Quarks and pions at finite chemical potential

A study of the QCD phase diagram with Dyson-Schwinger equations

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JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

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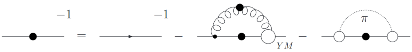
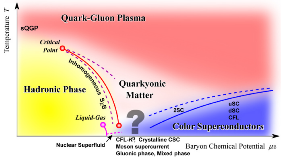
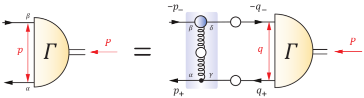
Outline

1. Introduction and Motivation
2. QCD phase diagram
3. Quarks at finite chem. potential
4. Pions at finite chem. potential
5. Summary and Outlook







QCD phase diagram

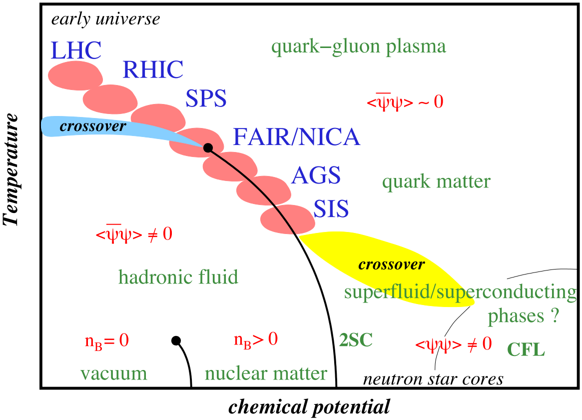
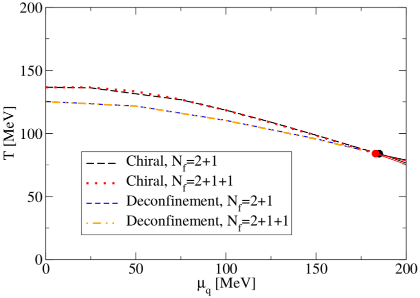


Figure: Schaefer and Wagner, Prog.Part.Nucl.Phys. 62 (2009) 381

Previous Studies

QCD phase diagram with $N_f = 2+1$ and $N_f = 2+1+1$ quark flavors:

Luecker and Fischer, Prog.Part.Nucl.Phys. 67(2), 200–205 (2012)
Fischer, Luecker and Welzbacher, Phys.Rev. D 90(3), 34022 (2014)



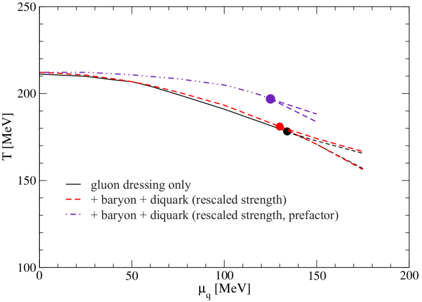
Previous Studies

Influence of baryonic effects in two-color-version of QCD:

Strodthoff, Schaefer and von Smekal, Phys.Rev. D 85, 074007 (2012)

Baryon effects on the location of QCD's CEP in DS approach:

Eichmann, Fischer and Welzbacher, Phys.Rev. D 93, 034013 (2016)

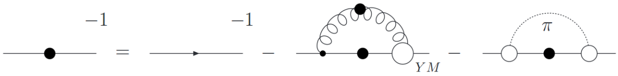


Motivation

Mesonic back-coupling effects in vacuum and finite T in DS (Dyson-Schwinger) approach:

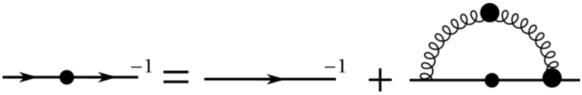
Fischer, Nickel and Wambach Phys.Rev. D 76, 094009 (2007)
Fischer and Williams Phys.Rev. D 78, 074006 (2008)
Fischer and Mueller Phys.Rev. D 84, 054013 (2011)

Open question: Influence of mesonic back-coupling effects onto QCD phase diagram and CEP



Goal of thesis: Investigation of mesons at finite chemical potential

Quark DSE



$$S^{-1}(p) = S_0^{-1}(p) + g^2 C_F Z_{1F} \int_q \gamma_\sigma S(q) \Gamma_\nu(p,q,k) D_{\sigma\nu}(k)$$

Missing components: Dressed quark-gluon vertex Γ_ν
Dressed gluon propagator $D_{\sigma\nu}$

Bare quark propagator (vacuum)

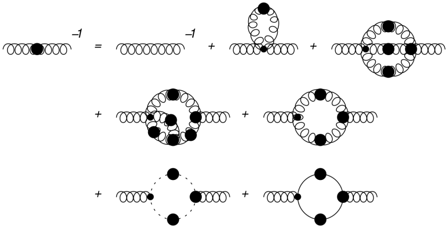
$S_0^{-1}(p) = Z_2(i\not p + \mathbb{1} Z_m m_r)$

Dressed propagator (vacuum)

$S^{-1}(p) = i\not p A(p) + \mathbb{1} B(p)$

Scalar B and vector A dressing function

Gluon DSE



Dressed gluon propagator (vacuum, Landau gauge)

$$D_{\sigma\nu}(k) = P^{\mathscr{T}}_{\sigma\nu}(k)\frac{Z(k)}{k^2}$$

Projector:

$$P^{\mathscr{T}}_{\sigma\nu}(k) = \left(\delta_{\sigma\nu} - \frac{k_\sigma k_\nu}{k^2}\right)$$

Gluon DSE:

$$D^{-1}_{\sigma\nu}(k) = D^{-1}_{0,\sigma\nu}(k) + \Pi^{YM}_{\sigma\nu}(k) + \Pi^{QL}_{\sigma\nu}(k)$$

$$\Pi^{QL}_{\sigma\nu}(k) = -\frac{g^2 Z_{1F}}{2}\sum_f^{N_f}\int_q tr_D\left[\gamma_\sigma S(q)\Gamma_\nu(p,q,k)S(p)\right]$$

Introducing the medium

Heat bath: T and μ_q introduce assigned direction $u = \left(\vec{0}, 1\right)$

- Quark vector dressing function splits up in spatial A and heat bath C part ($\not{p} \to \vec{p}\vec{\gamma},\, \tilde{\omega}_p\gamma_4$):

Dressed quark propagator (medium)

$$S^{-1}(p) = i\vec{p}\vec{\gamma}A(\omega_p, \vec{p}) + i\tilde{\omega}_p\gamma_4 C(\omega_p, \vec{p}) + \mathbb{1}B(\omega_p, \vec{p}) + \cancel{\vec{p}\vec{\gamma}\tilde{\omega}_p\gamma_4 D(\omega_p, \vec{p})}$$

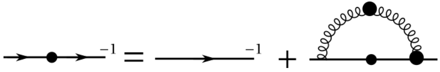
- Gluon splits up into a part transversal and a part longitudinal to heat bath ($P^{\mathscr{T}}_{\sigma\nu}(k) \to P^{T}_{\sigma\nu}(k),\, P^{L}_{\sigma\nu}(k)$)

Dressed gluon propagator (medium)

$$D_{\sigma\nu}(k; T) = \left(P^{T}_{\sigma\nu}(k)\frac{Z_T(k;T)}{k^2} + P^{L}_{\sigma\nu}(k)\frac{Z_L(k;T)}{k^2}\right)$$

Order parameter

Chirality:



Quark condensate

$$\langle \bar{\Psi}\Psi \rangle^f = -Z_m Z_2 \int_q tr_{DC} \left[S^f(p) \right]$$

Regularized quark condensate

$$\Delta_{f'f} = \langle \bar{\Psi}\Psi \rangle^{f'} - \frac{m_B^{f'}}{m_B^f} \langle \bar{\Psi}\Psi \rangle^f$$

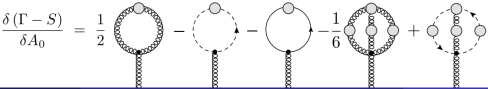
Deconfinement: $\langle L[A] \rangle \propto e^{-\frac{F_q}{T}}$, static quark free energy F_q

Dressed Polyakov loop[1]

$$\Sigma = -\int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i\varphi} \langle \bar{\Psi}\Psi \rangle_\varphi$$

Polyakov loop potential[2]

$$L[A] := \frac{1}{N_c} tr_C \left(\mathcal{P} e^{i \int d\tau A_0(\vec{x},\tau)} \right)$$

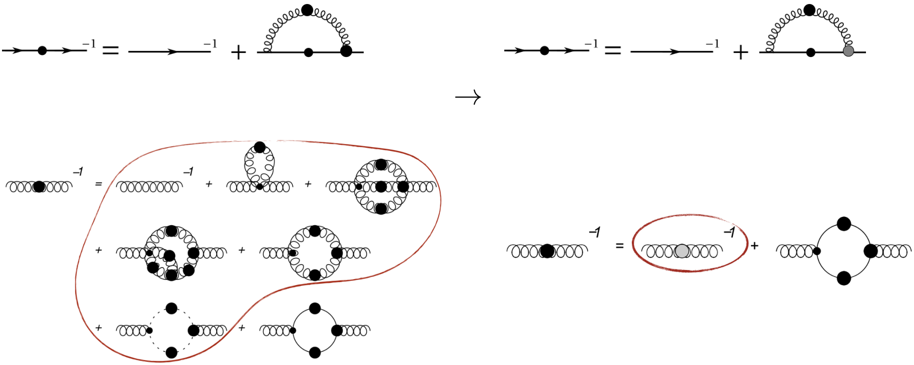


1 Synatschke, Wipf, Wozar, PRD 75, 114003 (2007); Bilgici, Bruckmann, Gattringer, Hagen, PRD 77 094007 (2008); CF, PRL 103 052003 (2009);

2 Braun, Gies, Pawlowski, PLB 684, 262 (2010); Braun, Haas, Marhauser, Pawlowski, PRL 106 (2011); Fister, Pawlowski, PRD 88 045010 (2013); CF, Fister, Luecker, Pawlowski, PLB 732 (2013)

Truncation scheme one: quark loop included

Gluon truncation:

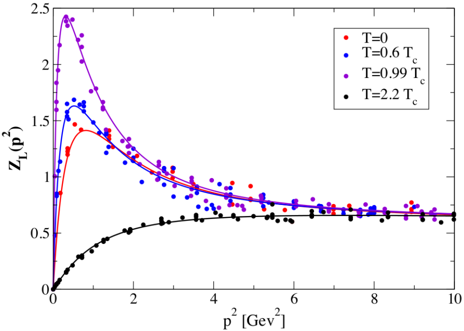


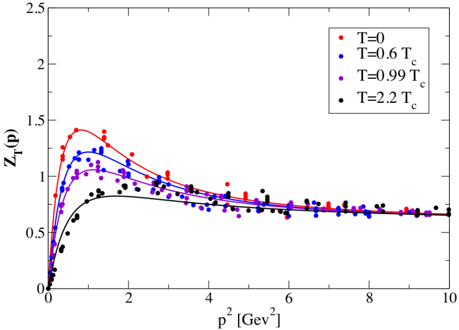
$$D^{-1}_{\sigma\nu}(k) = \left[D^{\text{lat. que}}_{\sigma\nu}(k)\right]^{-1} + \Pi^{QL}_{\sigma\nu}(k)$$

Fischer, Maas and Müller, Eur.Phys.J. C (2010) 68: 165

Truncation scheme one: quark loop included

T-dependent gluon propagator from quenched lattice simulations:





Crucial difference between transversal and longitudinal gluon $T_c = 277\,\text{MeV}$

Cucchieri, Maas, Mendes, PRD 75 (2007)
CF, Maas, Mueller, EPJC 68 (2010)
Aouane et al., PRD 85 (2012) 034501

Cucchieri, Mendes, PoS FACESQCD 007 (2010)
Silva, Oliveira, Bicudo, Cardoso, PRD 89 (2014) 074503
FRG: Fister, Pawlowski, arXiv:1112.5440

Truncation scheme one: quark loop included

Vertex truncation: STI and perturbative behavior at large momenta constrain vertex

$$\Gamma^f_\nu(p,q,k) = \gamma_\nu \Gamma(k^2)(\delta_{\nu,s}\Sigma_A + \delta_{\nu,4}\Sigma_C)$$

$$\Gamma(k^2) = \frac{d_1}{d_2 + k^2} + \frac{k^2}{\Lambda^2 + k^2}\left(\frac{\beta_0 \alpha(\mu'') \ln[k^2/\Lambda^2 + 1]}{4\pi}\right)^{2\delta}$$

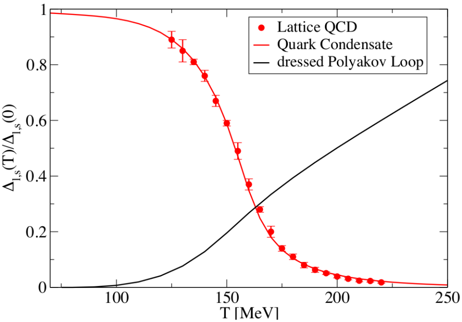
Considers first Ball-Chiu structure: $\Sigma_X = \frac{X(\vec{p}^2,\omega_p)+X(\vec{q}^2,\omega_q)}{2}$, $X \in \{A, B\}$

Abelian WTI: from approximated STI
Perturbation theory
Infrared ansatz: d_2 fixed to match gluon input, d_1 fixed via quark condensate

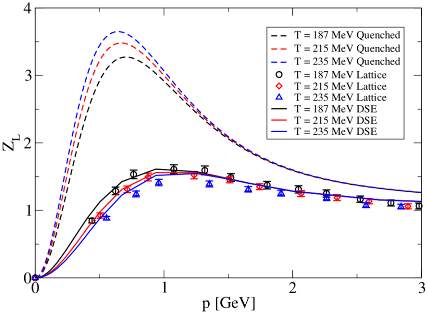
Fischer and Mueller Phys.Rev. D 80, 074029 (2009)

Truncation scheme one: quark loop included

Determination of d_1 and prediction for unquenched gluon:



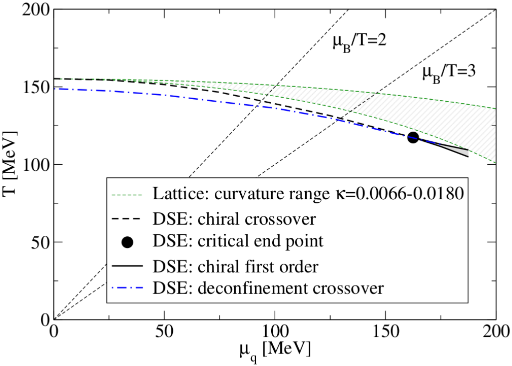
Lattice: Borsanyi et al. [Wuppertal-Budapest], JHEP 1009(2010) 073
DSE: CF, Luecker, PLB 718 (2013) 1036, CF, Luecker, Welzbacher, PRD 90 (2014) 034022

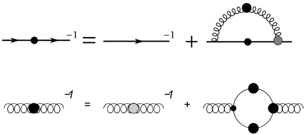


Lattice: Aouane, Burger, Ilgenfritz, Müller-Preussker, Sternbeck, PRD D87 (2013), [arXiv:1212.1102]
DSE: CF, Luecker, PLB 718 (2013) 1036, [arXiv:1206.5191]

Quantitative agreement: DSE results verified by lattice

Phase diagram for included quark loop





CEP at large μ_q

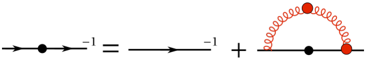
CF, Luecker, PLB 718 (2013) 1036,
CF, Fister, Luecker, Pawlowski, PLB 732 (2014) 273
CF, Luecker, Welzbacher, PRD 90 (2014) 034022

Extrapolated curvature from lattice

Kaczmarek at al. PRD 83 (2011) 014504,
Endrodi, Fodor, Katz, Szabo, JHEP 1104 (2011) 001
Cea, Cosmai, Papa, PRD 89 (2014), PRD 93 (2016)
Bonati et al., PRD 92 (2015) 054503
Bellwied et al. PLB 751 (2015) 559

FRG and DSE results combined:
CEP above $\mu_B/T > 2$

Truncation scheme two: simple model

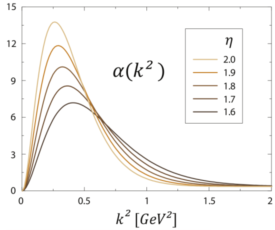


$$\alpha(k^2) = \alpha_{IR}(k^2/\Lambda^2, \eta) + \alpha_{UV}(k^2)$$

$$\Lambda = 0.74, \ \eta = 1.85 \pm 0.2$$

Combining vertex Γ and gluon Z to renormalization-group invariant effective coupling

$$\alpha(\mu) D_{\sigma\nu}(k) \Gamma^f_\nu(p,q,k) \propto \alpha(k^2) \frac{P^{\mathcal{T}}_{\sigma\nu}(k)}{k^2} \gamma_\nu$$

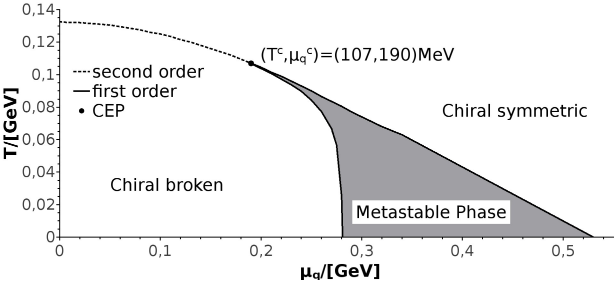


Maris-Tandy ansatz:

Simple ansatz, quark flavor decouple

Maris and Tandy, Phys.Rev. C 60, 055214 (1999)

Phase diagram for simple model

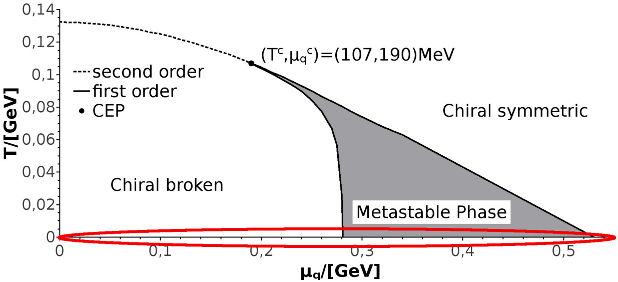


effective model for interaction, **chiral limit**, quark flavor decouple

Precise determination of CEP numerically challenging,
agreement for coexistence curves with previous calculations

Qin, Chang, Chen, Liu and Roberts, Phys.Rev.Lett. 106, 172301 (2011)

Phase diagram for simple model



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Precise determination of CEP numerically challenging,
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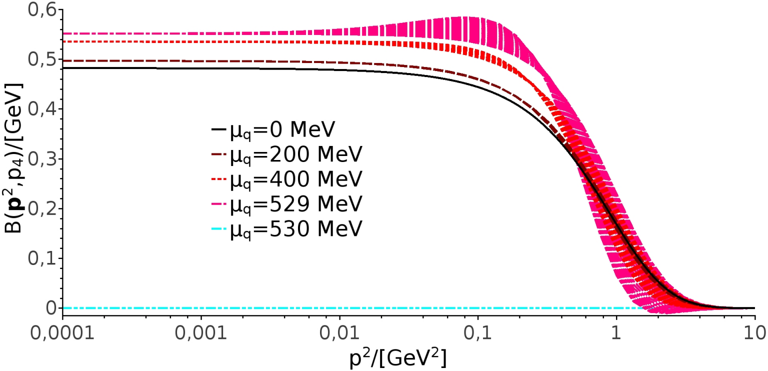
Qin, Chang, Chen, Liu and Roberts, Phys.Rev.Lett. 106, 172301 (2011)

Results for quark propagator at finite chemical potential

Scalar dressing function:



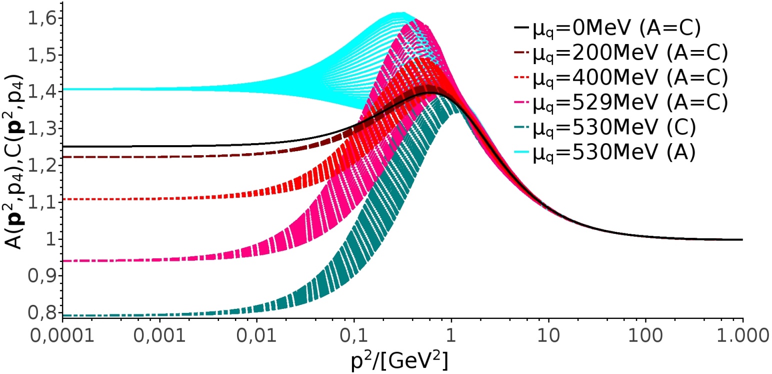
$$S^{-1}(p) = i\vec{p}\vec{\gamma}A(\vec{p}^2, p_4) + i\tilde{p}_4\gamma_4 C(\vec{p}^2, p_4) + \mathbb{1}B(\vec{p}^2, p_4)$$

$$\langle\bar{\Psi}\Psi\rangle \propto \int_q \frac{B(\vec{p}^2, p_4)}{D(\vec{p}^2, p_4)}$$

Chiral phase transition point $\mu_q^c = 530\,\text{MeV}$

Results for quark propagator at finite chemical potential

Vector dressing functions:



$$S^{-1}(p) = i\vec{p}\vec{\gamma}A(\vec{p}^2, p_4) + i\tilde{p}_4\gamma_4 C(\vec{p}^2, p_4) + \mathbb{1}B(\vec{p}^2, p_4)$$

Degeneration of vector dressing function only in chiral limit

Silver Blaze property

Silver Blaze property:

$T = 0$, $\mu_q \neq 0$: Partition function and observables independent from μ_q

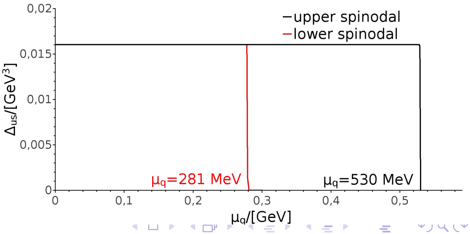
$\Leftrightarrow \mu_q <$ mass gap of the system δ

T. D. Cohen, Phys. Rev. Lett. 91 , 222001 (2003)

T. D. Cohen, arXiv:hep-ph/0405043 (2004)

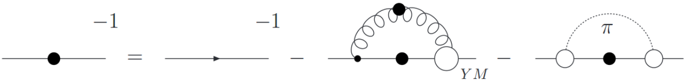
$$\delta = \begin{cases} \frac{m_B}{3} & m_B = \text{lightest baryon} \\ \frac{m_M}{2} & m_M = \text{lightest meson} \end{cases}$$

Reg. quark condensate at $\mu_q \neq 0$:



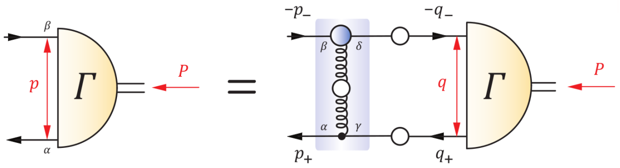
Reminder

Wanted: Influence of Pion back-coupling onto QCD phase diagram and CEP



First step: Investigate Pion at finite chemical potential

Homogeneous BSE



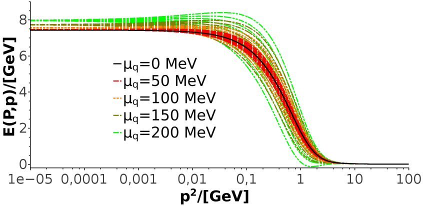
In Rainbow-ladder approximation with effective interaction $\alpha(k^2)$:

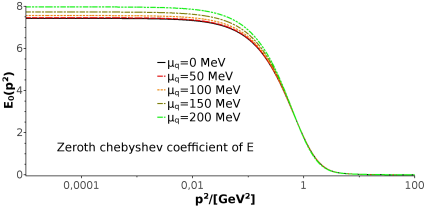
$$\Gamma_\pi(P,p) = -4\pi Z_2^2 C_F \int_q \frac{\alpha(k^2)}{k^2} P^{\mathcal{T}}_{\mu\nu}(k)\gamma^\mu S(q_+)\Gamma_\pi(P,q)S(-q_-)\gamma^\nu$$

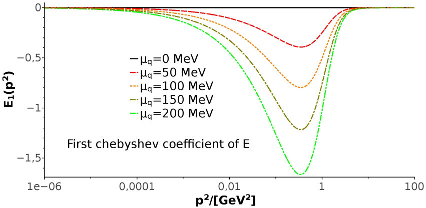
Pion amplitude in vacuum

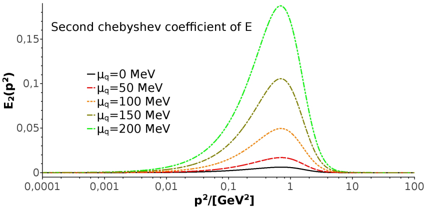
$$\Gamma_\pi(P,p) = \gamma_5\left[-iE(P,p) + \not{P}F(P,p) + \not{p}(Pp)G(P,p) + \left[\not{P},\not{p}\right]H(P,p)\right]$$

Pion amplitude





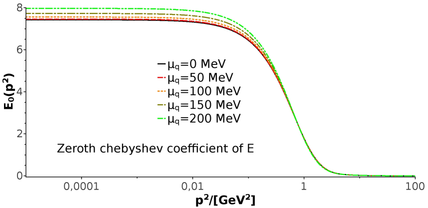


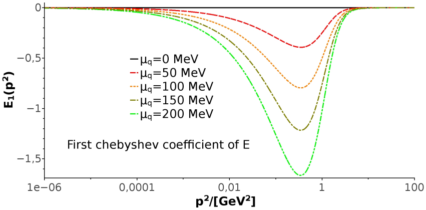


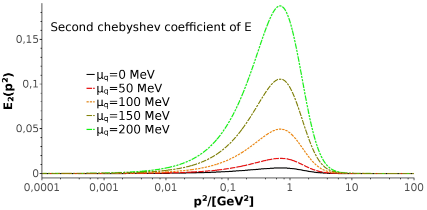
Pion amplitude

Chebyshev expansion:

$$E(P^2, p^2, \hat{P}\hat{p}) \approx \sum_{j=0}^{3} E^j(P^2, p^2)\, T_j(\hat{P}\hat{p})$$







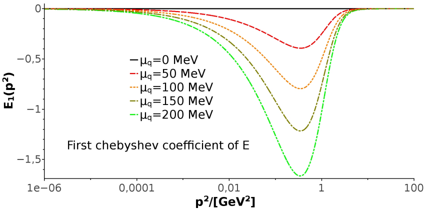
Pion amplitude

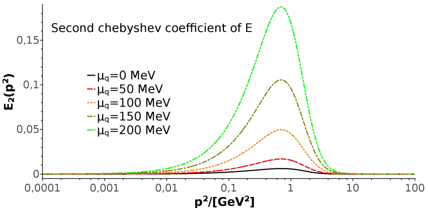
Chebyshev expansion:

$$E(P^2, p^2, \hat{P}\hat{p}) \approx \sum_{j=0}^{3} E^j(P^2, p^2)\, T_j(\hat{P}\hat{p})$$

Charge-conjugated pion amplitude:

$$\bar{\Gamma}_\pi(P,p) = \left[C\Gamma_\pi(P,-p)C^{-1}\right]^T$$

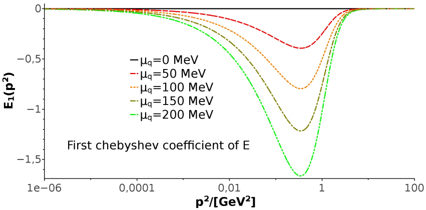




Pion amplitude

Chebyshev expansion:

$$E(P^2, p^2, \hat{P}\hat{p}) \approx \sum_{j=0}^{3} E^j(P^2, p^2)\, T_j(\hat{P}\hat{p})$$



Charge-conjugated pion amplitude:

$$\bar{\Gamma}_\pi(P, p) = \left[C \Gamma_\pi(P, -p) C^{-1} \right]^T$$

Pion: C-Parity $\wedge$ $J^{PC} = 0^{-+}$

$\rightarrow$ odd chebyshev coefficients vanish

$\rightarrow$ μ_q breaks C-Parity

Pion decay constant

Pion decay constant in vacuum

$$f_\pi P^\mu = Z_2 N_c \int_q tr_D \left[\gamma_5 \gamma^\mu S(q_+)\Gamma_\pi(q,P)S(-q_-)\right]$$

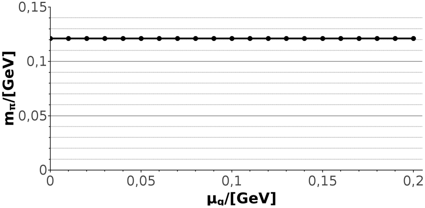
From vacuum to finite chemical potential:

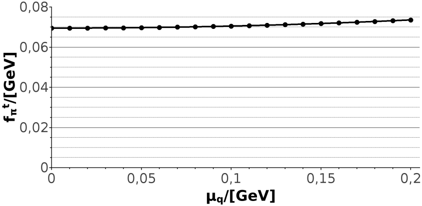
$$f_\pi P^\mu \xrightarrow{\mu_q>0} \left(f_\pi^t P_{\mu\nu}^{\mathscr{L}}(v) + f_\pi^s P_{\mu\nu}^{\mathscr{T}}(v) \right) P^\nu$$

Longitudinal projector $P_{\mu\nu}^{\mathscr{L}}(v) = v_\mu v_\nu$ with $v = (\vec{0}, 1)$

Pion properties

$\Gamma_\pi(P,p) = \gamma_5 E(P,p)$, quark mass $m_R = 3.7$ MeV at $\mu = 19\,\text{GeV}$:



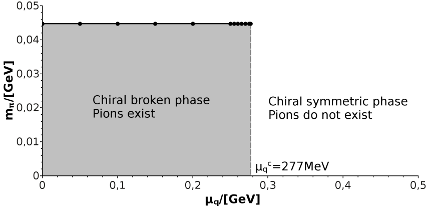


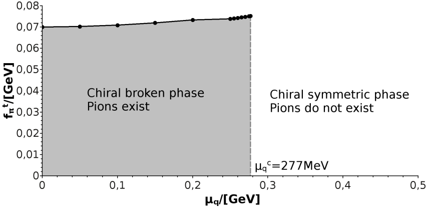
Full tensor structure, quark mass $m_R = 3.7$ MeV at $\mu = 19\,\text{GeV}$:
$m_\pi(\mu_q = 0) = 137.4$ MeV and $f_\pi(\mu_q = 0) = 92$ MeV

Silver Blaze property fulfilled: mass gap $= \frac{m_M}{2} = 60.5$ MeV

Pion properties

$\Gamma_\pi(P,p) = \gamma_5 E(P,p)$, quark mass $m_R = 0.5$ MeV at $\mu = 19\,$GeV:





$\mu_q^c = 277$ MeV corresponds also to phase transition point of quark condensate

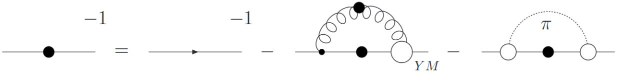
Qualitative agreement with (simpler) truncations:

Roberts, Phys.Part.Nucl. 30:223-257 (1999)
Roberts and Schmidt, Prog.Part.Nucl.Phys. 45(1) 1-103 (2000)
Liu, Chao, Chang and Wei, Chinese Physics Letters, Volume 22, Number 1

Summary and outlook

Summary:

- QCD phase diagram for $N_f = 2 + 1$ quark flavors
- Quark calculation for $\mu_q \neq 0$
  - Quark condensate fulfills Silver Blaze property
- Pion calculation for $\mu_q \neq 0$, one tensor structure
  - Introduction of chemical potential breaks C-Parity of pions
  - Pion mass and decay constant fulfill Silver Blaze property
  - Pion do not exist above chiral phase transition point



Outlook:

Thank you for your attention!

Origin of Dyson-Schwinger equations

Generating functional:

$$Z[J] = \mathcal{N} \int \mathcal{D}\varphi e^{i(S[\varphi] - \int_x \varphi(x) J(x))}$$

Local translational invariance: $\varphi(x) \to \varphi'(x) = \varphi(x) + \epsilon(x)$

Master-DSE for 1PI Green-functions:

$$\Gamma'_x(\tilde{\varphi}) = \frac{\delta S}{\delta \varphi(x)} \left(\tilde{\varphi} + \int_y \Delta_{.y}[\tilde{\varphi}] \frac{i\delta}{\delta \tilde{\varphi}(y)} \right)$$

Meaning of DSE: Quantum equations of motion for n-point function
Infinite tower of coupled integral equations
Ab initio, if solved completely and self-consistently

Connection to thermodynamics

Grand canonical partition function:

$$Z_{GC}(\beta,\mu_q) = tr\left(e^{-\beta(\hat{H}-\mu_q\hat{N})}\right) = \int_x \langle x|\, e^{-\beta(\hat{H}-\mu_q\hat{N})}\,|x\rangle$$

Particle number operator: $\hat{N} = \int d^4x \hat{\bar{\Psi}}\gamma_0\hat{\Psi}$

Generating functional for finite space-time:

$$Z(x',\tau';x,\tau) = \langle x'|\, e^{-\hat{H}(\tau'-\tau)}\,|x\rangle = \mathcal{N}\int_{x(\tau)}^{x'(\tau')}\mathscr{D}x(\tau'')e^{-S_E(\tau,\tau')}$$

In-medium generating functional:

$$Z_{GC}(\beta,\mu_q) = \mathcal{N}\int_{x(0)=x(\beta)}\mathscr{D}x(\tau)e^{-S_E(0,\beta)+\mu_q\int_0^\beta d\tau\int d^3x\hat{\bar{\Psi}}\gamma_0\hat{\Psi}}$$

Connection to thermodynamics

Finite integration interval and different periodicity conditions

$$\begin{aligned} \Psi(x,\tau) &= -\Psi(x,\tau+\beta) \qquad \text{fermions} \\ \Phi(x,\tau) &= +\Phi(x,\tau+\beta) \qquad \text{bosons} \end{aligned}$$

yield discreet four momentum components

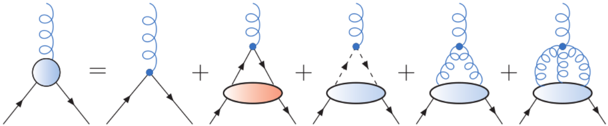
$$p \rightarrow (\vec{p}, \tilde{\omega}_p = \pi T(2n_p + \eta) + i\mu_q), \qquad \eta = \begin{cases} 1 & \text{fermions} \\ 0 & \text{bosons} \end{cases}$$

and a sum over the so called Matsubara-frequencies

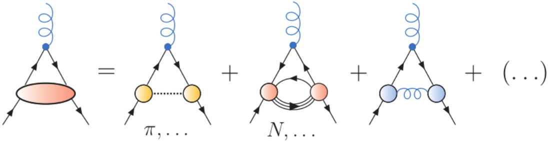
$$\int \frac{d^4q}{(2\pi)^4} \rightarrow T \sum_{n_q} \int \frac{d^3q}{(2\pi)^3}$$

Skeleton expansion

DSE of the fully dressed quark-gluon vertex:



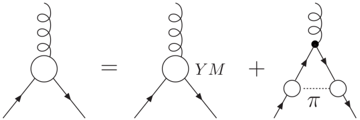
Skeleton expansion in terms of hadronic contributions:



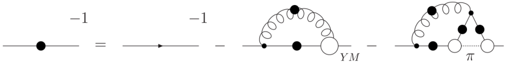
$\longrightarrow$ Separation of hadronic terms and Yang-Mills terms

Skeleton expansion

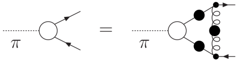
Only mesonic contributions:



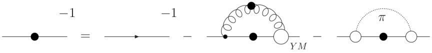
Inserting vertex into quark:

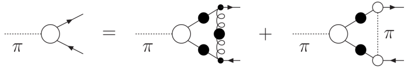


BSE:



Assumption: Only Yang-Mills part present in BSE ⇒ rewrite Quark DSE by inserting DSE into second diagram





Approximation justified if BSE vertex function with and without pion interaction term do not differ strongly

Fischer, Nickel and Wambach, Phys.Rev. D 76(9) (2007)

Lattice fit functions

Gluon:

$$Z_{T,L}^{\text{lat que}}(k^2) = \frac{x}{(x+1)^2}\left[\left(\frac{\hat{c}}{x+a_{T,L}(T)}\right)^{b_{T,L}(T)} + x\left(\frac{\beta_0\alpha}{4\pi}\ln(1+x)\right)^{\gamma}\right]$$

Parameters: $x = \frac{k^2}{\Lambda^2}$, $\beta_0 = \frac{11N_c - 2N_f}{3}$, $\hat{c} = 5.87$, $\Lambda = 1.4\,\text{GeV}$, $\gamma = -\frac{13}{22}$,
$\alpha(\mu'') = \frac{g^2}{4\pi} = 0.3$

$$a_L(t) = \begin{cases} 0.595 - 0.9025 \cdot t + 0.4005 \cdot t^2 & \text{if } t < 1 \\ 3.6199 \cdot t - 3.4835 & \text{if } t > 1 \end{cases},$$

$$a_T(t) = \begin{cases} 0.595 + 1.1010 \cdot t^2 & \text{if } t < 1 \\ 0.8505 \cdot t - 0.2965 & \text{if } t > 1 \end{cases}$$

$t = \frac{T}{T_c}$, $T_c = 277\,\text{MeV}$

Lattice fit functions

Gluon:

$$Z_{T,L}^{\text{lat que}}(k^2) = \frac{x}{(x+1)^2}\left[\left(\frac{\hat{c}}{x+a_{T,L}(T)}\right)^{b_{T,L}(T)} + x\left(\frac{\beta_0\alpha}{4\pi}\ln(1+x)\right)^{\gamma}\right]$$

Parameters: $x = \frac{k^2}{\Lambda^2}$, $\beta_0 = \frac{11N_c - 2N_f}{3}$, $\hat{c} = 5.87$, $\Lambda = 1.4\,\text{GeV}$, $\gamma = -\frac{13}{22}$, $\alpha(\mu'') = \frac{g^2}{4\pi} = 0.3$

$$b_L(t) = \begin{cases} 1.355 - 0.5741 \cdot t + 0.3287 \cdot t^2 & \text{if } t < 1 \\ 0.1131 \cdot t + 0.9319 & \text{if } t > 1 \end{cases},$$

$$b_T(t) = \begin{cases} 1.355 + 0.5548 \cdot t^2 & \text{if } t < 1 \\ 0.4296 \cdot t + 0.7103 & \text{if } t > 1 \end{cases}$$

$t = \frac{T}{T_c}$, $T_c = 277\,\text{MeV}$

Lattice fit functions

Vertex:

$$\Gamma_\nu^f(p,q,k) = \gamma_\nu \Gamma(x)\left(\delta_{\nu,s}\Sigma_A + \delta_{\nu,4}\Sigma_C\right)$$

$$\Gamma(x) = \frac{d1}{d2+k^2} + \frac{x}{1+x}\left(\frac{\beta_0\alpha(\mu'')\ln[x+1]}{4\pi}\right)^{2\delta}$$

$$\Sigma_X = \frac{X(\vec{p}^2,\omega_p) + X(\vec{q}^2,\omega_q)}{2}$$

Parameters: $x = \frac{k^2}{\Lambda^2}$, $\beta_0 = \frac{11N_c - 2N_f}{3}$, $\alpha(\mu'') = \frac{g^2}{4\pi} = 0.3$, $\delta = \frac{-9N_c}{44N_c - 8N_f}$,
$\Lambda = 1.4\,\mathrm{GeV}$, $d_2 = 0.5\,\mathrm{GeV}^2$,

$$d_1 = \begin{cases} 4.6 & \text{quenched theory} \\ 8.05 & \text{unquenched theory with } N_f = 2+1 \text{ quark flavors} \end{cases}$$

Thermal mass

Regularized quark loop:

$$\Pi^{reg}_{\sigma\nu}(k) = \left[\delta_{\sigma\alpha}\delta_{\nu\beta} - \delta_{\sigma\nu}P^{\mathscr{L}}_{\alpha\beta}(k)\right]\Pi^{QL}_{\alpha\beta}(k)$$

$$\Pi^{reg}_{T/L}(k) = \frac{1}{2/1k^2}\Pi^{reg}_{\sigma\nu}(k)P^{T/L}_{\sigma\nu}(k)$$

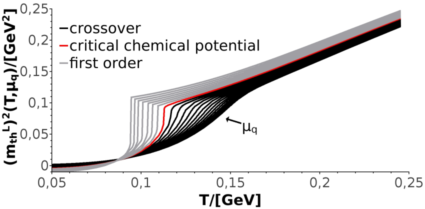
Separation of regular part and thermal mass:

$$\Pi^{reg}_{T/L}(k) = \Pi^{regular}_{T/L}(k) + \frac{2\left[m^{th}_{T/L}(T,\mu_q)\right]^2}{k^2}$$

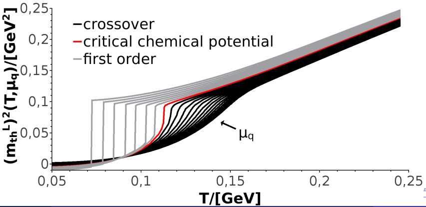
$$\left[m^{th}_{T/L}(T,\mu_q)\right]^2 := \frac{1}{2}\Pi^{reg}_{T/L}(k)\vec{k}^2\Big|_{\omega_k=0,\vec{k}^2\longrightarrow 0}$$

Thermal mass

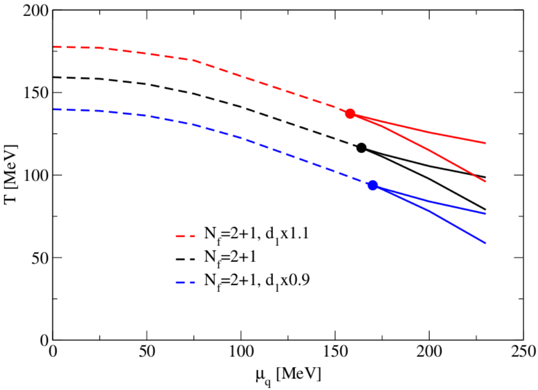
Upper spinodal:



Lower spinodal:



d_1 dependency



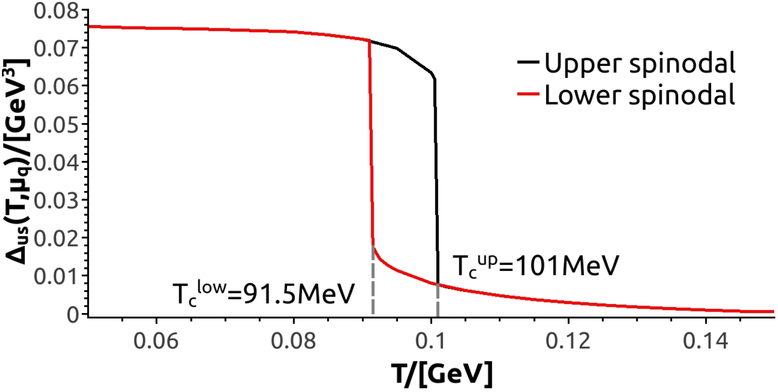
Maris-Tandy interaction

$$\alpha(k^2) = \pi \frac{\eta^7}{\Lambda^4} k^4 e^{-\eta^2 \frac{k^2}{\Lambda^2}} + \frac{2\pi\gamma_m \left(1 - e^{-k^2/\Lambda_t^2}\right)}{\ln\left[e^2 - 1 + \left(1 + k^2/\Lambda_{QCD}^2\right)^2\right]}$$

Parameters: $\gamma_m = \frac{12}{11N_c - 2N_f}$, $\Lambda_t = 1\,\mathrm{GeV}$, $\Lambda_{QCD} = 0.234\,\mathrm{GeV}$

- Scale Λ adjusted to observables like f_π
- Quark masses m_u, m_u, m_s from m_π, m_K
- α_{UV} from perturbative theory

Spinodals


0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
Δus(T,μq)/[GeV³]
Upper spinodal
Lower spinodal
Tc^up=101MeV
Tc^low=91.5MeV
0.06
0.08
0.1
0.12
0.14
T/[GeV]

Curvature

Curvature $\kappa =$ first coefficient in taylor series expansion of transition line in terms of $\frac{\mu_q}{T}$

$$\frac{T^c(\mu_q)}{T_0^c} = 1 - \kappa \left(\frac{\mu_q}{T_0^c}\right)^2 + O\left[\left(\frac{\mu_q}{T_0^c}\right)^4\right]$$

$T_0^c =$ transition temperature for $\mu_q = 0$

Remark: Curvature depends on choice of pesudo-critical temperature definition in crossover region

Calculation method

Until now only truncation scheme two (effective interaction)

Changes from medium to finite chemical potential:

- $(\vec{p}, \tilde{\omega}_p = \omega_p + i\mu_q) \longrightarrow (\vec{p}, \tilde{p}_4 = p_4 + i\mu_q)$
- $\frac{T}{(2\pi)^3} \sum_{\omega_q} \int d\vec{q}^2 \int d\Omega_{3D} \rightarrow \begin{cases} \frac{1}{(2\pi)^4} \int dq_4 \int d\vec{q}^2 \int d\Omega_{3D} & \text{Method A} \\ \frac{1}{(2\pi)^4} \int dq^2 \int d\Omega_{4D} & \text{Method B} \end{cases}$

Method A: Separate integration for spacial and temporal part
Vacuum and medium limit does not work

Method B: Uses hyperspherical coordinates instead
Vacuum and medium limit work perfectly

Silver Blaze Property

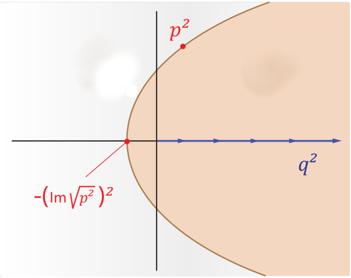
$$\langle\bar{\Psi}\Psi\rangle \sim \int_q S(\vec{q}, q_4 + i\mu_q) \overset{q_4 \to q_4 + i\mu_q}{=} \int_q S(\vec{q}, q_4) \sim \langle\bar{\Psi}\Psi\rangle_{vac}$$

Substitution possible $\Leftrightarrow$ no singularity between 0 and $i\mu_q$ in complex-p_4-plane

Complex quark

$$S^{-1}(p \pm i\frac{m_\pi}{2}) = S_0^{-1}(p \pm i\frac{m_\pi}{2}) + Z_2^2 C_F \int_q \gamma_\sigma S(q) \gamma_\nu P_{\sigma\nu}^{\mathscr{T}}(\tilde{k}) \frac{\alpha(\tilde{k}^2)}{\tilde{k}^2}$$

Complex gluon momentum: $\tilde{k} = k \mp i\frac{m_\pi}{2}$



$$p_\pm^2 = \left(p \pm \frac{P}{2}\right)^2 \quad \text{and} \quad P = \left(\vec{0}, im_\pi\right)$$

$$\longrightarrow \quad p_\pm^2 = p^2 - \frac{m_\pi^2}{4} \pm im_\pi \sqrt{p^2}$$

$$f(p_0) = \frac{\oint_\gamma \frac{f(p)}{p-p_0} dp}{\oint_\gamma \frac{1}{p-p_0} dp}$$

Parametrization of the pion

Momentum parametrization:

$$P = (0, 0, im_\pi, 0)$$
$$p = (\ |\vec{p}|(\ 0\ ,\ 0\ ,\ 1\ )\ ,\ p_4\ )$$
$$q = (\ |\vec{q}|(\ 0\ ,\ \sin(\Psi_q)\ ,\ \cos(\Psi_q)\ )\ ,\ q_4\ )$$

$$|\vec{p}| = |p| \sin(\theta_p)$$
$$p_4 = |p| \cos(\theta_p)$$

Integral parametrization:

$$\int \frac{d^4q}{(2\pi)^4} = \frac{1}{16\pi^3} \int_{\sigma^2}^{\Lambda^2} dq^2 q^2 \int_0^\pi d\Psi_q \sin(\Psi_q) \int_0^\pi d\theta_q \sin^2(\theta_q)$$

Chebyshev expansion

Pion amplitude: $\Gamma_\pi(P,p) = \sum_k^4 f_k(P,p)\tau_k(P,p)$

Chebyshev expansion of dressing function $f_k(P^2, p^2, z_p)$**:**

$$f_k(P^2, p^2, z_p) \approx \sum_{j=0}^{\tilde{N}}{}' f_k^j(P^2, p^2) T_j(z_p)(i)^j$$

Chebyshev polynomials: $T_n(z_p) = \cos(n\theta_p) = \cos(n \arccos(z_p))$ with $z_p \in [-1, 1]$

Recursive formula: $T_j(z_p) = 2z_p T_{j-1}(z_p) - T_{j-2}(z_p)$ with first polynomials $T_0(z_p) = 1$ and $T_1(z_p) = z_p$

C-Parity

Pion amplitude: $\Gamma_\pi(P,p) = \sum_k^4 f_k(P,p)\tau_k(P,p)$

Charge-conjugated pion amplitude: $\bar{\Gamma}_\pi(P,p) = \left[C\Gamma_\pi(P,-p)C^{-1}\right]^T$

C-Parity: $\bar{\Gamma}_\pi(P,p) = \eta_c\Gamma_\pi(P,p)$ with eigenvalue $\eta_c = \pm 1$

$$\begin{aligned}\bar{\tau}_k(P,p) &= \left[C\tau_k(P,-p)C^{-1}\right]^T = \xi_k\tau_k(P,p) \qquad \longrightarrow \qquad \eta_c = \xi_k\tilde{\xi}_k \quad \forall k\\ \bar{f}_k(P,p) &= \left[Cf_k(P,-p)C^{-1}\right]^T = \tilde{\xi}_k f_k(P,p)\end{aligned}$$

Pion: $\eta_c = +1$

Cheby. exp. $\longrightarrow$ $\bar{f}_k(P^2,p^2,z_p) = f_k(P^2,p^2,-z_p) \overset{!}{=} f_k(P^2,p^2,z_p)$

C-Parity of the pion

Chebyshev expansion of dressing function $E(P^2, p^2, \hat{P}\hat{p})$**:**

$$E(P^2, p^2, \hat{P}\hat{p}) \approx \sum_{j=0}^{3} E^j(P^2, p^2) T_j(\hat{P}\hat{p})$$

Charge-conjugated pion amplitude: $\bar{\Gamma}_\pi(P, p) = \left[C\Gamma_\pi(P, -p)C^{-1}\right]^T$

$$\Gamma_\pi = -i\gamma_5 E(P^2, p^2, \hat{P}\hat{p}) \quad \Rightarrow \quad \bar{\Gamma}_\pi = -i\gamma_5 E(P^2, p^2, -\hat{P}\hat{p})$$

Pion properties: $J^{PC} = 0^{-+} \quad \Rightarrow \quad \bar{\Gamma}_\pi(P, p) = \Gamma_\pi(P, p)$

$$E^{(2j+1)}(P^2, p^2) \stackrel{!}{=} 0$$

Power-iteration

BSE:

$$\hat{K}(P^2)\,|\Gamma_n(P,p)\rangle = \lambda_n(P^2)\,|\Gamma_{n+1}(P,p)\rangle$$

On-shell condition: $P^2 = -M_j^2 \quad \longrightarrow \quad \lambda(P^2) = 1$

Iteration number: n

Eigenvalue:

$$\lambda_n(P^2) = \frac{\langle \Gamma_n(P,p) |\, \Gamma_{n+1}(P,p)\rangle}{\langle \Gamma_n(P,p) |\, \Gamma_n(P,p)\rangle}$$

$\lambda_n(P^2) \xrightarrow{n\to\infty} \lambda(P^2)$

Pion propagator

Pion velocity:

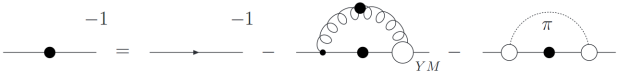
$$u^2 = \left(\frac{f_\pi^s}{f_\pi^t}\right)^2$$

Pion dispersion relation:

$$\omega^2 = u^2 \left(\vec{P}^2 + m_\pi^2\right)$$

Pion propagator:

$$D_\pi(P) = \frac{1}{P_4^2 + u^2\left(\vec{P}^2 + m_\pi^2\right)}$$



Vacuum: $f_\pi^t \overset{\mu_q \to 0}{=} f_\pi^s \Rightarrow u \to 1$

Son and Stephanov Phys.Rev. D, 66(7) (2002)
Fischer and Mueller Phys.Rev. D 84, 054013 (2011)