Phase transitions in finite systems: from possible signals in heavy ion collisions to their rigorous theoretical treatment II

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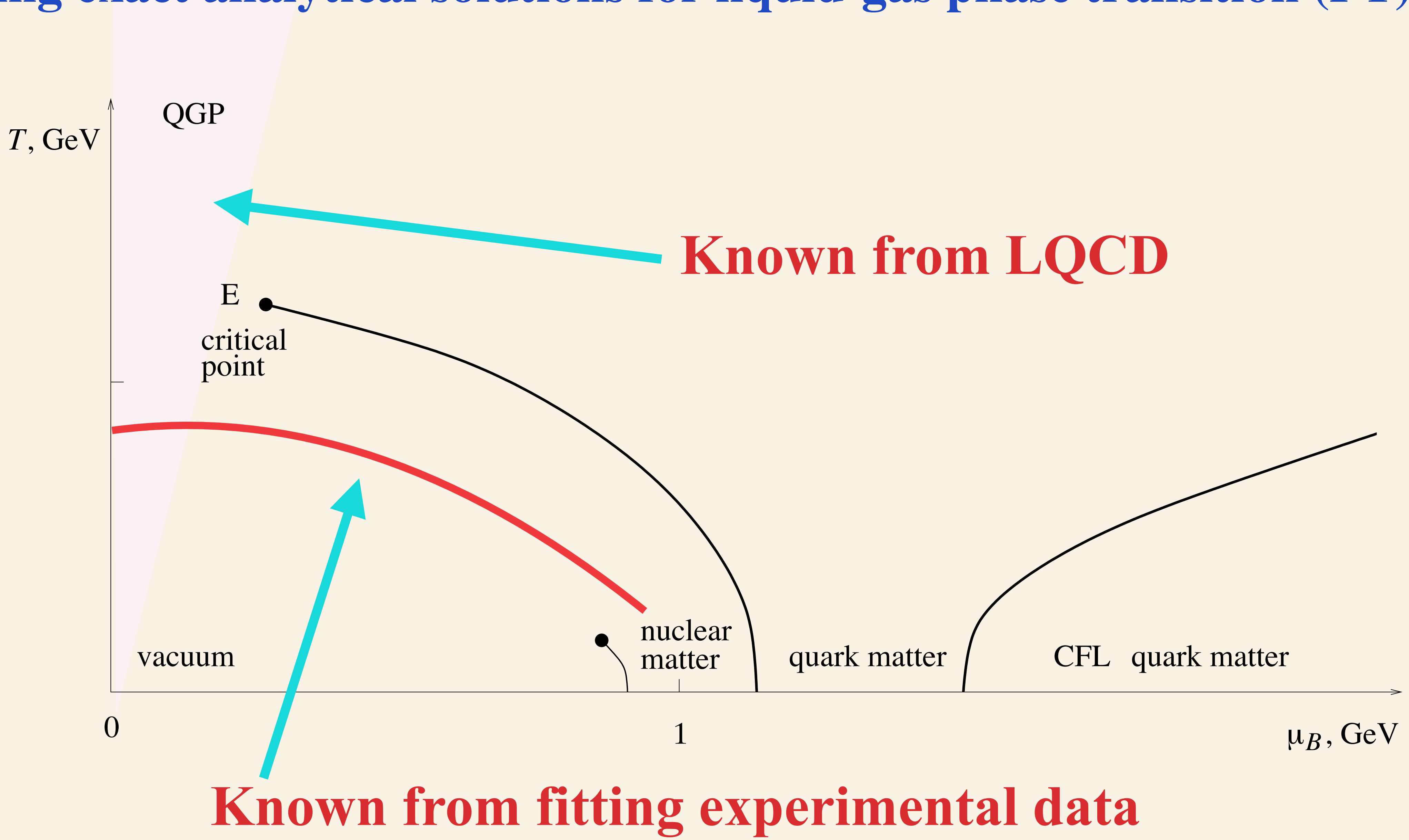
Dubna, July 13, 2017

Outline

- **Motivation**
- **Reminder of basic elements of exactly solvable statistical models of cluster type**
- **Mechanism of phase transition (PT) and cross-over generation in these models**
- **Source of negative eigen surface tension in cluster models**
- **Solution of statistical models of cluster type for finite V**
- **Conclusions**

Motivation

- **Practical purpose:** using exactly solvable model, the input from LQCD and from description of hadronic multiplicities at chemical freeze-out to get location of QCD (3)CEP
- **Academic purpose:** rigorously define analogs of phases in finite systems using exact analytical solutions for liquid-gas phase transition (PT)



Statistical Approach: Gas of Bags Model

- **1965 Hagedorn model invention**

  R. Hagedorn, Nuovo Cimento Suppl. **3**, 147 (1965).

- **1981 J. Kapusta formulated the Gas of Bags Model. Interior pressure of bags corresponds to the MIT bag model.**

  J. I. Kapusta, Phys. Rev. D **23**, 2444 (1981).

- 1981 An exact analytical solution of the Gas of Bags Model (GBM) is found. Roughly it is Hagedorn model with finite volume fireballs. Between fireballs there is hard core repulsion a la VDW. GBM employs the eigen volumes of bags and not their excluded volumes. M.I. Gorenstein, V.K. Petrov and G.M. Zinovjev, Phys. Lett. B 106 (1981) 327.

- 1982-84 Several works on GBM. Major result: mass-volume spectrum of MIT Bag Model is derived from

$$\rho(m,v) \simeq C\, v^{\gamma}(m-Bv)^{\delta}\, \exp\left[\frac{4}{3}\,\sigma_Q^{1/4}\, v^{1/4}\,(m-Bv)^{3/4}\right]$$

Interaction in the Gas of Bags Model

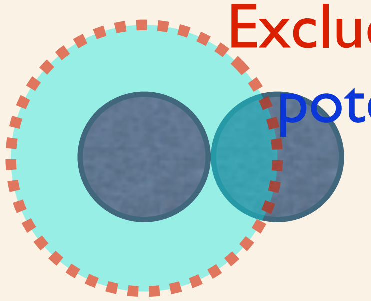
Attraction: is accounted by many sorts of clusters (= hadrons and bags) being in chemical equilibrium.

Repulsion:

Low density approximation!

Interaction: Hard core repulsion a la VDW

Excluded Volume (per particle) of hard core potential of radius R is 4 eigen volumes!



High densities!

Eigen volume approximation means that bags move inside some cells!
It is good for high densities!

Interaction in the Gas of Bags Model

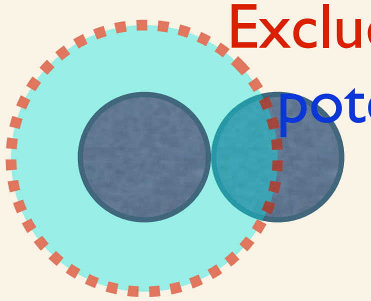
Unfortunately, in GBM there was no mechanisms

for (3)Critical Endpoint and for Cross-over

Low density approximation!

Interaction: Hard core repulsion a la VDW

Excluded Volume (per particle) of hard core potential of radius R is 4 eigen volumes!



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It is good for high densities!

Well Established Statistical Models

- Fisher Droplet Model (FDM) describes the condensation of gaseous clusters (droplets of all sizes) into liquid. M.E. Fisher, Physics 3 (1967) 255.
- FDM was used to describe the nucleation of real fluids, the compressibility factor of real fluids, clusters of d=2,3 Ising model, percolation clusters e.t.c.

- Statistical Multifragmetation Model (SMM) describes the low excitation energy nuclear reactions with large nuclei. J.P. Bondorf et al Phys. Rep. 257 (1995) 131.
- An exact analytical solution of a simplified SMM found by K.A.B. et al Phys. Rev. C 62 (2000) 044320
- It predicted a very narrow range for tau exponent P. T. Reuter and K.A.B., Phys. Lett. B 517 (2001) 233 tau = 1.825 +/- 0.025, which was observed experimentally by ISiS and EoS Collaborations and could not be explaind by FDM.

Free Energy in Statistical Models

- In FDM and SMM the FREE ENERGY of a V-volume cluster has the Bulk, V**1, Surface, V**(2/3), and Topological, $-\tau$ T ln (V), parts.

  τ is Fisher exponent

- At the phase equilibrium the Bulk part of free energy vanishes (equal pressures).

- At the (tri)critial point the Surface part of free energy vanishes (the energy and entropy gaps between gaseous and liquid phases disappear; recall the critical opalescence).

These properties are known from the experiments on ordinary liquids.

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IMPORTANT CONCLUSION: the GBM lacks the surface free energy!

The Van der Waals Repulsion

The Grand canonical partition (GCP) of n hadronic bags
with the hard-core repulsion of the Van der Waals type ($\mu_B = 0$)

$$\mathbf{Z}(V,T) = \sum_{\{N_k\}} \left[\prod_{k=1}^{n} \frac{\left[(V - v_1 N_1 - \ldots - v_n N_n)\, \phi_k(T)\right]^{N_k}}{N_k!} \right] \theta\,(V - v_1 N_1 - \ldots - v_n N_n)\ ,$$

the particle density of bags of mass m_k and eigen volume v_k and degeneracy g_k

$$\phi_k(T) \equiv g_k\ \phi_k(T) \equiv \frac{g_k}{2\pi^2} \int_0^\infty p^2 dp\ e^{-\frac{(p^2\ +\ m_k^2)^{1/2}}{T}} = g_k \frac{m_k^2 T}{2\pi^2}\ K_2\left(\frac{m_k}{T}\right)$$

Using the standard Laplace transformation with respect to volume V,
one gets the **isobaric partition with the simple pole:**

$$\hat{Z}(s,T) \equiv \int_0^\infty dV \exp(-sV)\ Z(V,T) = \frac{1}{[s - F(s,T)]}$$

describes hard core repulsion in GC ensemble

$$\text{with}\quad F(s,T) \equiv \sum_{j=1}^{n} \exp\left(-v_j s\right) g_j \phi(T, m_j)\ .$$

- The Θ function is VERY important because ensures that bags do not overlap!

Basic Ingredients of QGBST Model

If the number of bag kinds is infinite, there may appear an essential singularity of the Isobaric Partition. This is used in GBM and QGBST to generate PT. This can be seen as follows (also for non-zero μ):

For $V \to \infty$ the whole analysis is reduced to the analysis of the Singularities of IP!

After Inverse Laplace transform GCP becomes

$$Z(V,T,\mu) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} ds\; Z(s,T,\mu)\; e^{s\,V} =$$

$$\sum_{s_i^*} Res\left(Z(s_i^*,T,\mu)\;\; e^{s_i^*\,V}\right) \longrightarrow e^{V\,\max\,(s_i^*)}$$

Comparing with

$$Z(V,T,\mu) \longrightarrow e^{\frac{p\,V}{T}} \Longrightarrow p(T,\,\mu) = T\,\max\,(s_i^*)\;,$$

Equation for Singularities:

$$s^*(T) \;=\; F(s^*,T)$$

where $\sigma > \max\,Re(s_i^*)$ - **the rightmost singularity.**

- PT happens, if two singularities coincide.

In other words, the Gibbs criterion follows automatically!

QGBST Model

Volume spectrum of bags in isobaric ensemble K.A.B., PRC 76 (2007)

discrete = hadrons **continuous = QG bags**

$$F(s,T,\mu_B) \equiv F_H(s,T,\mu_B) + F_Q(s,T,\mu_B) =$$

$$\sum_{j=1}^{n} g_j e^{\frac{\mu_j}{T} - v_j s} \phi(T,m_j) + u(T) \int_{V_0}^{\infty} dv \, \frac{\exp\left[(s_Q(T,\mu_B) - s)\, v - \Sigma(T,\mu_B)\, v^{\varkappa}\right]}{v^{\tau}}$$

thermal particle density of bags of mass m_k and eigen volume v_k and degeneracy g_k

$$g_k\, \phi(T,m_k) \equiv \frac{g_k}{2\pi^2} \int_0^{\infty} p^2 dp \; e^{-\frac{(p^2 + m_k^2)^{1/2}}{T}}$$

$$V_0 \simeq 1 \text{ fm}^3$$

F_H has no s-singularities at any T and generates a simple pole only!

The bag spectrum $F_Q(s,T)$ is chosen to give an essential singularity $s_Q(T) \equiv \frac{p_Q(T)}{T}$.

The finite width of bags is neglected here!
But it can be accounted by additional mass integration.

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Degeneracy of QG bags = kind of Hagedorn spectrum

Main parameters: $s_Q(T,\mu_B)$ defines pressure of QG bags $p_Q = T s_Q(T,\mu_B)$

$\Sigma(T,\mu_B)$ is reduced surface tension coefficient

$\varkappa = \frac{2}{3}$, Fisher exponent $\tau > 1$

QGBST Model incorporates the best features of Hadron Gas Model, Bag Model and Fisher droplet model

Surface Tension Parameterization

$$\Sigma(T,\mu_B) = \begin{cases} \Sigma > 0, & \text{for} \quad T \to T_\Sigma(\mu_B) - 0 \\ 0, & \text{for} \quad T = T_\Sigma(\mu_B) \\ \Sigma < 0, & \text{for} \quad T \to T_\Sigma(\mu_B) + 0 \end{cases}$$

Sign of $\Sigma(T,\mu_B)$ determines the singularities of Isobaric Partition in the complex s-plane

Case I: $\Sigma(T,\mu_B) > 0$ is similar to GBM **=> 1-st order PT**

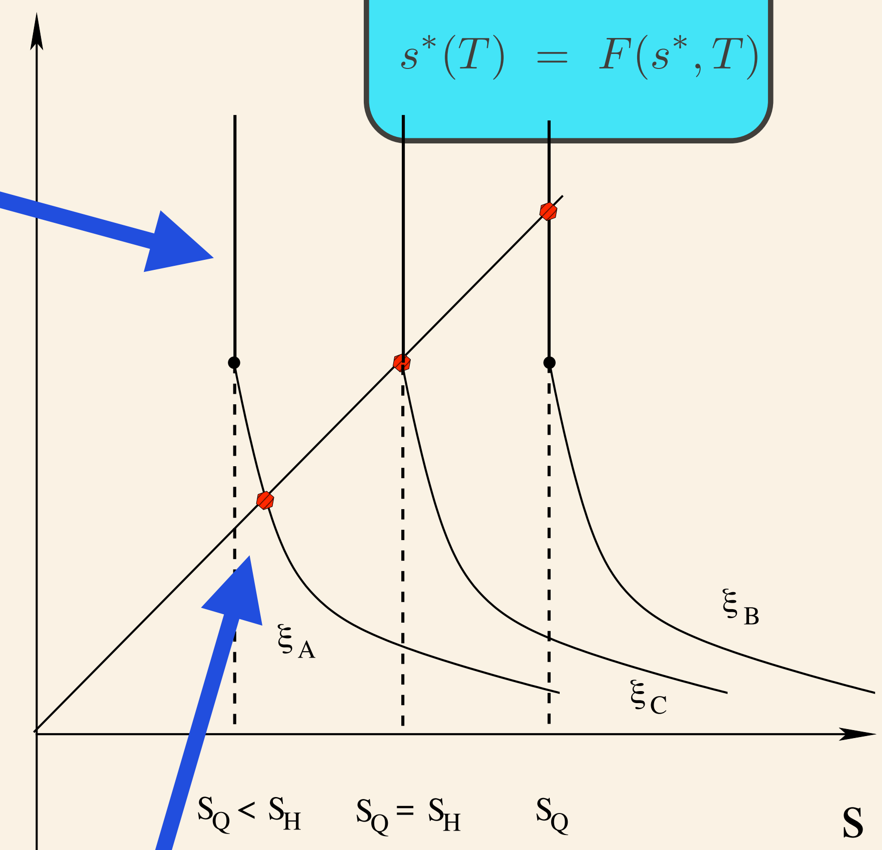
$$F_Q(s,T,\mu_B) = u(T) \int_{V_0}^{\infty} dv \, \frac{\exp\left[(s_Q(T,\mu_B) - s)\, v - \Sigma(T,\mu_B)\, v^{\varkappa}\right]}{v^{\tau}}$$

Defines essential singularity

fix μ_B then

Equation for Singularities:

$$s^*(T) = F(s^*,T)$$



QGP pressure $p_Q = T s_Q(T,\mu_B)$ can be chosen in several ways. For definiteness we use the MIT Bag model pressure

example

$$p_Q = \frac{\pi^2}{90} T^4 \left[\frac{95}{2} + \frac{10}{\pi^2}\left(\frac{\mu_B}{T}\right)^2 + \frac{5}{9\pi^4}\left(\frac{\mu_B}{T}\right)^4\right] - B$$

$u(T,\mu_B), B$ should obey the **sufficient conditions** for a PT existence:

$$F(s_Q(T,\mu_B=0)+0,\ T,\ \mu_B=0) > s_Q(T,\mu_B=0)\,,$$
$$F(s_Q(T,\mu_B)+0,\ T,\ \mu_B) < s_Q(T,\mu_B)\,, \quad \text{for all} \quad \mu_B > \mu_A.$$

$$F_H(s,T,\mu_B) = \sum_{j=1}^{n} g_j e^{\frac{b_j \mu_B}{T} - v_j s} \phi(T,m_j)$$

Defines simple pole

The Role of Surface Tension

Case II: $\boldsymbol{\Sigma(T, \mu_B) = 0}$ **is similar to GBM too,**

but PT order is defined by Fisher exponent $\boldsymbol{\tau}$

Structure of singularities for $\boldsymbol{\tau \leq 2}$ **is also similar to GBM**

Can be shown from second derivative that $\boldsymbol{2^{nd}}$ **order PT** exists for $\frac{3}{2} < \tau \leq 2$.

In general for $(n+1)/n \leq \tau < n/(n-1)$ $(n = 3, 4, 5, ...)$ there is a n^{th} order phase transition

$$
\begin{aligned}
s_H(T_c) &= s_Q(T_c)\ , \quad s'_H(T_c) = s'_Q(T_c)\ , \ \ldots \\
s_H^{(n-1)}(T_c) &= s_Q^{(n-1)}(T_c)\ , \quad s_H^{(n)}(T_c) \neq s_Q^{(n)}(T_c)\ ,
\end{aligned}
$$

with $s_H^{(n)}(T_c) = \infty$ for $(n+1)/n < \tau < n/(n-1)$ and

with a finite value of $s_H^{(n)}(T_c)$ for $\tau = (n+1)/n$.

The Role of Surface Tension

Case III: $\Sigma(T, \mu_B) < 0$ is principally different from GBM

It is able to explain the cross-over existence above (3)CEP not only in QCD, but in other liquid-gas PTs!

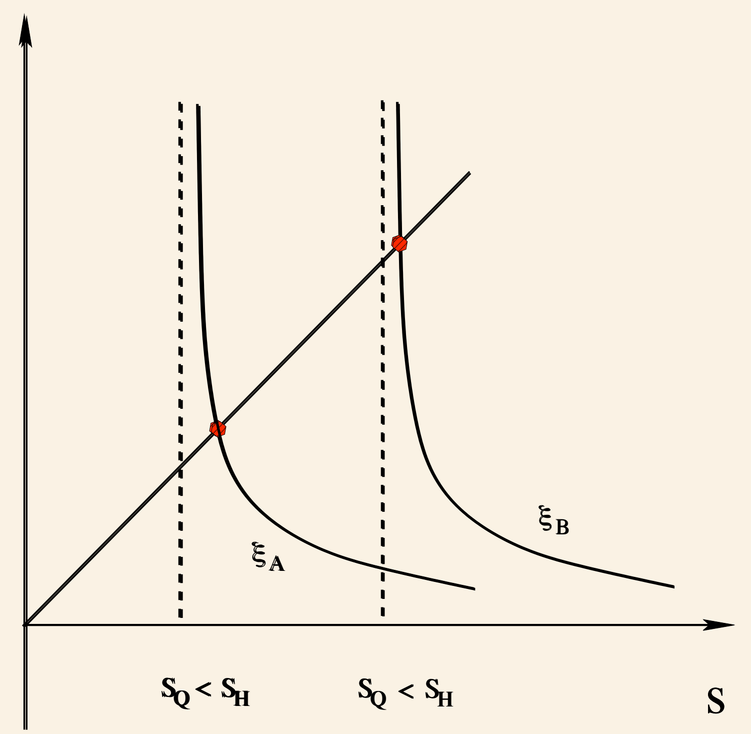
fix μ_B then

Equation for Singularities:

$$s^*(T) = F(s^*, T)$$

Has a simple pole only!

Then s^* can approach s_Q at $T \to \infty$



Physics: negative surface tension prevents formation of large and heavy QG bags!

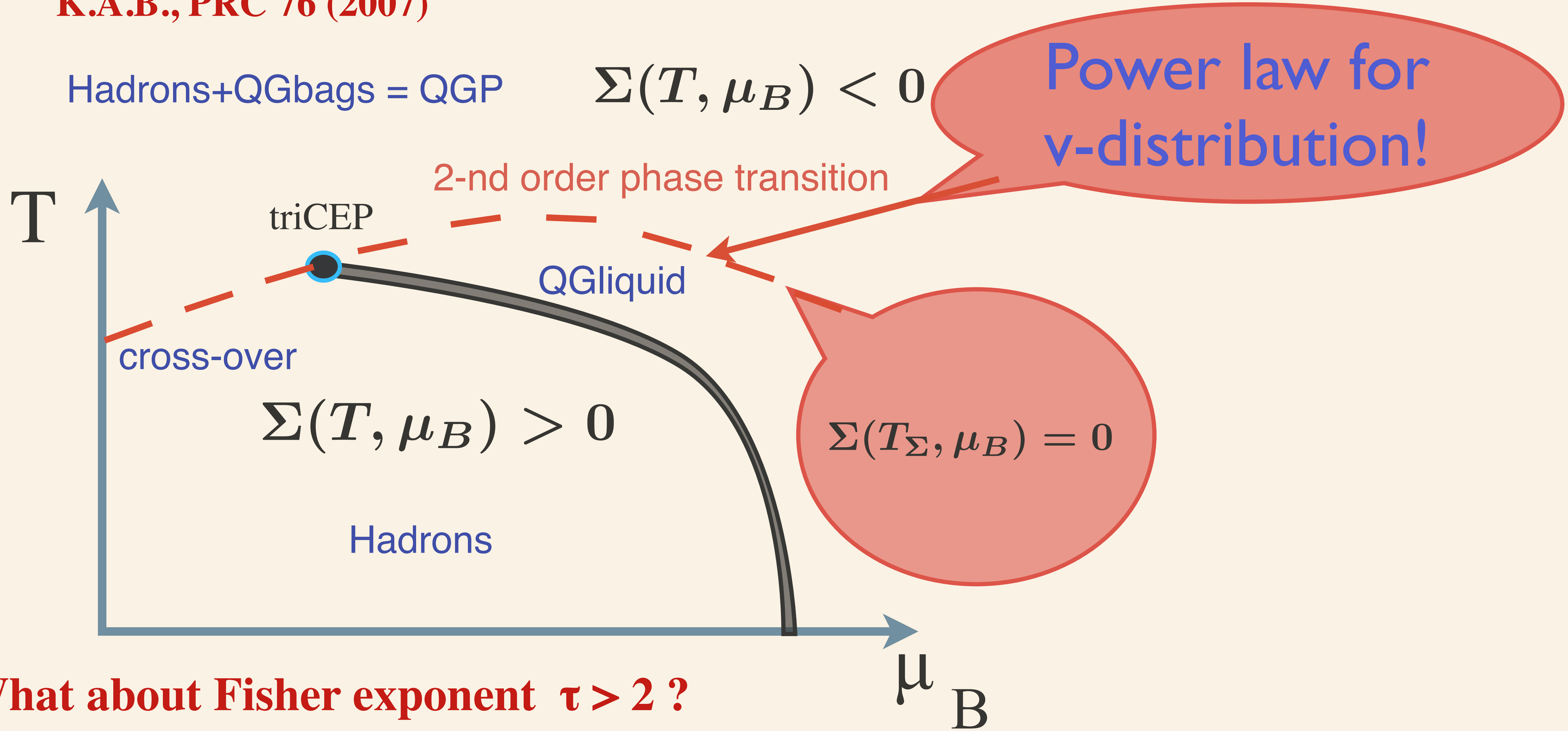
Results for TriCEP

Our group has calculated the critical indices for this case and found that the phase diagram must look like shown below

A. Ivanytskyi, NPA(2012) 880

Exists for Fisher exponent τ: 1< τ < 2 only!

K.A.B., PRC 76 (2007)



What about Fisher exponent τ > 2 ?

What about the critical endpoint?

CEP Generation

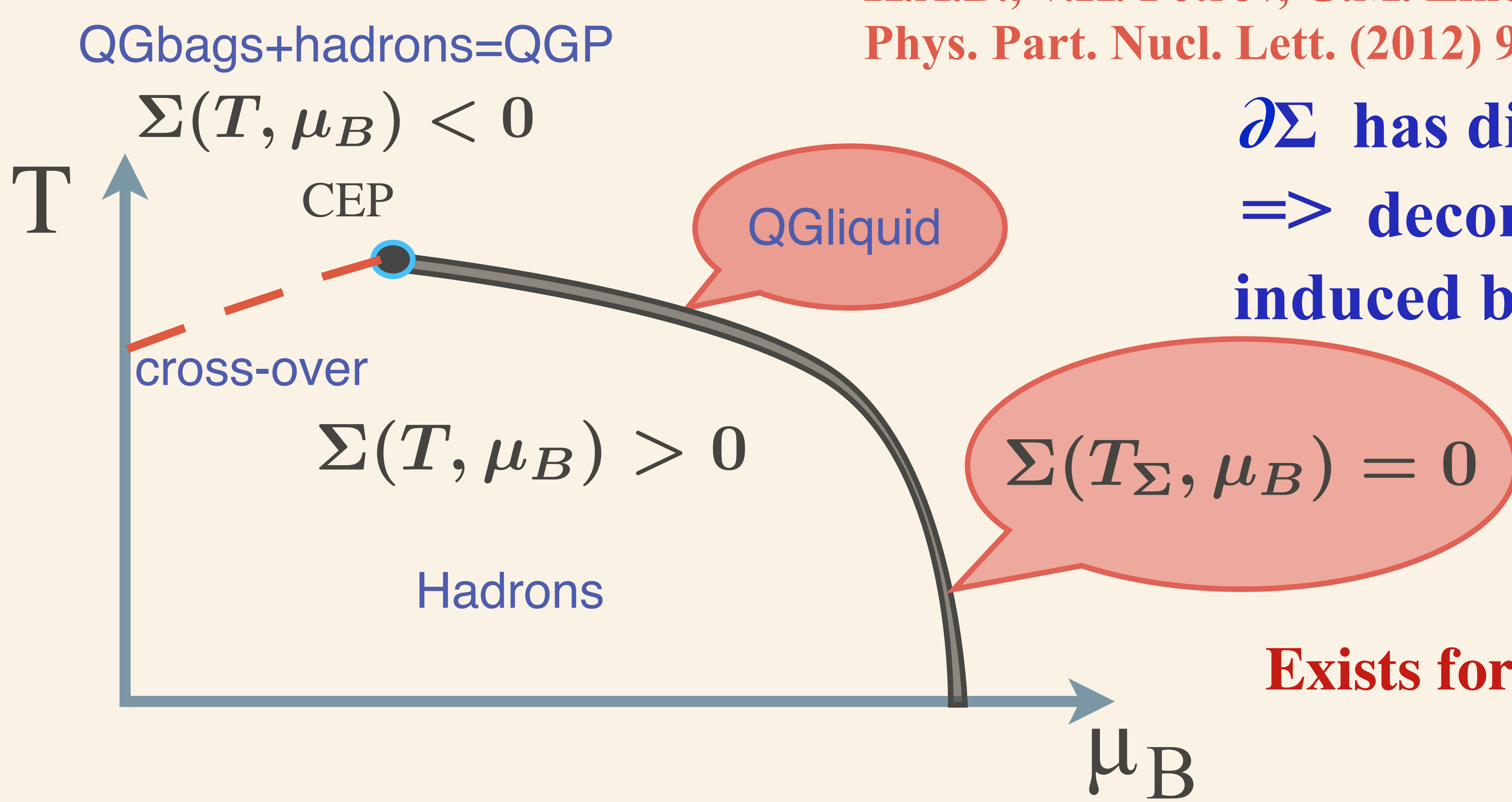
Main idea:

to match the curves of deconfinement PT and $\Sigma = 0$!

Prediction:

the power law in V-distribution of bags will be not just at CEP as one would expect, but in the mixed phase with $\Sigma = 0$!

K.A.B., V.K. Petrov, G.M. Zinovjev,
Phys. Part. Nucl. Lett. (2012) 9



$\partial\Sigma$ has discontinuity at $T_\Sigma(\mu_B)$
=> deconfinement PT is induced by a surface tension!?

Exists for Fisher exponent $\tau > 2$!

Aftermath: Unanswered Questions

1. What is a reason for a kink in Surface Tension?
2. Is something wrong with negative values of Surface Tension coefficient?

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Preliminary answer on 1-st question:

A. Our recent analysis of geometrical clusters formed by Polyakov loops in SU(2) gluodynamics shows that the kink in Surface Tension exists!

see A.I. Ivanytskyi et al., Nucl. Phys. A 960 (2017) 90

B. My personal belief is that in contrast to ordinary liquids the kink in surface is due to chiral symmetry restoration.

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Recall A. Ivanystkyi talk on EoS beyond the Van der Waals approximation

Induced surface tension coefficient Σ_k of a particle with hard-core radius R_k

$$\Sigma_k = p_k R_k \exp\left[-4\pi R_k^2 \cdot (\alpha - 1)\frac{\Sigma_{tot}}{T}\right],$$

with $\Sigma_{tot} \equiv \sum_k \Sigma_k$

Change of cluster mass m_k =>

change of partial pressure p_k =>

change of partial surface tension Σ_k

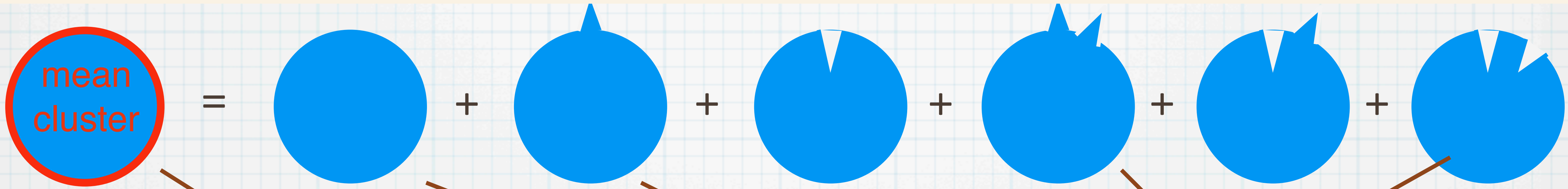
Should be studied!

Eigen Surface Free Energy: F = E -TS

To find eigen surface F one has to count for ALL surface deformations together with energy costs

Can be exactly done within Hills and Dales Model for v-volume cluster:

K.A.B. et al, PRE 72 (2005)



$$\underbrace{\exp\left[-\frac{\sigma_o v^{2/3}}{T}\right]}_{Energy\ part}\ \underbrace{\exp(S)}_{Entropy\ part} = \underbrace{\exp\left[-\frac{\sigma_o v^{2/3}}{T}\right]}_{Sphere's\ Energy} \times \left\{1 + \left(\underbrace{w_H N_H}_{1\ Hill} + \underbrace{w_D N_D}_{1\ Dale}\right) \exp\left[-\frac{\sigma_o \Delta S_1}{T}\right] + 2,\ 3,\ \text{etc deformations}\right\}$$

$$= \underbrace{\exp\left[-\frac{\sigma_o v^{2/3}}{T}\right]}_{Energy\ part}\ \underbrace{\exp\left[+\frac{\sigma_o v^{2/3}}{T_c}\right]}_{Entropy\ part}$$

Simplest case (M. Fisher)

Also one can find supremum and infimum for surface F and surface partition

$$\sigma_0(1-\lambda_L T)\, v^{\frac{2}{3}} \quad \geq \quad F \quad \geq \quad \sigma_0(1-\lambda_U T)\, v^{\frac{2}{3}}\,, \quad \lambda_L \approx 0.28\, T_c^{-1}, \quad \lambda_U \approx 1.06\, T_c^{-1}$$

K.A.B. & Elliott, UJP 52 (2007)

Thus, there is NOTHING wrong, if surface F < 0 above critical T!
This means only that entropy dominates!

QGBST Model for Finite Volume

Main problem: one cannot simply replace upper limit of integration by a finite volume V, since this is a Laplace transform with respect to V!

$$F_Q(s,T,\mu_B) = u(T) \int_{V_0}^{\infty} dv \, \frac{\exp\left[(s_Q(T,\mu_B) - s)\, v - \Sigma(T,\mu_B)\, v^{\varkappa}\right]}{v^{\tau}}$$

Fortunately, this can be done within the formalism developed in

see K.A.B., Acta. Phys. Polon. B 36 (2005) and K.A.B., P.T. Reuter, Ukr. J. Phys. 52 (2007) 489

Replace V-integral by a K-sum over volumes of bags V_k : with max $K \equiv K(V)$

$$F_Q(s,T,\mu_B) \longrightarrow \mathcal{F}_Q(\lambda, V) \equiv \sum_{k=1}^{K(V)} \phi_k(T) \exp\left[(s_Q(T,\mu_B) - \lambda) V_k\right]$$

Note that surface tension and Fisher terms are hidden in $\phi_k(T)$

For convenience make a regular mesh over volumes $V_k = k\, b$, where b is minimal volume

$$\mathcal{F}_Q(\lambda, V) = \sum_{k=1}^{K(V)} \phi_k(T) \exp\left[\frac{\nu\, k}{T}\right] \quad \text{with effective chem. potential } \nu = (s_Q(T,\mu_B) - \lambda) b$$

Evidently, for small b one can make irregular mesh as well by setting some of $\phi_k(T) = 0$

QGBST Model for Finite Volume

Main problem: one cannot simply replace upper limit of integration by a finite volume V, since this is a Laplace transform with respect to V!

$$K(V) \text{ is number of species } \Rightarrow K(V) \le K_{max} < \infty, \text{ if no PT exists}$$

$$\Rightarrow K(V) \to \infty, \text{ if PT exists in the limit } V \to \infty$$

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QGBST Model for Finite Volume II

V-independent hadronic mass-volume spectrum can be considered as a single term

$$F_H(s,T,\mu_B) = \sum_{j=1}^{n} g_j e^{\frac{b_j \mu_B}{T} - v_j s} \phi(T, m_j) \;\equiv\; \boldsymbol{\phi_0(T,\mu_B)}$$

Then the total mass-volume spectrum is $\mathcal{F}(\lambda, V) \equiv \phi_0(T,\mu_B) + \mathcal{F}_Q = \sum_{k=0}^{K(V)} \phi_k \exp\left[\frac{\nu k}{T}\right]$

The GCE partition can be written as

$$\mathcal{Z}(V,T,\mu) = \sum_{\{\lambda_n\}} e^{\lambda_n V} \left[1 - \frac{\partial \mathcal{F}(\lambda_n, V)}{\partial \lambda_n}\right]^{-1}$$

For finite V one has to account for ALL singularities λ_n in a complex plane!

In our case λ_n (n =0, 1, ...) are simple poles of Isobaric Partition and are defined by

$$\lambda_n = \mathcal{F}(\lambda_n, V)$$

λ_0 is the only real solution, while $\lambda_{n\geq 1}$ come in complex conjugate pairs

GCE Pressure for Finite Volume

Each $p_n = T Re(\lambda_n)$ is the partial thermal pressure of state n

For finite V the mechanical pressure cannot be expressed in terms of $T\lambda_n$

$$p_{mech} \equiv T \frac{\partial \ln [\mathcal{Z}(V,T,\mu)]}{\partial V} = \frac{T}{\mathcal{Z}(V,T,\mu)} \sum_{\{\lambda_n\}} \left[\frac{\lambda_n}{\left[1 - \frac{\partial \mathcal{F}(\lambda_n, V)}{\partial \lambda_n}\right]} + \frac{\frac{\partial^2 \mathcal{F}(\lambda_n, V)}{\partial V \, \partial \lambda_n}}{\left[1 - \frac{\partial \mathcal{F}(\lambda_n, V)}{\partial \lambda_n}\right]^2} \right] e^{\lambda_n V}$$

Hence for finite V it differs from the weighted thermal pressure

$$p_{thermal} \equiv \frac{T}{\mathcal{Z}(V,T,\mu)} \sum_{\{\lambda_n\}} \frac{\lambda_n \, e^{\lambda_n V}}{\left[1 - \frac{\partial \mathcal{F}(\lambda_n, V)}{\partial \lambda_n}\right]} \neq p_{mech}$$

It is not completely clear how to deal with the finite V corrections for finite systems, although T. Hill discussed this problem in his books and articles.

The point is that T. Hill discussed this problem for weakly interacting systems

I believe that Lattice QCD pressure is the thermal one, but maybe there is a proof ?

Singularities of Isobaric Partition

GCP is reduced to sum over all singularities λ_n $(n = 0, 1, 2, ..)$ of **Isobaric Partition** $\Leftrightarrow$ **collective states of the same GCP**

K.A.B., Acta. Phys. Polon. B 36 (2005)

- Simple poles: $\lambda_n = R_n + i I_n$

$$\mathcal{Z}(V,T,\mu) = \sum_{\{\lambda_n\}} e^{\lambda_n V} \left[1 - \frac{\partial \mathcal{F}(V,\lambda_n)}{\partial \lambda_n}\right]^{-1}$$

$$\lambda_n = \mathcal{F}(V, \lambda_n) \Rightarrow \begin{cases} R_n = \sum_{k=0}^{K(V)} \phi_k(T)\; e^{\frac{Re(\nu_n)k}{T}} \cos(I_n b k)\,, \\ I_n = - \sum_{k=0}^{K(V)} \phi_k(T)\; e^{\frac{Re(\nu_n)k}{T}} \sin(I_n b k)\,. \end{cases}$$

- Effective chemical potential $\nu_n \equiv \nu(\lambda_n)$
- Reduced distribution $\phi_k(T)$ **has no bulk free energy**
- **Real root meaning** $(R_0; I_0 = 0)$:

  (i) Root R_0 exists for any $(T,\ \mu)$

  (ii) $T R_0$ is constrained grand canonical gas pressure:

$$T R_0 = \sum_{k=0}^{K(V)} \phi_k(T) \exp\left[(s_Q(T,\mu_B) - R_0) b k\right]$$

Singularities of Isobaric Partition II

Eqs. for simple poles: effective number of D.O.F.

with $I_n \neq 0$: $\lambda_n = R_n + iI_n \Rightarrow \begin{cases} R_n = \sum_{k=0}^{K(V)} \phi_k(T)\, e^{\frac{Re(\nu_n)k}{T}} \cos(I_n bk)\,, \\ I_n = -\sum_{k=0}^{K(V)} \phi_k(T)\, e^{\frac{Re(\nu_n)k}{T}} \sin(I_n bk)\,. \end{cases}$

Since Eq. $|\cos(I_n bk)| = 1$ cannot hold for all k: $1 \leq k \leq K(V)$

$\Rightarrow$ Gas Singularity is the Rightmost One: $R_0 > R_{n>0}$

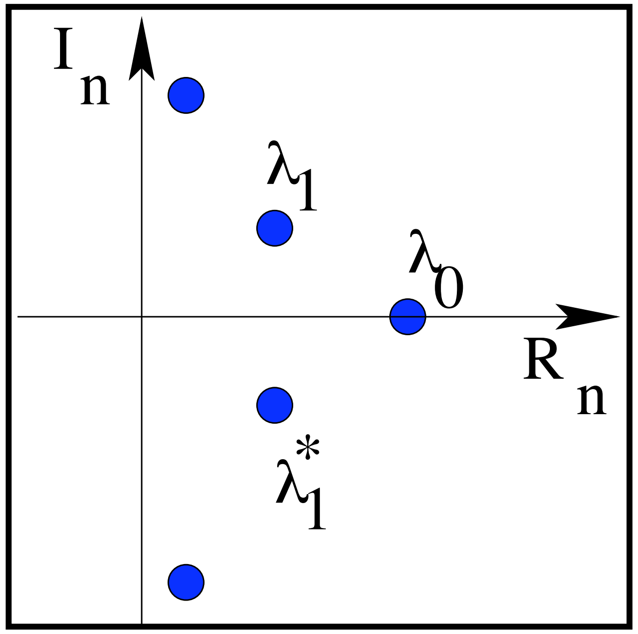
For $\nu \leq 0$ and any T there is a gaseous pole only, i.e. $\lambda = R_0$

- **Complex λ-plane** :

For $Re(\nu) > 0$ there appear some number of complex conjugate poles

- Since both sides of Eq. for I_n are odd functions of I_n $\Rightarrow$:
if λ_n is a root, then λ_n^* is a root too.
$\Rightarrow$ **Partition is always real**

$\Rightarrow$ **The gas pressure $T\,R_0$ is the largest pressure**



For finite V the number of solutions defines the analog of phase:
gas has 1 solution,
mixed phase has 3, 5 or more solutions

Meaning of Complex Roots

- **From GCP** $\mathcal{Z}(V,T,\mu) = \sum_{\{\lambda_n\}} e^{\lambda_n V} \left[1 - \frac{\partial \mathcal{F}(V,\lambda_n)}{\partial \lambda_n}\right]^{-1},$

$\Rightarrow$ $-Re(\lambda_n)VT = -R_n VT$ **is free energy of** λ_n **state.**

$\Rightarrow$ **Gaseous state is always stable since** $-R_0 VT < -R_{n>0} VT$

$\Rightarrow$ $n > 0$ **states are metastable for finite** V**!**

- **From correspondence:**

Statistical Operator $\Longleftrightarrow$ Evolution Operator

$e^{-\frac{\hat{H}}{T}} \Longleftrightarrow e^{-\frac{iHt}{\hbar}}$

$\Rightarrow$ $\frac{1}{T}$ **is complex time** t

$\Rightarrow$ $\cos\left(\frac{I_n bTk}{T}\right) = \cos\left(\frac{itk}{\tau_n}\right) = \frac{1}{2}\left[e^{-k\frac{t}{\tau_n}} + e^{k\frac{t}{\tau_n}}\right]$

$\Rightarrow$ $\sin\left(\frac{I_n bTk}{T}\right) = \sin\left(\frac{itk}{\tau_n}\right) = \frac{1}{2i}\left[e^{-k\frac{t}{\tau_n}} - e^{k\frac{t}{\tau_n}}\right]$

$\Rightarrow$ $\tau_n = \pm\frac{1}{|I_n| bT}$
is formation/decay time

$\Rightarrow$ **Gaseous phase**
is stable because $\tau_0 = \pm\infty$

Example Re(ν) >> T and Finite K(V)

Consider limit $Re(\nu_n) \gg T$**:**

then R_n is defined by the largest fragment $k = K(V) = Const$

$$I_n = \frac{\pi n + \delta_n}{K b}, \quad |\delta_n| \ll \pi, \quad \delta_n \approx \frac{(-1)^{n+1}\pi n}{K b\ \phi_K(T)} \ \text{exp}\left[-\frac{Re(\nu_n)\,K}{T}\right],$$

$$R_n \approx (-1)^n\ \phi_K(T)\ \ \text{exp}\left[\frac{Re(\nu_n)\,K}{T}\right].$$

Important: $R_{n>0}$ **can have either sign in this limit!**

$$R_n \approx s_Q(T,\mu) - \frac{1}{K(V)b}\,\ln\left|\frac{R_n}{\phi_K(T)}\right| = s_Q(T,\mu) - \frac{1}{K(V)b}\,\ln\left|\frac{R_n}{\phi_K(T)}\right|$$

$\Rightarrow$ For $Re(\nu_n) \gg T$ **some** R_n **approach liquid phase singularity!**

- for $R_n \to +\infty$ we compressed the system to the densest state.

  So, it is metastable liquid, indeed! Since $I_{n>0}$ do not vanish.

- for $R_n \to -\infty$ in the GCP there may appear (with vanishing probability!) a largest fragment alone.

- for $\infty > Re(\nu) \geq Re(\nu_1(T))$: **analog of mixed phase**

Example Re(ν) >> T and Finite K(V)

Consider limit $Re(\nu_n) \gg T$:

In fact, the same is true for $Re(\nu_n) > T$ and $V \to \infty$

Important: $R_{n>0}$ **can have either sign in this limit!**

$$R_n \approx s_Q(T,\mu) - \frac{1}{K(V)b} \ln \left| \frac{R_n}{\phi_K(T)} \right| = s_Q(T,\mu) - \frac{1}{K(V)b} \ln \left| \frac{R_n}{\phi_K(T)} \right|$$

$\Rightarrow$ For $Re(\nu_n) \gg T$ **some** R_n **approach liquid phase singularity!**

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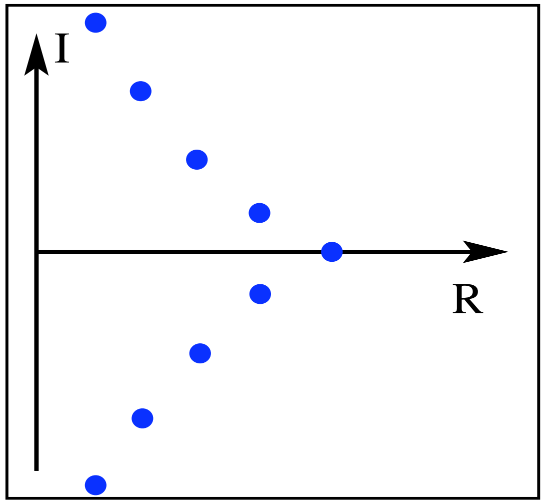
- for $\infty > Re(\nu) \geq Re(\nu_1(T))$: **analog of mixed phase**

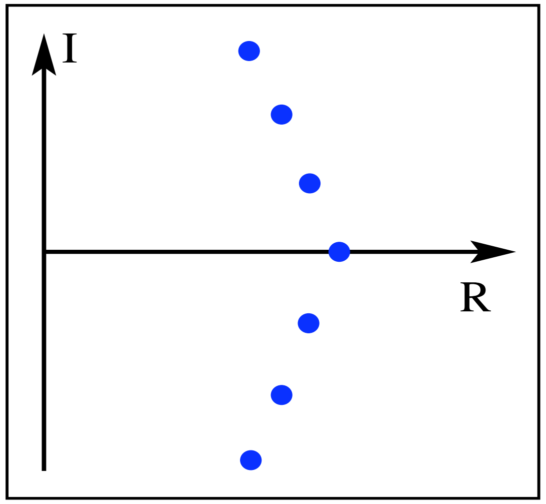
Isobaric Partition Poles for Large V

Without PT

Im part is fixed and equidistant



Small volume



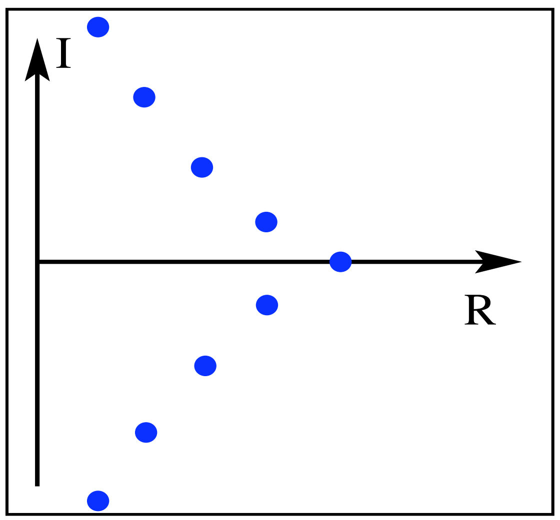
Large volume

$$R_0 > R_1 > R_2 > \ldots > R_n$$

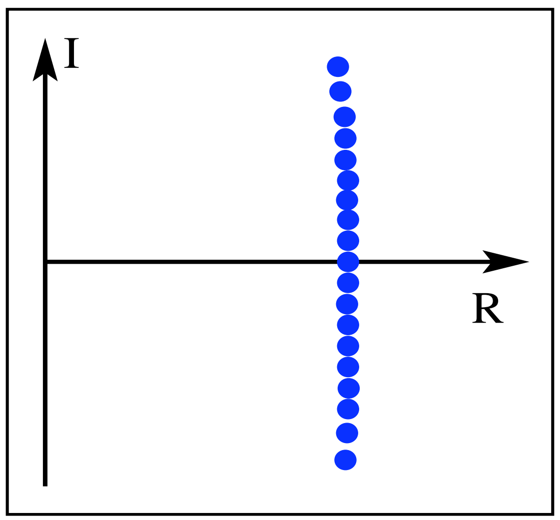
$$I_{n>0} \approx \pm \frac{\pi\, n}{K_{max} b}$$

With PT

Im part is equidistant, but moves to 0



Small volume



Large volume

Thus, this approach can distinguish the case with PT from the one without PT, but what about experiments? Is there any evidence for such states?

What About (3)CEP Analog for Finite V?

Honestly, at the moment there is no convincing definition.

In fact, result depends on how do we define (3)CEP!

1. **If (3)CEP corresponds to a power law in mass distribution of clusters (=bags or nuclear fragments or droplets of liquid), then such a point of vanishing surface tension belongs to a gas.**

K. A. B., A. I. Ivanytskyi, V. V. Sagun, D. R. Oliinychenko, Phys. Part. Nucl. Lett.10, 832 (2013)

2. **If (3)CEP is defined as a position of the maximum of specific heat capacity (or maximum of isothermal compressibility), then its location depends on EoS of gaseous and liquid phases!**

3. **In general, these two maxima can have different locations for finite V!**

Conclusions

- **New mechanisms of PT and (3)CEP models for QCD are required**

- **Rigorous theory of surface tension of ordinary liquids and QCD clusters is necessary**

- **Statistical thermodynamics of finite systems should be developed**

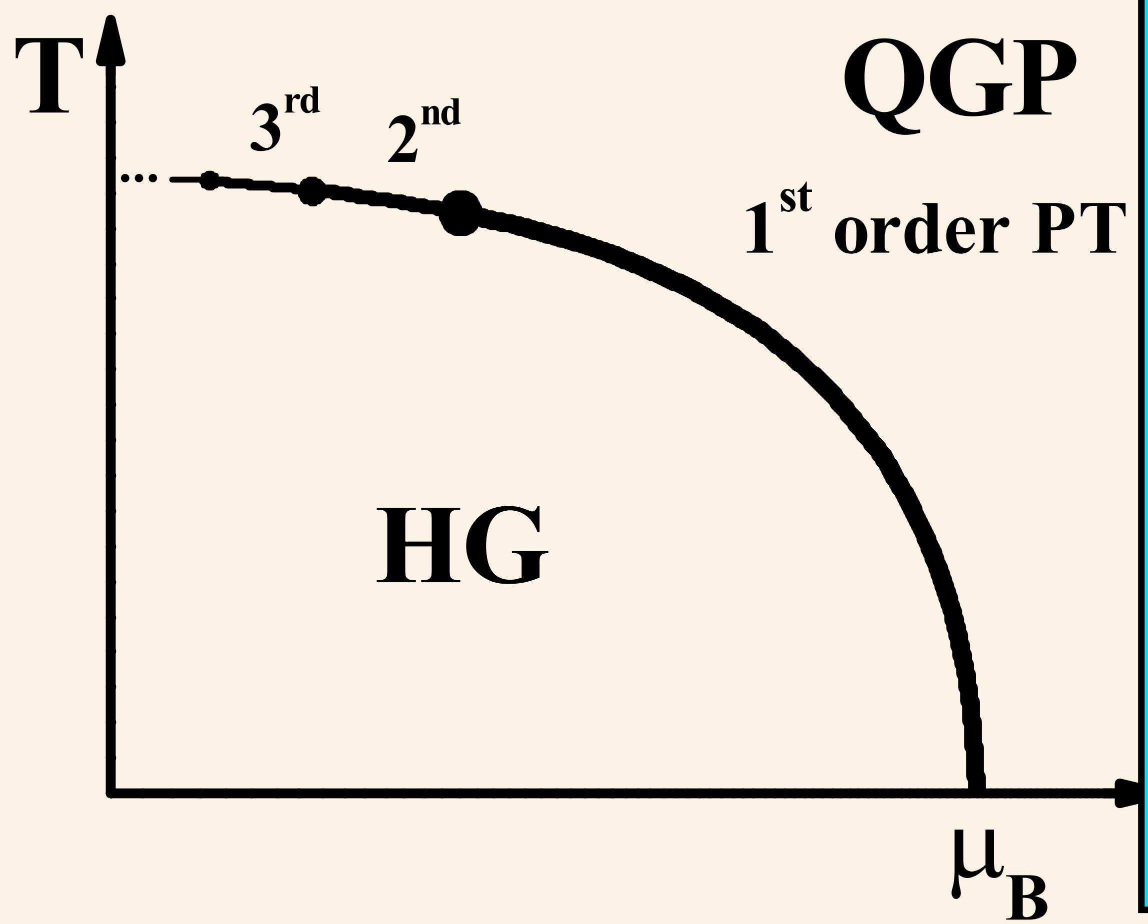
- **A lot of interesting work related to NICA and FAIR experiments awaits for us!**

Thank You for Your Attention!

Back Up Slides

Problems of the Gas of Bags Model

* 2005 A new and EXTRAVAGANT idea to revitalize the GBM: in order to get the CEP and cross-over M.I. Gorenstein, M. Gazdzicki and W. Greiner, Phys. Rev. C 72 (2005) 024909, suggested a line along which the PT order gradually decreases.

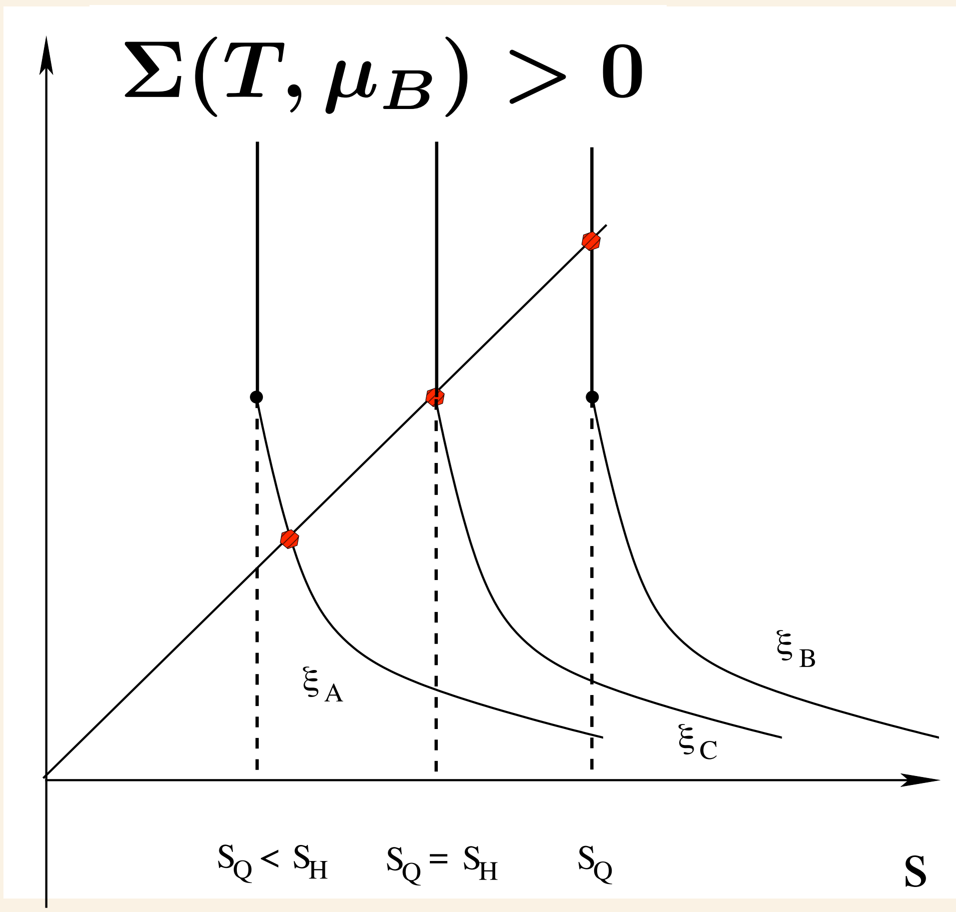


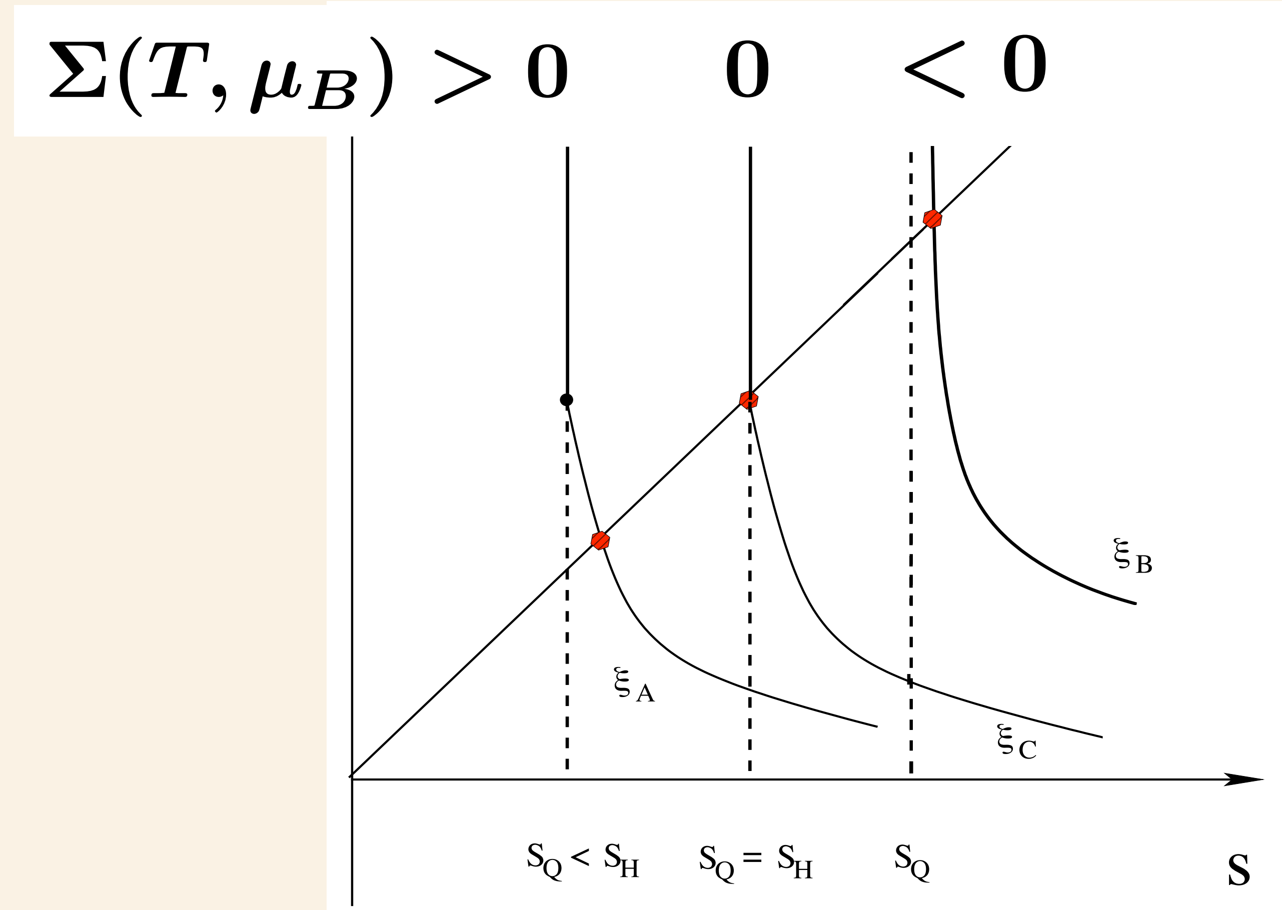
There are tenth of thousand substances with extremely complicated phase diagrams known in physics, but such a pathological diagram has never been seen! There are no causes known for such a line! It contradicts to the whole concept of critical phenomena!

- Consequently, such a formulation of GBM lacks an important physical input and has to be modified.

Structure of singularities for CEP

- Thus, for the CEP case the rightmost singularity below and above PT line is a SIMPLE POLE!





Case of triCEP PRC 76 (2007)

Case of CEP **PEPANLett 9 (2012)**

Parameter ξ can be either T or μ_B.

For example, if ξ is T, then $\xi_A < T_c$, $\xi_c = T_c$ and $\xi_B > T_c$.