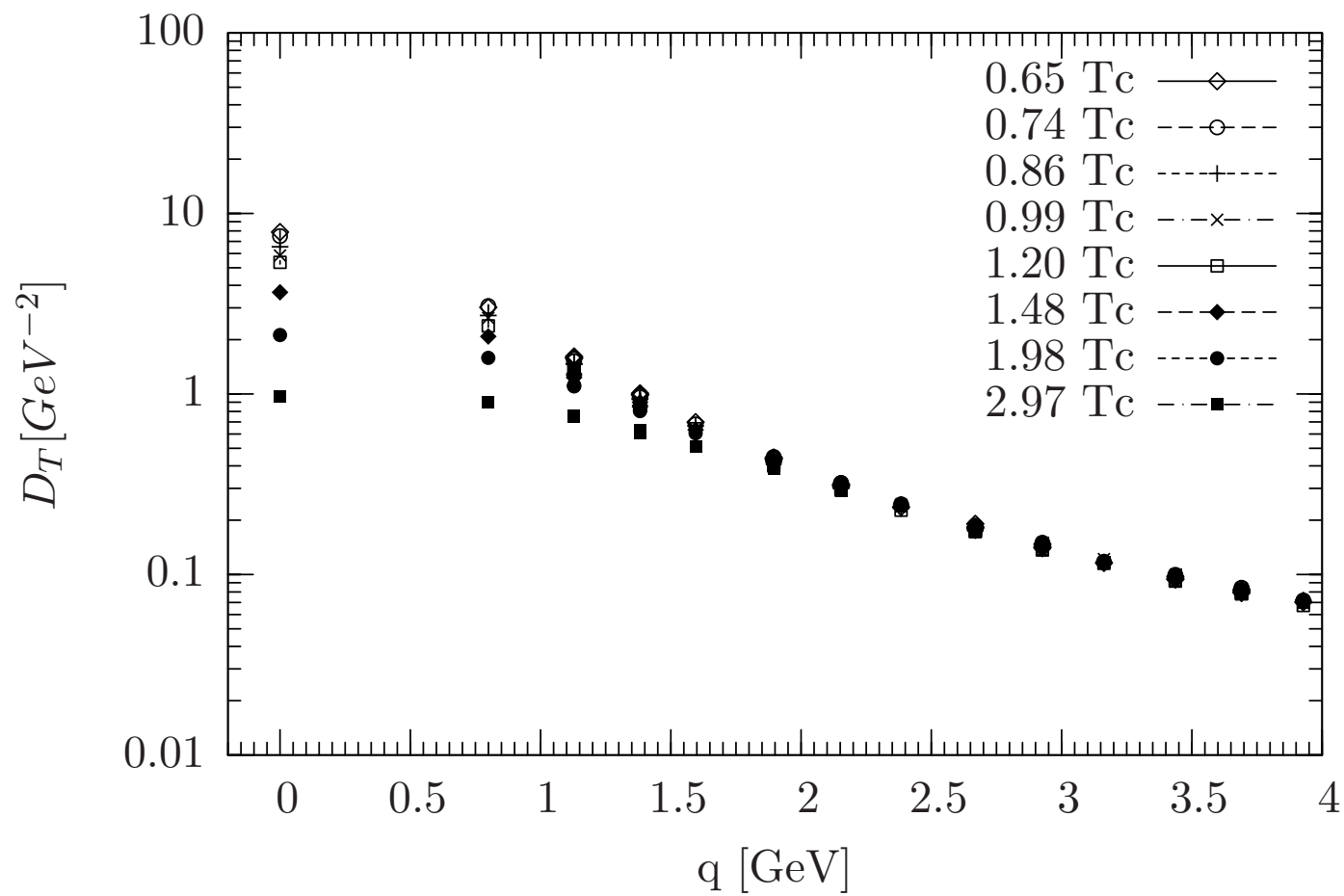
Important : the first order nature of the phase transition is evident.
More or less rapid changes are visible in the propagators !

Temperature dependence studied at fixed scale,
here shown only for $\beta = 6.337$, the finest lattice

T/T_c	N_τ	N_σ	β	$a(\text{GeV}^{-1})$	$a(\text{fm})$	n_{conf}	n_{copy}
0.65	18	48	6.337	0.28	0.055	150	1
0.74	16	48	6.337	0.28	0.055	200	1
0.86	14	48	6.337	0.28	0.055	200	1
0.99	12	48	6.337	0.28	0.055	200	1
1.20	10	48	6.337	0.28	0.055	200	1
1.48	8	48	6.337	0.28	0.055	200	1
1.98	6	48	6.337	0.28	0.055	200	1
2.97	4	48	6.337	0.28	0.055	210	1

long simulated annealing sequences, no copies !

q dependence of color-magnetic D_T for various temperatures
only zero Matsubara frequency



7.2 Phase structure from propagators ?

It is not too difficult to reconcile the first order phase transition despite the (merely) gradual changes of the propagators actually seen !

Our finite-temperature results for pure Yang-Mills theory have been used by K. Fukushima and K. Kashiwa, Phys. Lett. B 723 (2013) 360 arXiv:1206.0685v5

- for the effective potential of the Polyakov loop
- for reconstructing the Equation of State (EoS)

In leading order of the 2PI formalism, the thermodynamical potential can be approximated as follows in terms of both the gluon and ghost propagators :

$$\frac{1}{T}\Omega_{\text{glue}} \simeq -\frac{1}{2}\text{tr} \ln D_{\text{gl}}^{-1} + \text{tr} \ln D_{\text{gh}}^{-1}$$

For example, the inverse gluon propagator have been extracted from our data in the interval up to $T = 1.2 T_c$

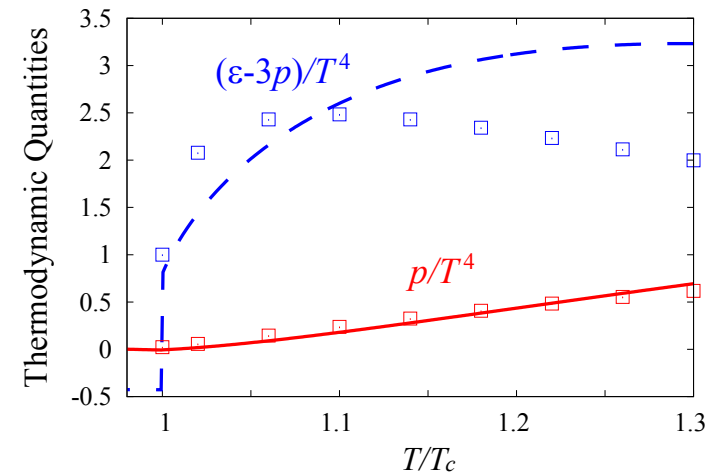
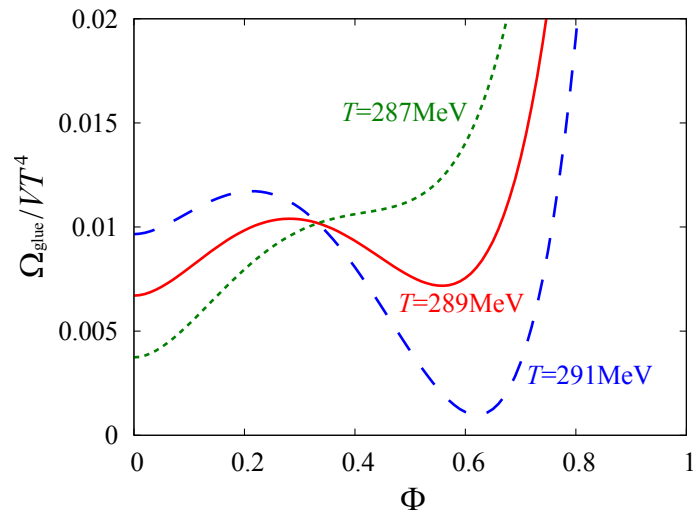
$$D_{\text{gl}}^{-1}(p^2) = \left[p^2 Z_T(p^2) T_{\mu\nu} + \xi^{-1} p^2 Z_L(p^2) L_{\mu\nu} \right] \delta^{ab}$$

Results :

- Order and transition temperature has been successfully reconstructed from our T -dependent propagator data.
- The pressure and trace anomaly are only qualitatively obtained.

Order parameter and EoS of pure Yang-Mills (Fukushima and Kashiwa)

Transition temperature and first rise of the pressure successfully reconstructed from our T -dependent propagator data for gluodynamics !



7.3 DSE prediction for unquenching the propagators

Can one obtain non-quenched propagators from the quenched ones without actually doing the non-quenched lattice simulation ?

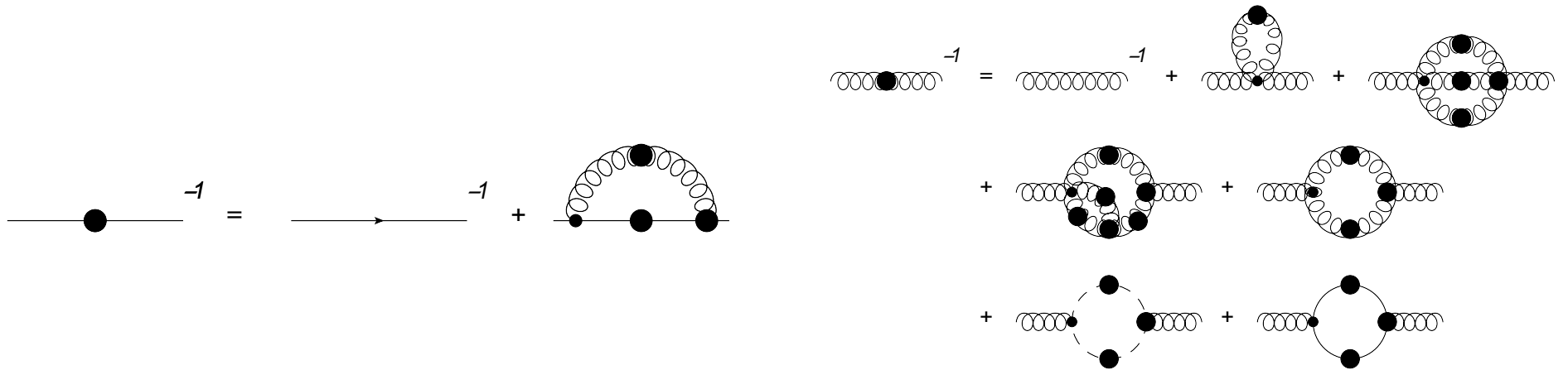
How good can DSE predict/postdict what will be/has been measured on the lattice in a non-quenched simulation ?

C. S. Fischer and J. Luecker, “Propagators and phase structure of $N_f = 2$ and $N_f = 2 + 1$ QCD”

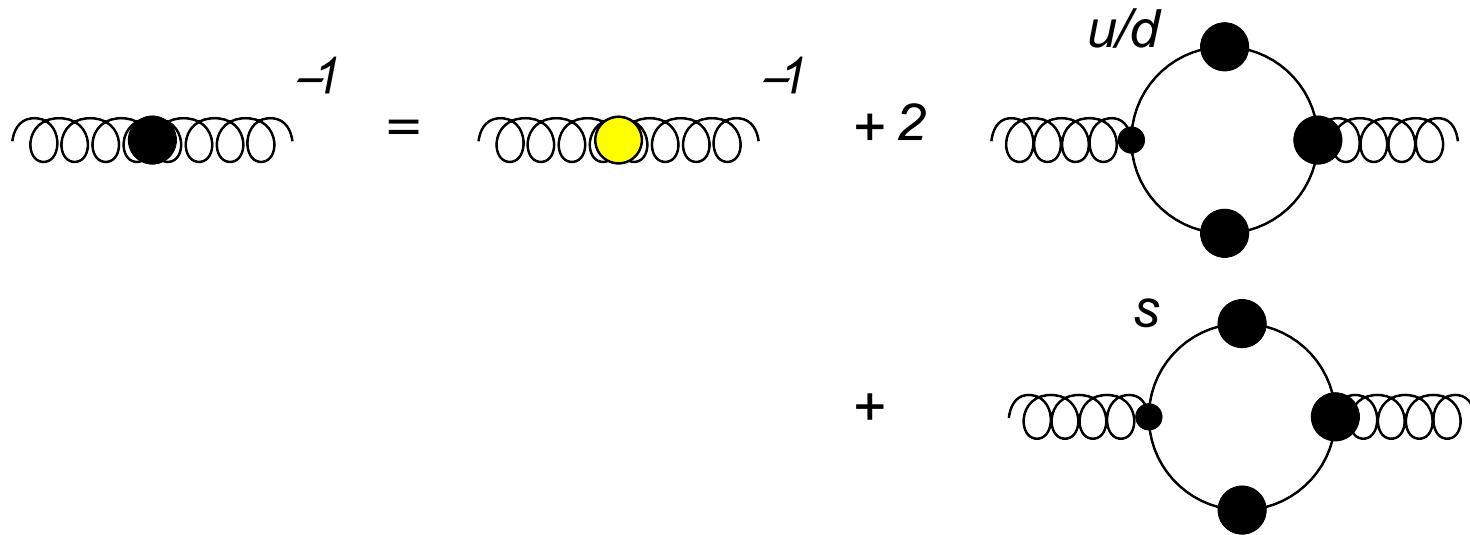
Physics Letters B 718 (2013) 1036, arXiv:1206.5191,
and arXiv:1306.6022

The full set of Dyson-Schwinger equations was used to predict the T dependence of full QCD propagators from the quenched ones, depending on m_π as a parameter to characterize the non-quenched simulations.

Full Dyson-Schwinger equations for the quark (left) and the gluon (right) propagator



Truncated gluon Dyson-Schwinger equation relating the quenched and the non-quenched gluon propagator (for u, d and evtl. s quarks)
 (yellow insert = quenched gluon propagator)



By-product of this study : quark propagator at $T \neq 0$
(so far not yet provided by us for twisted mass at $T \neq 0$)

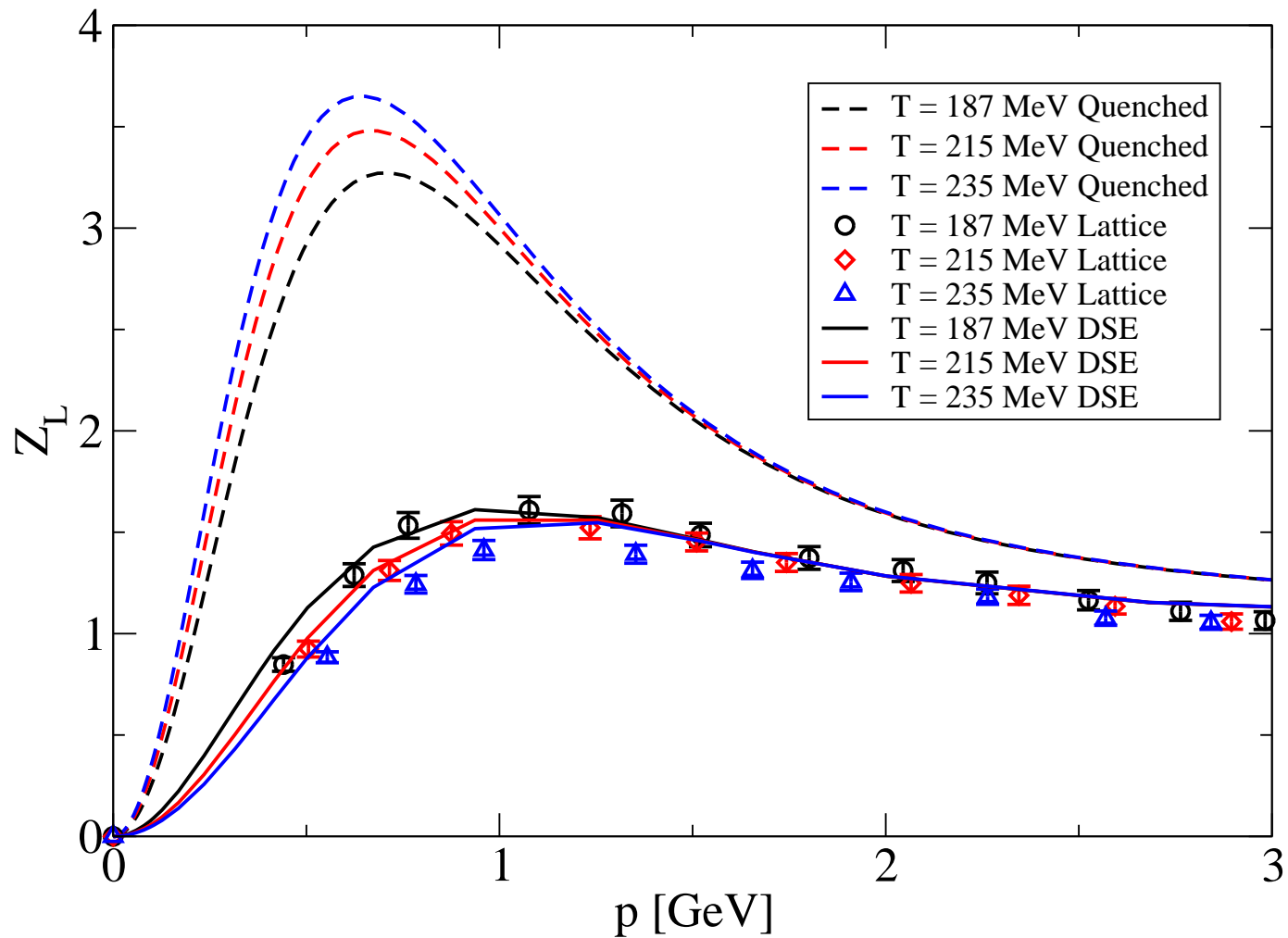
The quark propagator is planned to be measured in
future finite- T simulations (now with $N_f = 2 + 1 + 1$).

Will be interesting to compare with DSE predictions !

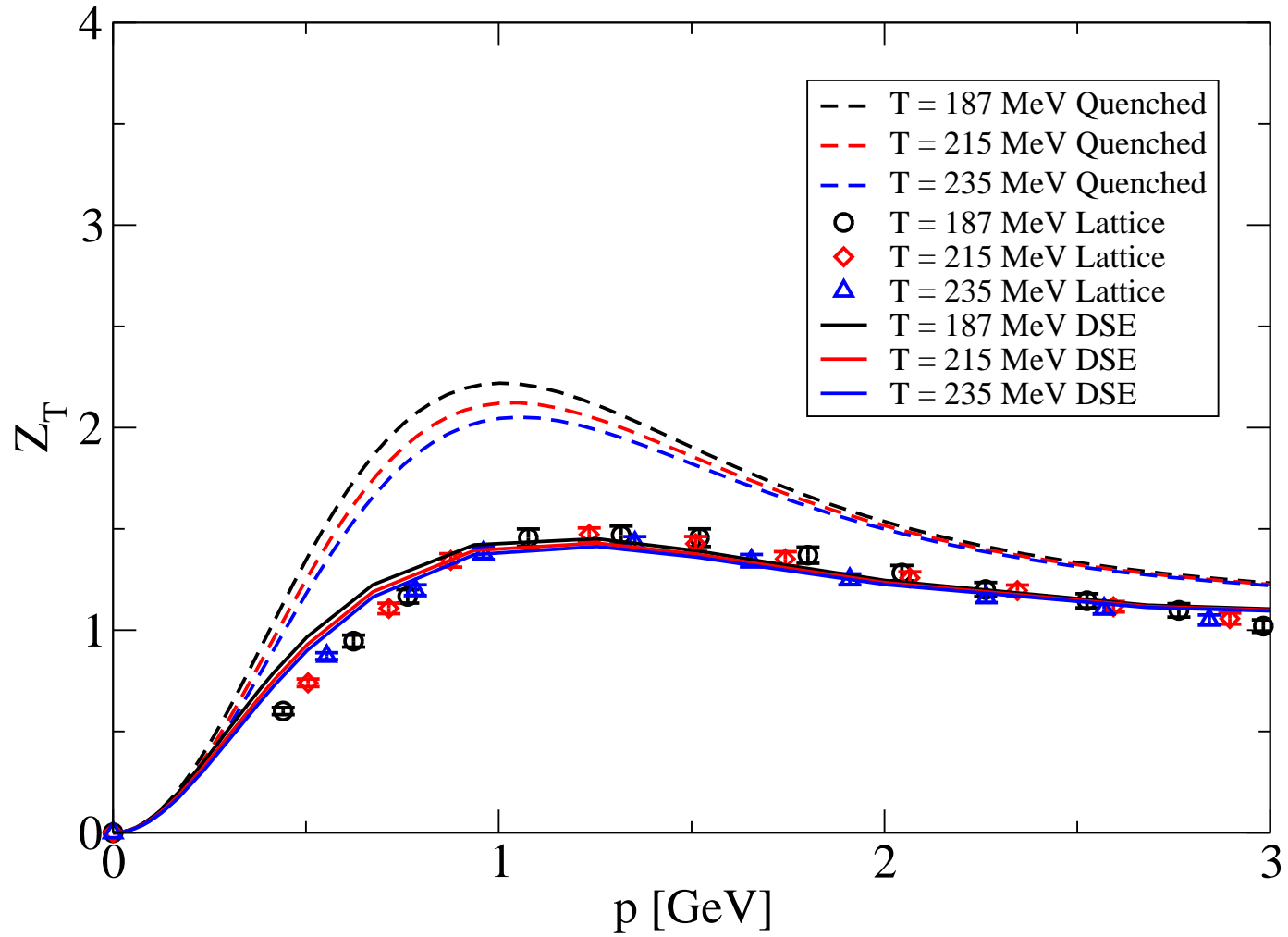
The group of Ch. Fischer (Giessen) is strongly interested
in future $N_f = 2 + 1 + 1$ data !

An *ab initio* study of the (momentum dependence) of
the quark-gluon vertex is presently of high interest !

q dependence and unquenching effect of Z_L for various temperatures



q dependence and unquenching effect of Z_T for various temperatures



Our next paper [Phys. Rev. D 87 \(2013\) 114502](#), [arXiv:1212.1102](#)

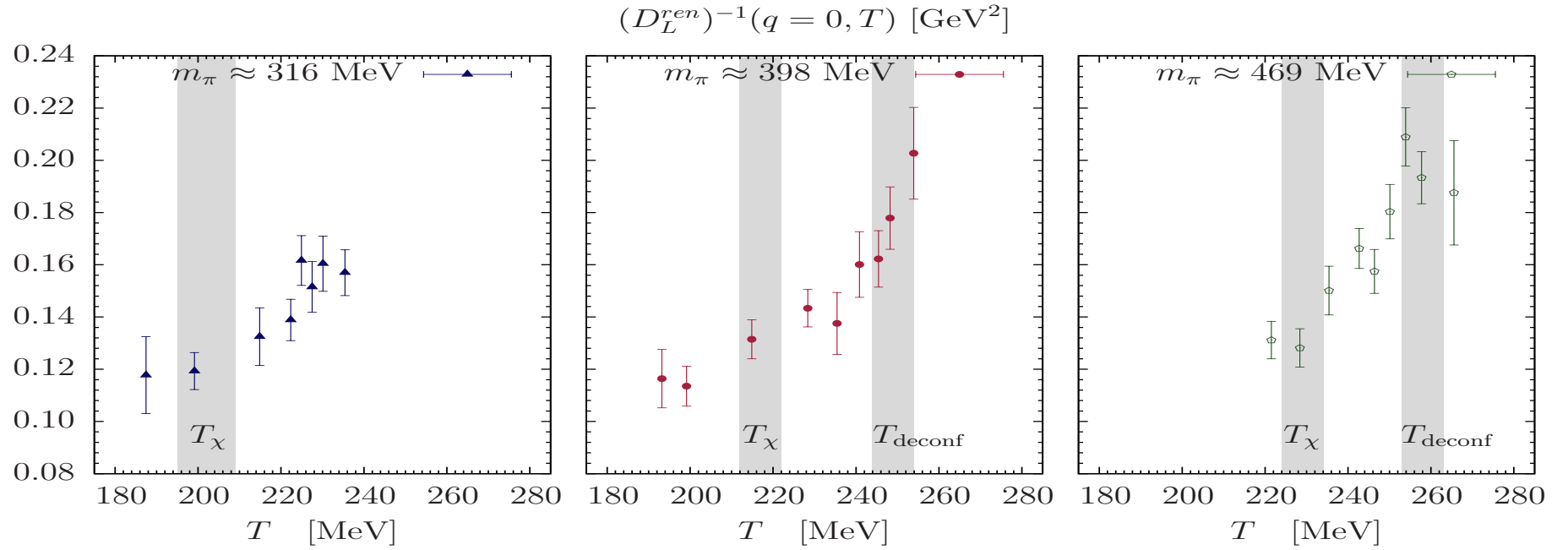
“Landau gauge gluon and ghost propagators from lattice

QCD with $N_f = 2$ twisted mass fermions at finite temperature”

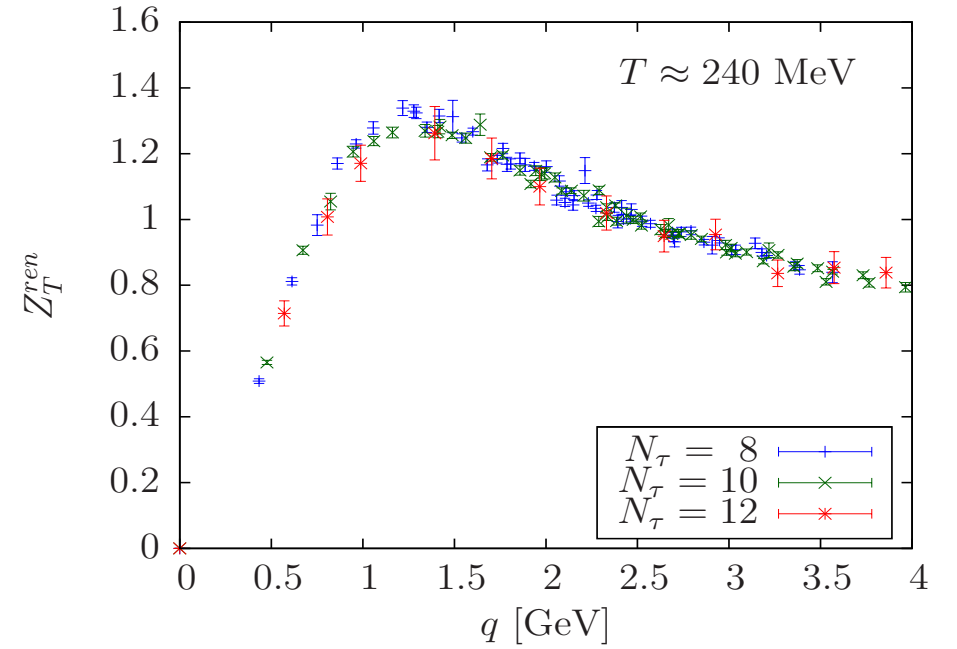
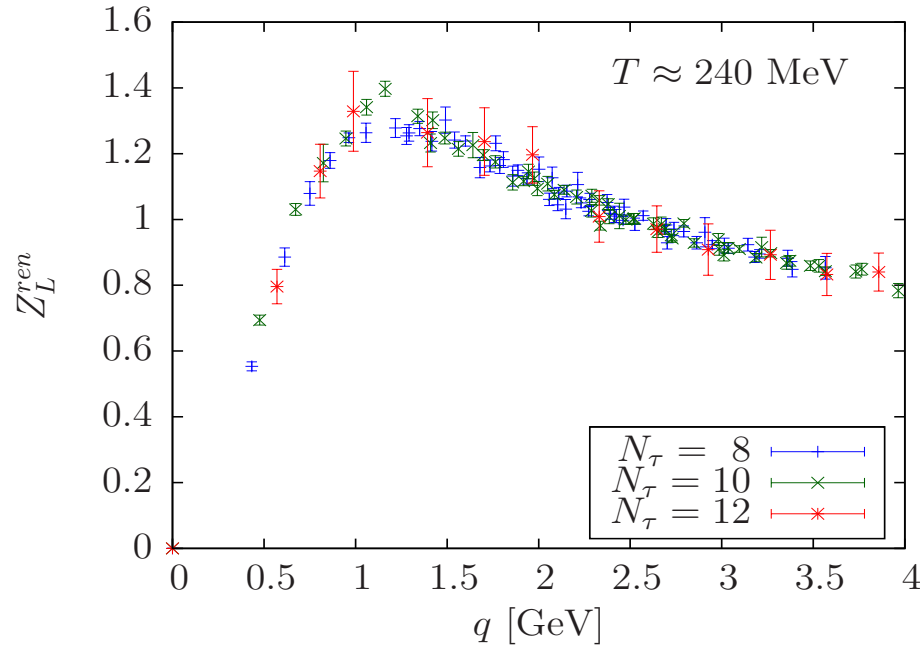
R. Aouane, F. Burger, E.-M. I., M. Müller-Preussker, A. Sternbeck

has provided the awaited unquenched propagators for twisted mass ensembles of the tmfT collaboration, in continuum parametrizations ready for comparison with DSE predictions in the momentum ranges :

- $0.4 \text{ GeV} < q < 3.0 \text{ GeV}$ for the gluon propagators (perfect fit !)
fitting parameter b^2 in the **Gribov-Stingl fit** compatible with zero
(no splitting in complex conjugate poles visible in this momentum range !)
- $0.4 \text{ GeV} < q < 4.0 \text{ GeV}$ for the ghost propagator (less good,
fit correct within few percent, a **mass term m_{gh} wouldn't help**).



The panels show the inverse renormalized longitudinal gluon propagator $(D_L^{ren})^{-1}$ at zero momentum for the three pion mass values indicated.



The left (right) panel shows a scaling check of the renormalized longitudinal (transversal) gluon dressing function at $T \simeq 240$ MeV by varying $N_\tau = 8, 10, 12$. (For $m_\pi \simeq 398$ MeV as an example !)

8. Perspectives of the Project: new Questions/new Partners for Dubna

1. Frascati: M. P. Lombardo, **one of the founders of tmfT !**

with recent contributions from A. Trunin (Dubna) and F. Burger (Berlin)

- closer characterization of the thermal crossover for $N_f = 2 + 1 + 1$
- topological susceptibility $\chi_{\text{top}}(T)$ for $N_f = 2 + 1 + 1$
- equation of state with the integral method for $N_f = 2 + 1 + 1$
- relation to axions as dark matter candidates (her talks/lectures at this workshop and at the Helmholtz Summer School 2017)

Emphasis on topology and EoS !

2. Heidelberg: J. M. Pawłowski and A. Rothkopf
with input (gauge fixing) from A. Trunin (Dubna)

- Usual Landau gauge gluon propagator for $N_f = 2 + 1 + 1$ (zero Matsubara frequency) was already evaluated for the ensembles with largest m_{π^\pm} .
- Then extended to non-zero Matsubara frequency, as input for
- .. spectral functions of gluons in Landau gauge, using a new Bayesian technique (Rothkopf)
- complex heavy-quark potential from Wilson-line correlators in Coulomb gauge (Rothkopf)

Emphasis on real-time properties of QGP !

3. Giessen and Jena : Ch. Fischer and A. Sternbeck

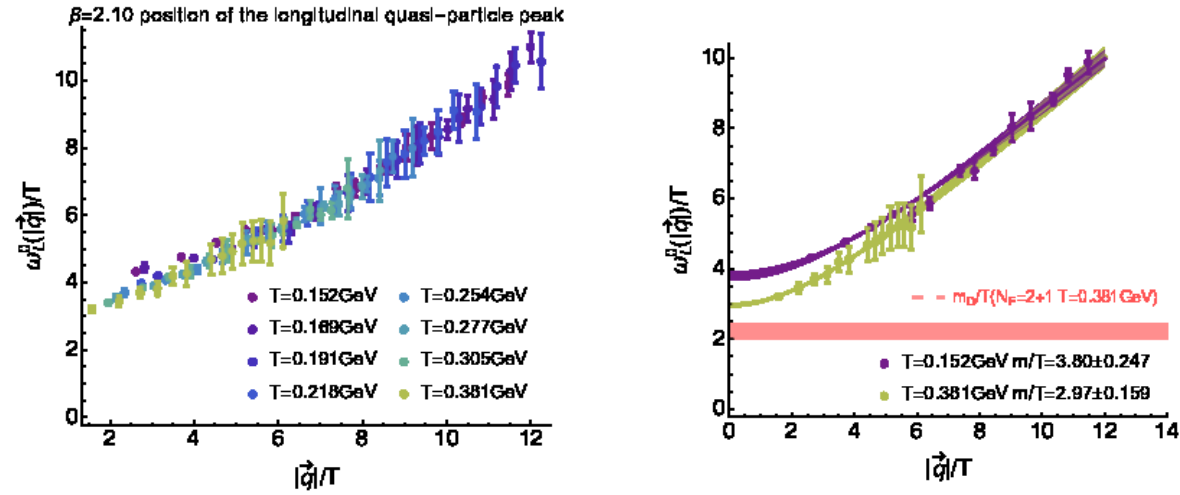
- exploring the phase structure through propagators and vertex functions
- effect of changing N_f via SDE
- extension to $\mu \neq 0$ through SDE
- thus finding the critical endpoint
- begin a systematic study of the quark propagator
- begin a systematic study of the gluon-quark vertex

Emphasis on phase structure for different N_f , for $\mu \neq 0$,
quark propagator and spectral function !

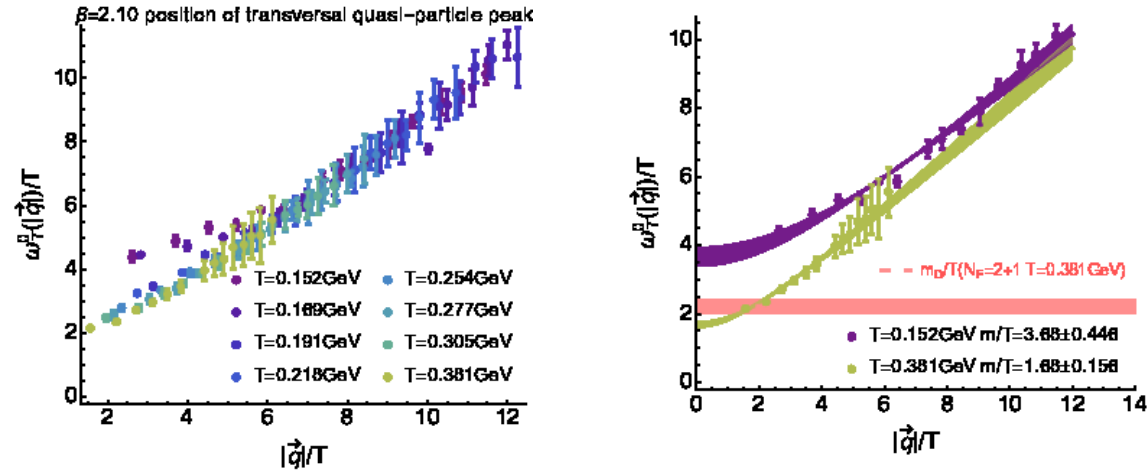
4. Giessen : L. von Smekal

- further studies of thermodynamic functions
- calculation of transport coefficients in QGP
- quark localization and spectral correlations among eigenvalues
(L. Holicki)
- meson screening functions and symmetry restoration
(B. Wellegehausen)

Empasis on the transition and properties inside the QGP !



Left: Momentum dependence of the **longitudinal quasi-particle peak** position at $\beta = 2.10$ (takes for small momenta a non-zero value). At the lowest temperatures within the hadronic phase one finds a larger intercept than in deconfinement. Right: Fit of the lowest and highest temperature curves with the ansatz $\omega_L^0(|\vec{q}|) = A\sqrt{B^2 + |\vec{q}|^2}$. (Quasiparticle mass defined as $m = AB$.) Debye mass from $N_f = 2+1$ lattice QCD is given for comparison.



Left: Momentum dependence of the **transversal quasi-particle peak** position at $\beta = 2.10$ (takes for small momenta a non-zero value). At the lowest temperatures within the hadronic phase one finds a larger intercept than in deconfinement. Right: Fit of the lowest and highest temperature curves with the ansatz $\omega_L^0(|\vec{q}|) = A\sqrt{B^2 + |\vec{q}|^2}$. (Quasiparticle mass defined as $m = AB$.) Debye mass from $N_f = 2+1$ lattice QCD is given for comparison.

9. Conclusions and Outlook

- The $N_f = 2$ crossover structure and its chiral limit are understood.
- The results are in fair agreement with other results with Wilson fermions and staggered fermions.
- As expected for a crossover, chiral and deconfinement crossover temperatures are not exactly coincident.
- The effect of unquenching on the gluon propagators (longitudinal and transversal) had been successfully predicted by DSE (Ch. Fischer et al.) on the basis of our previous quenched measurements.

- The effect of the first order transition in gluodynamics ($N_f = 0$) and of the crossover with $N_f = 2$, respectively, on the gluon propagators is manifest mainly in the longitudinal (electric) gluon propagator.
- A rapid drop of D_L at low momenta in the high-temperature phase starts not before $T > T_\chi$ (spans two orders of magnitude).
- A softer drop of D_T at low momenta begins already below T_χ (only one order of magnitude is spanned).
- The effect on the effective potential of the Polyakov loop (in the case of gluodynamics) results from the T dependence of both D_L and D_T
- An approx. calculation of the EoS has been tried, needs to be refined.
- Interesting is the effect of heavier quarks with the extension from $N_f = 2 \rightarrow N_f = 2 + 1 + 1$ (both available m_π from 470 MeV down to 210 MeV).

- Interesting cross-checks with Dyson-Schwinger (Ch. Fischer et al. concerning N_f dependence, extension to $\mu \neq 0$) and with other continuum approaches are possible through the propagators !
- Strange and charmed quarks are less important for finding the chiral transition temperature in the chiral limit ($m_u = m_d \rightarrow 0$).
- However, strange quarks influence the actual temperature of the crossover for (near to) physical light quark masses.
- Strange and charmed quarks are essential for the thermodynamics (EoS and other thermodynamic functions) of the high-temperature phase !
- Gluon and quark propagators should be studied in order to check the N_f dependence predicted by the Dyson-Schwinger approach. The prediction from $N_f = 0$ to $N_f = 2$ (unquenching) was successful.

- Color flux tubes, an object under study since the times of the quenched approximation, should be studied in order to study the influence of N_f (the number of dynamical quarks), in particular near to the deconfinement temperature.
- $N_f = 2 + 1 + 1$ configurations are under examination (together with quenched lattice configurations) in order to study the non-positive gluon spectral function in both cases (with J. Pawłowski and A. Rothkopf).
- $N_f = 2 + 1 + 1$ configurations are under examination to what extent the complex-valued $\bar{Q}Q$ potential can be extracted (with A. Rothkopf).
- The high-temperature dependence of the topological susceptibility $\chi_{\text{top}}(T)$ should be refined.
- Spacelike meson propagators (screening lengths in various channels).