Correlations and Critical Behavior in Lattice $SU(2)$ Gluodynamics

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September 04, 2018

- Phase transitions for pedestrains
- Phase transition in $SU(2)$
- Correlation between the asymmetry and the Polyakov loop
- Correlation between the longitudinal propagator and the Polyakov loop
- Regression analysis
- Evaluation of the critical exponents and amplitudes
- Conditional distributions of the longitudinal propagator
- Conclusions

Our main result:

$$D_L = D_L^C + C \cdot (T - T_c)^{0.326419(3)} \cdot (1 + \alpha(T))$$

- D_L - zero-momentum gluon propagator
- we consider $SU(2)$ lattice gauge theory in the Landau gauge
- $T_c \approx 297$ MeV
- $\lim_{T \to T_c} \alpha(T) = 0$

Ising model

$\sigma_n = \pm 1$

Finite-volume lattice: $\vec{n} = (n_1, ..., n_D), \quad 1 \le n_\mu \le L, \quad n_\mu \in \mathbb{N}$

Infinite-volume lattice: $\vec{n} \in \mathbb{Z}^D$,

$$H = -J \sum_{|\vec{i}-\vec{j}|=1} \sigma_{a\vec{i}} \sigma_{a\vec{j}} - h \sum_{\vec{i} \in \mathbb{Z}^D} \sigma_{a\vec{i}}$$

$$Z = \sum_{\sigma_n} e^{-H[\sigma]/T} = \exp\left(-\frac{F}{T}\right)$$

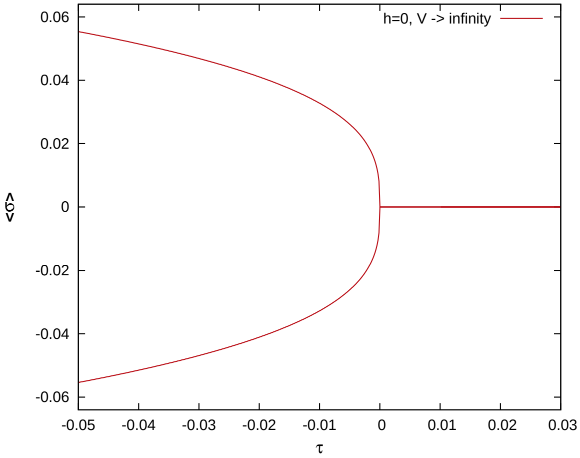
$$m = \frac{\partial F}{\partial h}, \quad S = -\frac{\partial F}{\partial T}, \quad \chi = \frac{\partial^2 F}{\partial h^2}, \quad c = -T\frac{\partial^2 F}{\partial T^2},$$

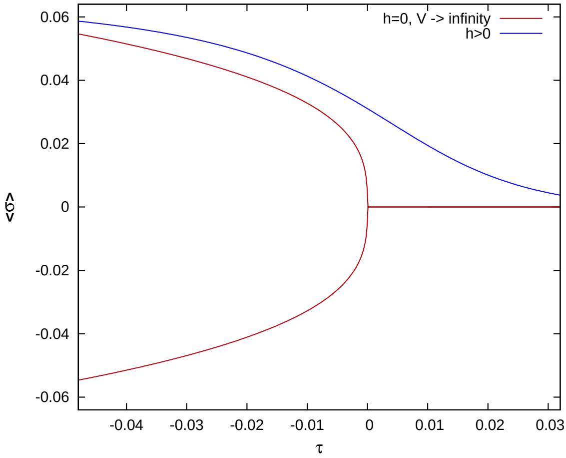
Phase transition: singular behavior of $F(T)$ at $T = T_c$

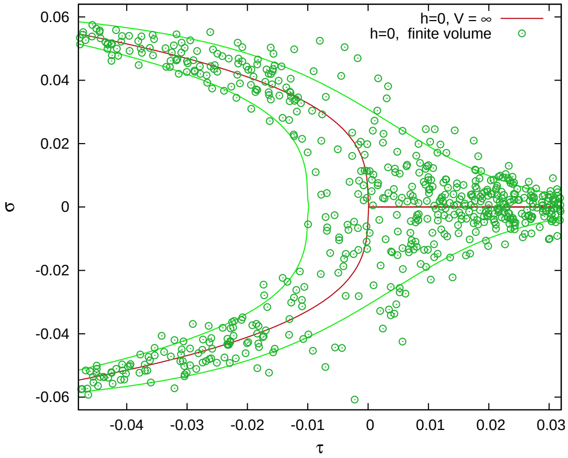
Critical exponents

$$\tau = \frac{T - T_c}{T_c}$$

$$\begin{aligned}
m|_{h=0} &\simeq C_\beta (-\tau)^\beta \quad (\tau < 0) \\
|m|_{\tau=0} &\simeq C_\delta |h|^{1/\delta} \quad (\tau = 0) \\
\chi = \frac{\partial m}{\partial h} &\simeq C_\gamma |\tau|^{-\gamma} \\
G(\vec{k})|_{\tau=0} = a^D \sum_{\vec{n} \in \mathbb{Z}^D} \langle \sigma_{a\vec{n}} \sigma_{\vec{0}} \rangle e^{ia\vec{n}\vec{k}} &\simeq \frac{C_\eta}{|\vec{k}|^{2-\eta}} \\
\text{correlation length } \xi : \quad \langle \sigma_{\vec{x}} \sigma_{\vec{0}} \rangle|_{\tau \neq 0} \sim e^{-|\vec{x}|/\xi} & \\
\xi &\simeq C_\xi |\tau|^{-\nu} \\
\text{heat capacity}: \quad c_{h=0} &\simeq C_c |\tau|^{-\alpha}
\end{aligned}$$


h=0, V -> infinity
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
<σ>
-0.05
-0.04
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
τ


h=0, V -> infinity
h>0
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
<σ>
-0.04
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
τ


h=0, V = ∞
h=0, finite volume
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
σ
-0.04
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
τ

We consider

Ising model	$\longrightarrow$	$SU(2)$ lattice gauge theory
magnetization	$\longrightarrow$	Polyakov loop

- Chromoelectric-chromomagnetic asymmetry
- Zero-momentum longitudinal gluon propagator

Lattices: $(\vec{x}, x_4) \in \Lambda(N_t \times N_s^3), \qquad N_t = 8, \quad 32 \leq N_s \leq 88$

$$L(\vec{x}) = \frac{1}{2} Tr \prod_{x_4=1}^{N_t} U(\vec{x}, x_4; \mu = 4)$$

Polyakov loop: $$\mathcal{P} = \frac{1}{N_s^3} \sum_{\vec{x}} L(\vec{x})$$

$$\langle L(\vec{x}) L(\vec{0}) \rangle \simeq A \exp\left(- \frac{|\vec{x}|}{\xi} \right), \quad |\vec{x}| \to \infty$$

$$G(\vec{p}) = \frac{1}{N_s^3} \sum_{\vec{x}} \langle L(\vec{x}) L(\vec{0}) \rangle e^{i \vec{p} \vec{x}}$$

Critical exponents and amplitudes

$$\tau = \frac{T - T_c}{T_c}; \quad \tau > 0 - \text{deconfinement}$$

$$\langle \mathcal{P} \rangle \simeq B\tau^{\beta}$$

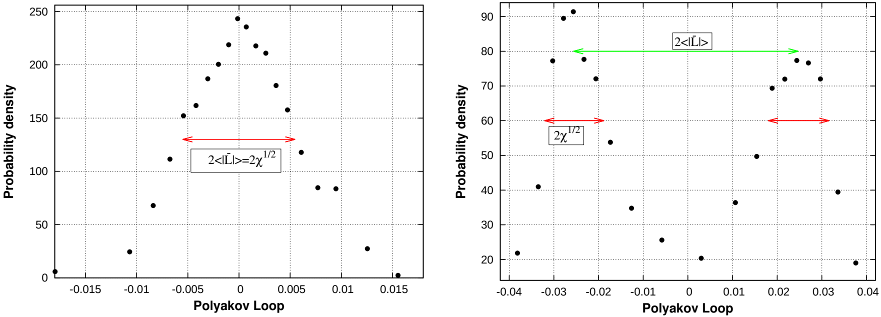
$$\xi \simeq \frac{f_{\pm}}{|\tau|^{\nu}}$$

$$\langle \mathcal{P}^2 \rangle - \langle \mathcal{P} \rangle^2 = G(\vec{0}) \simeq \frac{C_{\pm}}{N_s^3 |\tau|^{\gamma}}$$

$$\tau = 0 : \qquad G(\vec{p}) \simeq \frac{H}{|\vec{p}|^{2-\eta}}$$

Universality hypothesis:

$3D$ Ising model <==> $SU(2)$ in 3+1



Binder 1981 – Ising model;
Mitrjuskin, Zadorozhny 1986 – Lattice $SU(2)$;
Engels, Fingberg et al., 1990 – Binder cumulant

Critical exponents and amplitudes

$3D$ Ising
F.Kos, D.Poland et al.
JHEP (2016)

$SU(2), 4D$
J.Engels, T.Schiedeler 1998

$\beta = 0.326419(3)$
$\gamma = 1.237075(10)$
$\eta = 0.036298(2)$
$\nu = 0.629971(4)$

$B = 0.825(1)$
$C_+ = 0.0587(8),\ C_- = 0.01243(12)$

C_+/C_- is universal; for the 3D Ising universality class
$C_+/C_- = 4.75(3)$ [1998]

Conformal bootstrap

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|x-y|^{2\Delta_{\mathcal{O}}}}$$

$$\nu = \frac{1}{3-2\Delta_\epsilon}, \qquad \gamma = \frac{3-2\Delta_\sigma}{3-\Delta_\epsilon}$$

$$\langle A(x)B(y)C(z)\rangle = \frac{f_{ABC}}{|x-y|^{\Delta_A+\Delta_B-\Delta_C}|y-z|^{\Delta_B+\Delta_C-\Delta_A}|z-x|^{\Delta_C+\Delta_A-\Delta_B}}$$

$$\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle = \sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} C^{\Delta_\sigma}_{\Delta_{\mathcal{O}} l_{\mathcal{O}}}(x_1,x_2,x_3,x_4)$$

$$\sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} \left(C^{\Delta_\sigma}_{\Delta_{\mathcal{O}} l_{\mathcal{O}}}(x_1,x_2,x_3,x_4) - C^{\Delta_\sigma}_{\Delta_{\mathcal{O}} l_{\mathcal{O}}}(x_3,x_2,x_1,x_4)\right) = 0$$

Poland, Rychkov et al., Nature 2016

The Chromo-Electric-Magnetic Asymmetry

$$\begin{aligned} \langle A_E^2 \rangle &= g^2 \langle A_4^a(x) A_4^a(x) \rangle, \\ \langle A_M^2 \rangle &= g^2 \langle A_i^a(x) A_i^a(x) \rangle. \end{aligned} \quad (1)$$

The quantity of particular interest is the (color) electric-magnetic asymmetry introduced by Chernodub and Ilgenfritz in 2008:

$$\langle \Delta_{A^2} \rangle \equiv \langle A_E^2 \rangle - \frac{1}{3} \langle A_M^2 \rangle \ . \quad (2)$$

Later we will use the dimensionless quantity

$$\Delta_{A^2} = \frac{\langle A_E^2 \rangle - \frac{1}{3} \langle A_M^2 \rangle}{T^2} \ . \quad (3)$$

We work in the Landau gauge $\partial_\mu A_\mu^a = 0$

Definition of the longitudinal (L) and transverse (T) propagators:

$$D^{ab}_{\mu\nu}(p) = \delta_{ab}\left(P^T_{\mu\nu}(p)D_T(p) + P^L_{\mu\nu}(p)D_L(p)\right) ,$$

where $P^{T;L}_{\mu\nu}(p)$ - orthogonal transverse (longitudinal) projectors

$$D_L(p) = \frac{1}{3}\sum_{a=1}^{3}\langle A^a_0(p)A^a_0(-p)\rangle$$

$$D_T(p) = \left[\begin{array}{ll} \frac{1}{6}\sum_{a=1}^{3}\sum_{i=1}^{3}\langle A^a_i(p)A^a_i(-p)\rangle & p \neq 0 \\ \frac{1}{9}\sum_{a=1}^{3}\sum_{i=1}^{3}\langle A^a_i(p)A^a_i(-p)\rangle & p = 0 \end{array}\right.$$

chromoelectric screening mass $m_e = \dfrac{1}{\sqrt{D_L(0)}}$

We study critical behavior of the quantities

$$\mathcal{A} = \Delta_{A^2} - \Delta_{A^2}^{C} \tag{4}$$

and

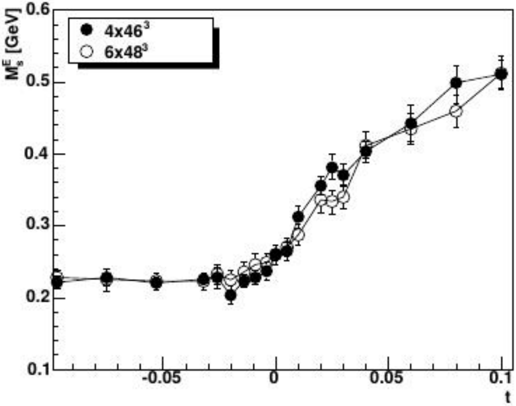
$$\mathcal{D} = D_L(0) - D_L^{C}(0) \ , \tag{5}$$

Asymptotic expansions of Δ_{A^2} and $\mathcal{D}$
in $\tau = \dfrac{T - T_c}{T_c}$ at $\tau \to 0_+$ have the form

$$\mathcal{A} \simeq B_{\mathcal{A}} \tau^{\beta_{\mathcal{A}}} \ , \tag{6}$$

$$\mathcal{D} \simeq B_{\mathcal{D}} \tau^{\beta_{\mathcal{D}}} \ , \tag{7}$$

We evaluate the critical exponents $\beta_{\mathcal{A}}$ and $\beta_{\mathcal{D}}$
and amplitudes $B_{\mathcal{A}}$ and $B_{\mathcal{D}}$.



$$M^E = \frac{1}{\sqrt{D_L(0)}}$$

A.Maas, J.Pawlowski, L von Smekal, D.Spielmann, 2011

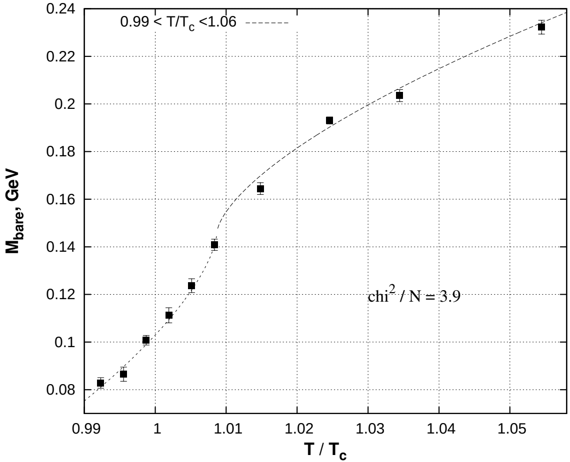
A.Maas *et al.*, 2011 (6×48^3):

$$M_E(\tau) = m_{gribov} + \theta(\tau)\mathcal{M}_+\tau^{\gamma_+/2} + \theta(-\tau)\mathcal{M}_-\tau^{\gamma_-/2} \quad (8)$$

$$m_{gribov} = 0.25^{+3}_{-2}; \quad \mathcal{M}_+ = 1.5^{+1}_{-3}, \;\; \mathcal{M}_- = -0.07^{+736}_{-17}\;; \quad (9)$$
$$\gamma_+ = 1.54^{-12}_{0.05}\;, \;\; \gamma_- = 0.6^{+45}_{+5}$$

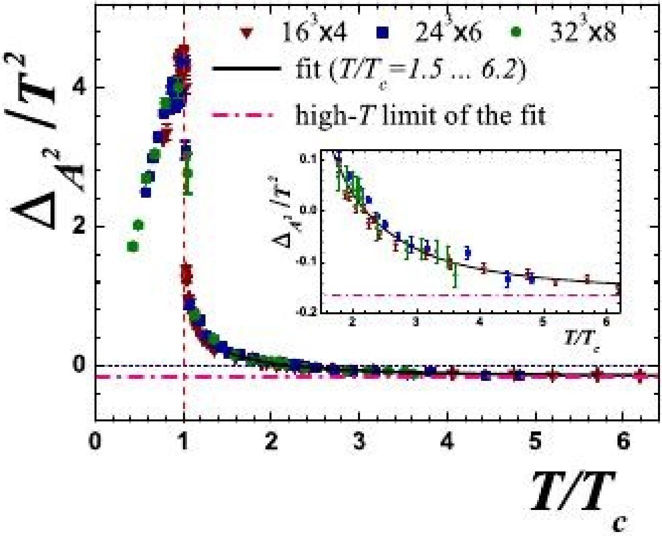
Our **previous** result 2015,
($N_t = 8$, extrapolation to the infinite-volume limit):

$$m_{gribov} = 0.217(3); \quad \mathcal{M}_+ = 0.93(11), \;\; \mathcal{M}_- = -1.23(19)\;; \quad (10)$$
$$\gamma_+ = \gamma_- = 0.63(3)$$

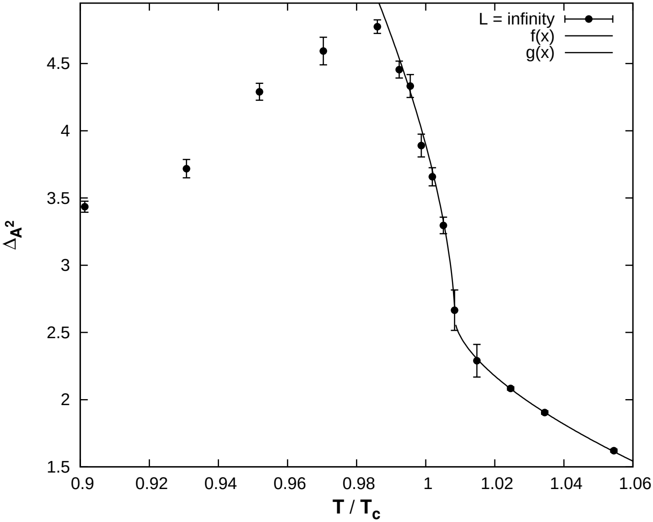


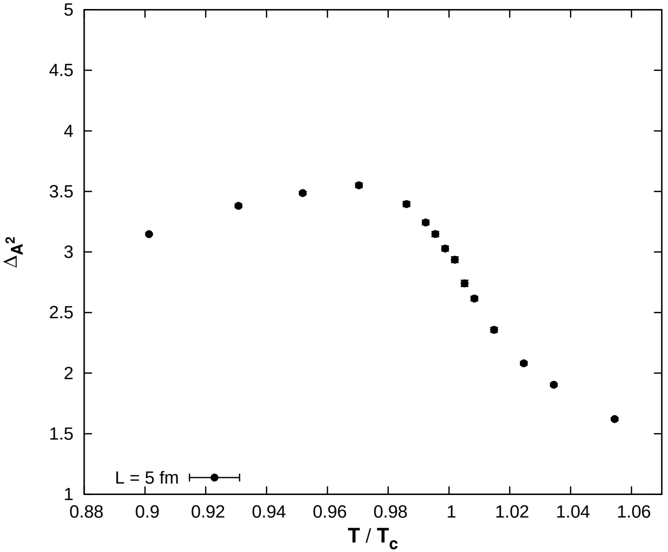
Unjustified assumptions:

- **Maas' et al.:** Correlator $\langle A_0(\vec{x})A_0(\vec{0})\rangle$ is associated with the same critical exponent (γ) as that of Polyakov loops
- **Our :** Negative Polyakov-loop sector can be safely ignored

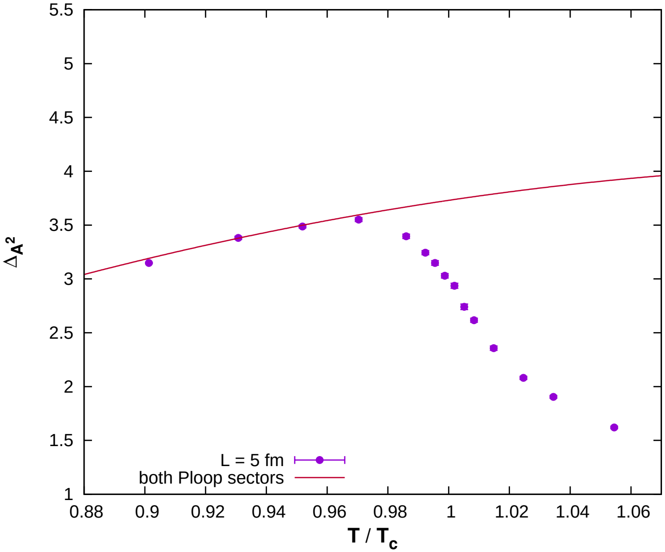


Chernodub, Ilgenfritz 2008

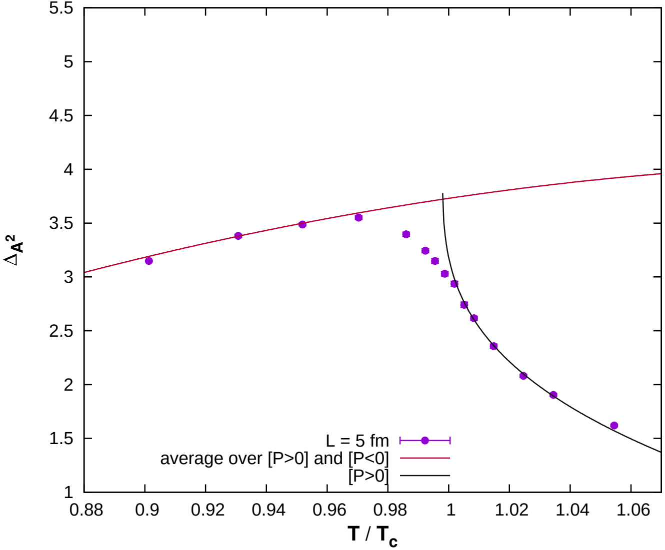

L = infinity
f(x)
g(x)
Δ_A^2
T / T_c
4.5
4
3.5
3
2.5
2
1.5
0.9
0.92
0.94
0.96
0.98
1
1.02
1.04
1.06



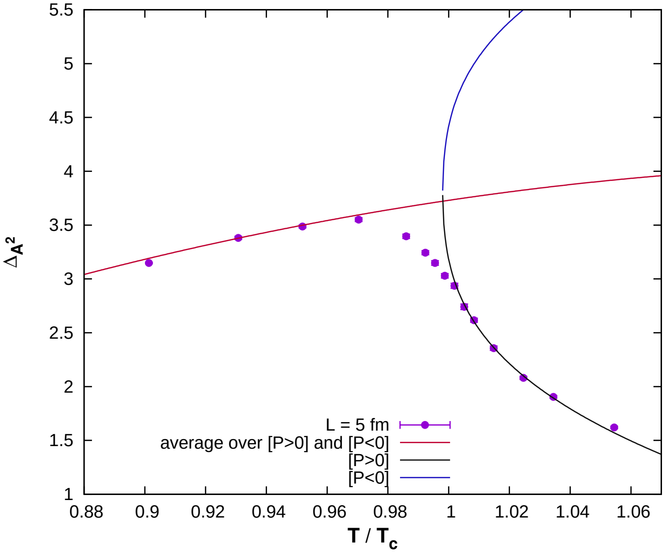
5
4.5
4
3.5
3
2.5
2
1.5
1
0.88
0.9
0.92
0.94
0.96
0.98
1
1.02
1.04
1.06
Δ_A^2
T / T_c
L = 5 fm



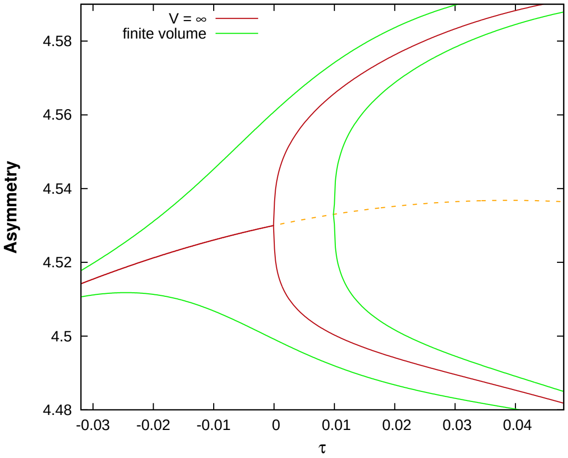
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0.88
0.9
0.92
0.94
0.96
0.98
1
1.02
1.04
1.06
Δ_A^2
T / T_c
L = 5 fm
both Ploop sectors

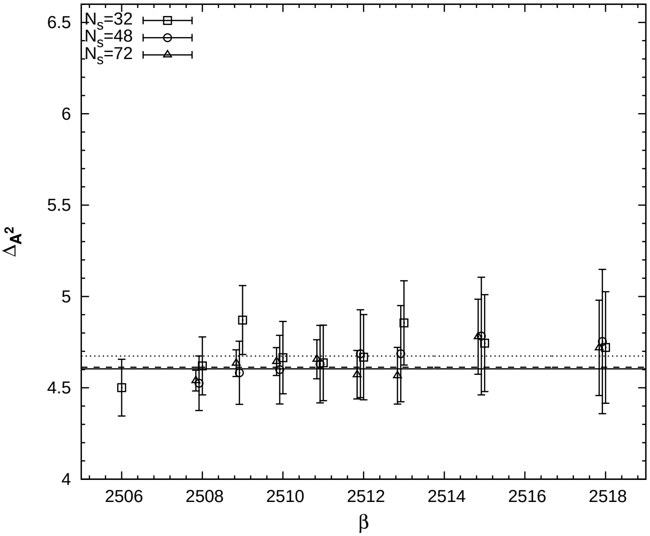


5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0.88
0.9
0.92
0.94
0.96
0.98
1
1.02
1.04
1.06
Δ_A^2
T / T_c
L = 5 fm
average over [P>0] and [P<0]
[P>0]

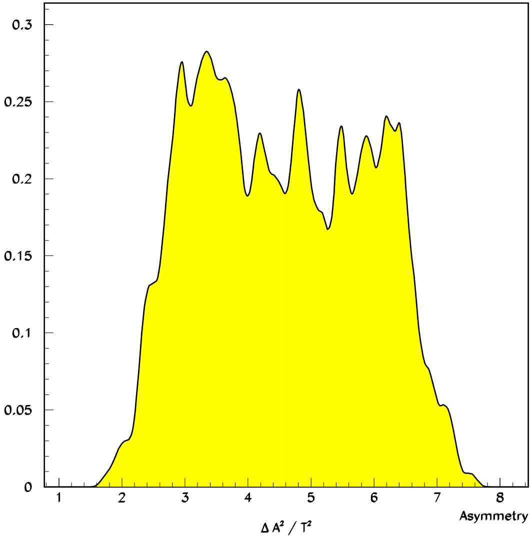


5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
Δ_A^2
0.88
0.9
0.92
0.94
0.96
0.98
1
1.02
1.04
1.06
T / T_c
L = 5 fm
average over [P>0] and [P<0]
[P>0]
[P<0]

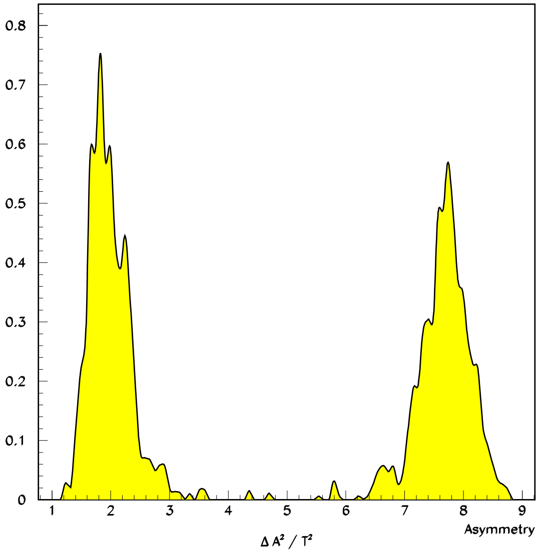

V = ∞
finite volume
Asymmetry
4.58
4.56
4.54
4.52
4.5
4.48
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
0.04
τ



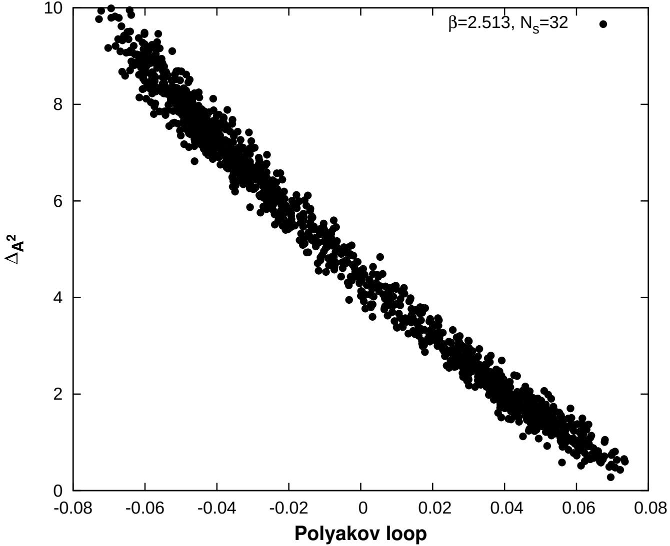
Average value of Δ_{A^2} versus T, $T_c \to \beta = 2.5104(2)$



$T/T_c = 0.9925$; $L = 6.0$ fm; $72^3 \times 8$

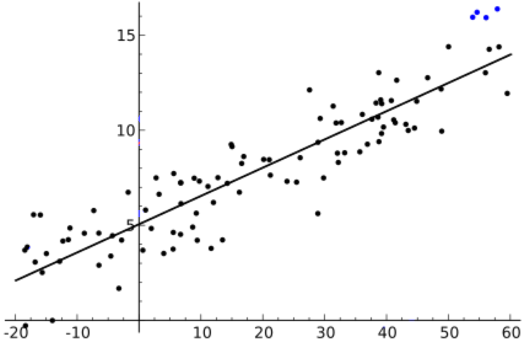


$T/T_c = 1.024$; $L = 5.8$ fm; $\quad 72^3 \times 8$



Scatter plot: asymmetry versus Polyakov loop; $\tau = 0.008$, $L = 2.7$ fm

Regression analysis



Y (regressand)
depends on
X (regressor)

The problem: to find the conditional expectation value of Y

as a function of X:
$E(Y|X) = f(X, \theta)$,

here $f(X, \theta) = \theta_0 + \theta_1 X$

Quantities under consideration

- The conditional cumulative distribution function (CDF)

$$F(\Delta|\mathcal{P}) \qquad\qquad \text{(on this page} \quad \Delta \equiv \Delta_{A^2}\text{)}$$

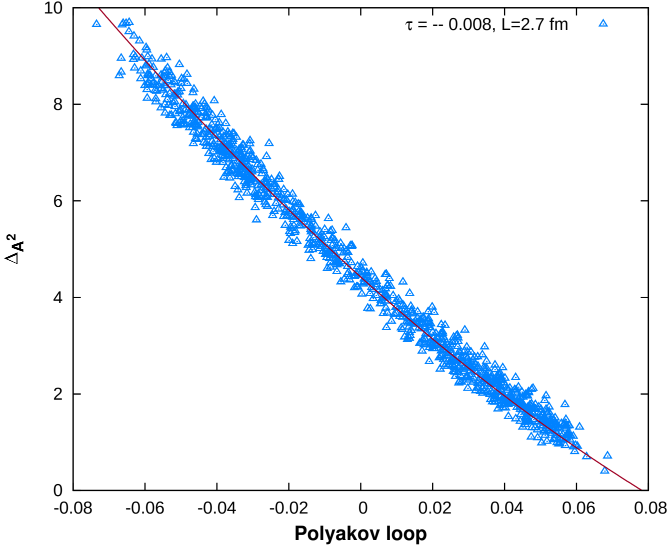
  describes the distribution of gauge-field configurations in the asymmetry for a fixed value $\mathcal{P}$ of the Polyakov loop.
- The conditional expectation

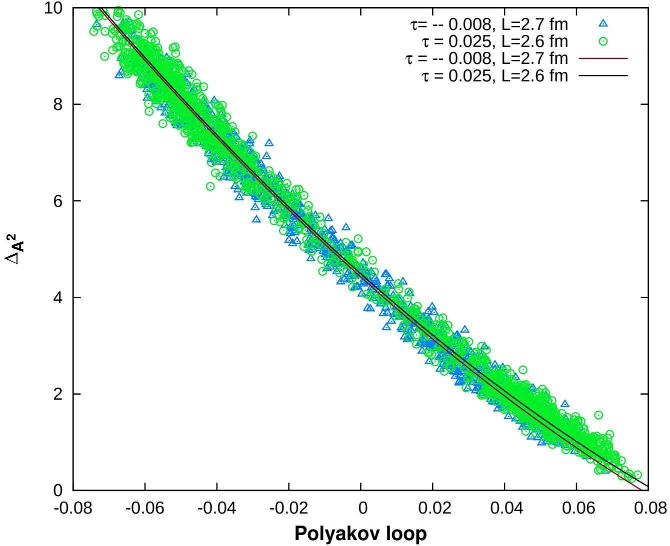
$$\langle\Delta\rangle_{\mathcal{P}} = E(\Delta|\mathcal{P}) = \int \frac{dF(\Delta|\mathcal{P})}{d\Delta}\,\Delta\,d\Delta.$$

- As $T \to T_{c+}$ (that is, at $\mathcal{P} \sim 0$) it can be fitted to a polynomial:

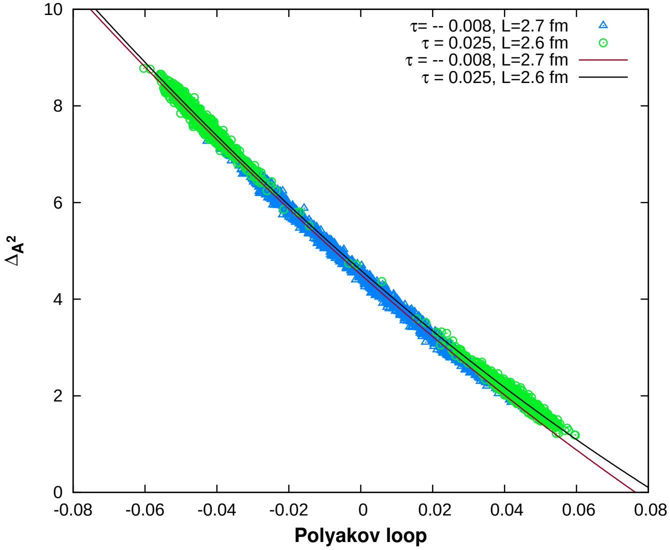
$$E(\Delta|\mathcal{P}) \simeq \Delta^{C} + \sum_{j=1}^{n} A_j \mathcal{P}^j\,.$$

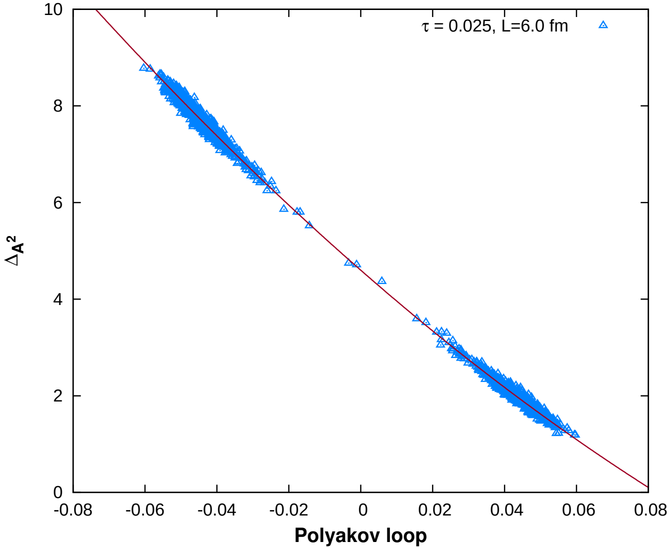
We employ the method of least squares to determine $\mathcal{A}_C$ and A_i


τ = -- 0.008, L=2.7 fm
Δ_A^2
Polyakov loop
10
8
6
4
2
0
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08



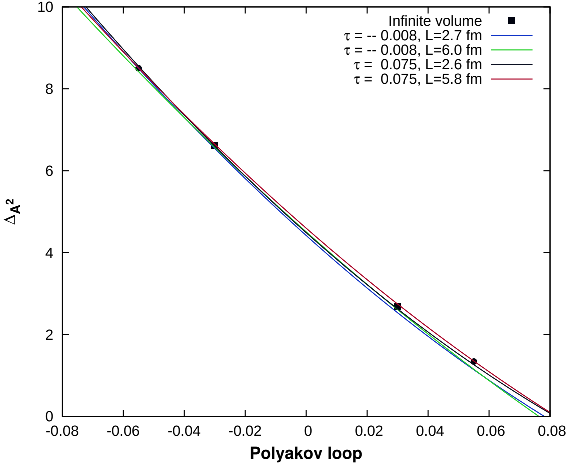
τ= -- 0.008, L=2.7 fm
τ = 0.025, L=2.6 fm
τ = -- 0.008, L=2.7 fm
τ = 0.025, L=2.6 fm
10
8
6
4
2
0
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
Polyakov loop


τ= -- 0.008, L=2.7 fm
τ = 0.025, L=2.6 fm
τ = -- 0.008, L=2.7 fm
τ = 0.025, L=2.6 fm
Δ_A^2
Polyakov loop
10
8
6
4
2
0
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08



τ = 0.025, L=6.0 fm
10
8
6
4
2
0
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
Δ_{A^2}
Polyakov loop

Infinite-volume limit



distributions tend to dots on the regression curve

Our main assumption

$$\Delta_{A^2} = \Delta^C_{A^2} + A_1 \mathcal{P} + \overline{o}(\mathcal{P})$$

where

$$\lim_{\mathcal{P}\to 0} \frac{\overline{o}(\mathcal{P})}{\mathcal{P}} = 0$$

$$\Longrightarrow$$

$$\beta_{\mathcal{A}} = \beta = 0.326419(3),$$
$$B_{\mathcal{A}} = A_1 B = -9.6(2.3)$$

$$\Delta_{A^2} = \Delta_{A^2}^{C} + A_1\mathcal{P} + A_2\mathcal{P}^2 + ...$$

From this expansion it follows that

$$\mathcal{A} = \Delta_{A^2} - \Delta_{A^2}^{C} \simeq A_1\mathcal{P} \simeq A_1 B\tau^{\beta} \ , \tag{11}$$

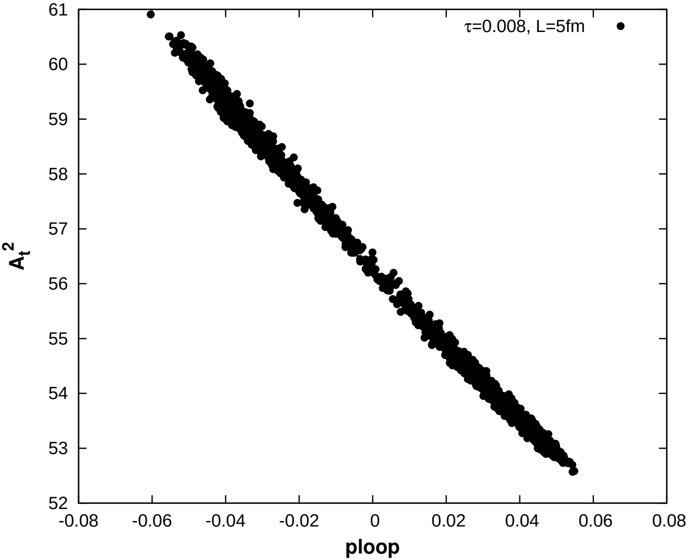
whereas, by definition,

$$\mathcal{A} \simeq B_{\mathcal{A}}\tau^{\beta_{\mathcal{A}}}. \tag{12}$$

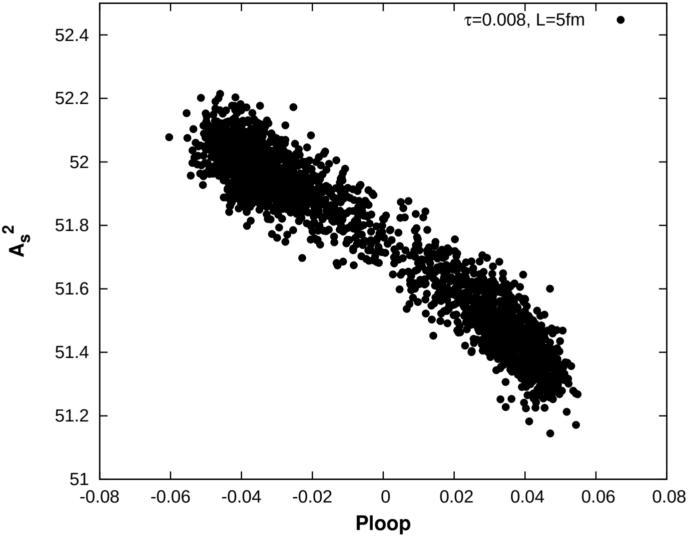
Therefore,

$$\beta_{\mathcal{A}} = \beta = 0.326419(3),$$
$$B_{\mathcal{A}} = A_1 B = -9.6(2.3)$$

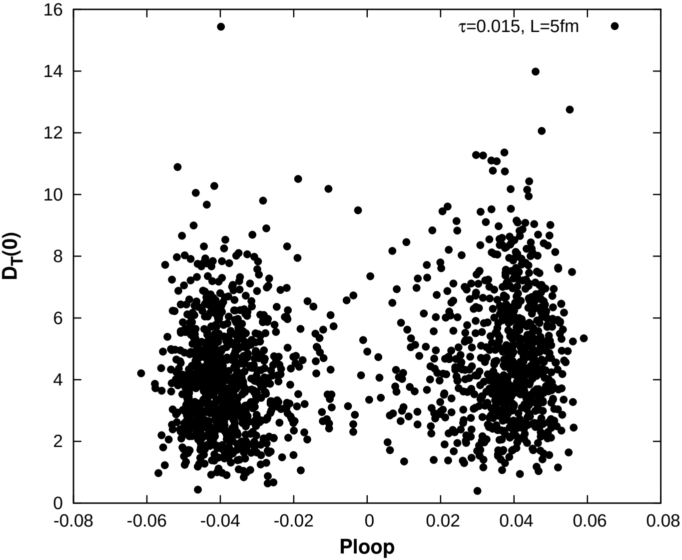
At $\tau < 0$ $\mathcal{A} \approx 0$ is a smooth function

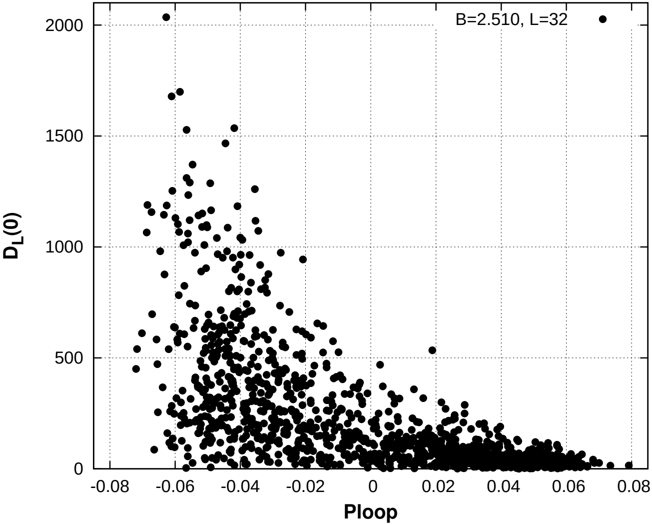


τ=0.008, L=5fm
A_t^2
ploop
61
60
59
58
57
56
55
54
53
52
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08

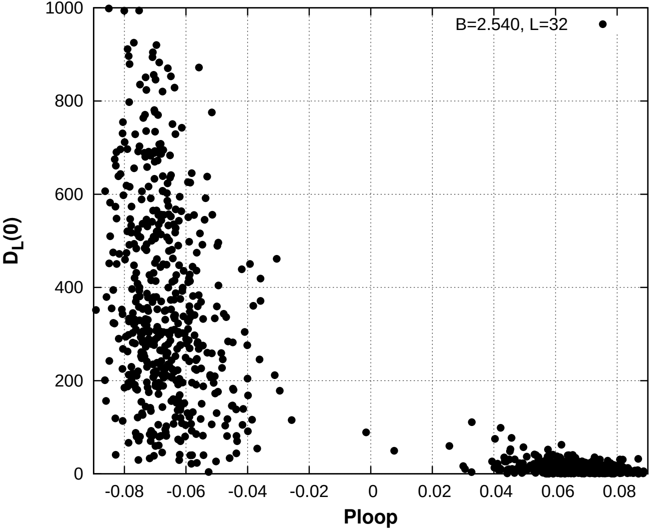


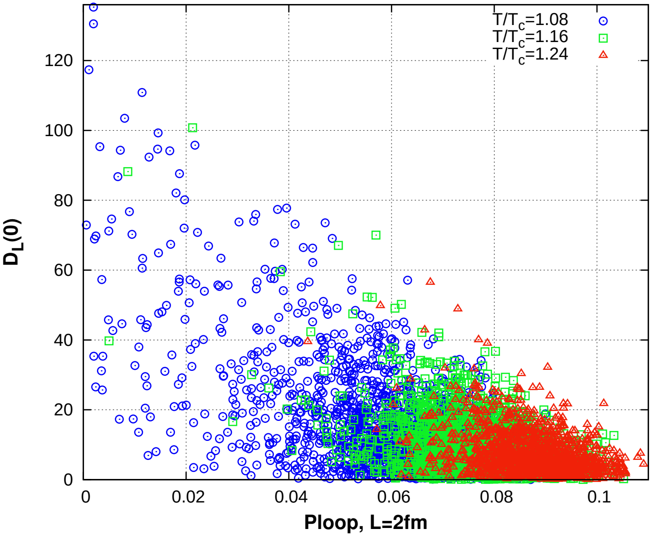
τ=0.008, L=5fm
52.4
52.2
52
51.8
51.6
51.4
51.2
51
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
Ploop
A_s^2


τ=0.015, L=5fm
D_T(0)
Ploop
16
14
12
10
8
6
4
2
0
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08

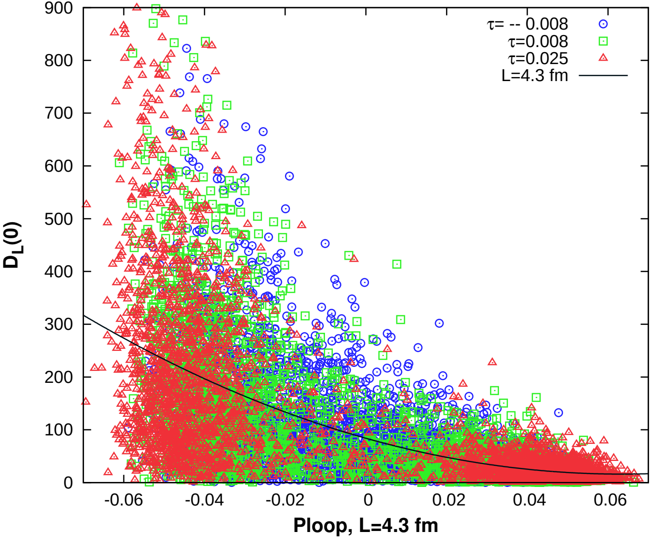


B=2.510, L=32
2000
1500
1000
500
0
$D_L(0)$
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
Ploop


B=2.540, L=32
$D_L(0)$
Ploop
1000
800
600
400
200
0
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08



Homoscedasticity is severely broken



τ= -- 0.008
τ=0.008
τ=0.025
L=4.3 fm
900
800
700
600
500
400
300
200
100
0
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
$D_L(0)$
Ploop, L=4.3 fm

Critical behavior of the bare longitudinal propagator

$$D_L(0) = D_L^C(0) + D_1\mathcal{P} + D_2\mathcal{P}^2 + ...$$

From this expansion it follows that

$$\mathcal{D} = D_L(0) - D_L^C(0) \simeq D_1\mathcal{P} \simeq D_1 B\tau^{\beta} \ , \qquad (13)$$

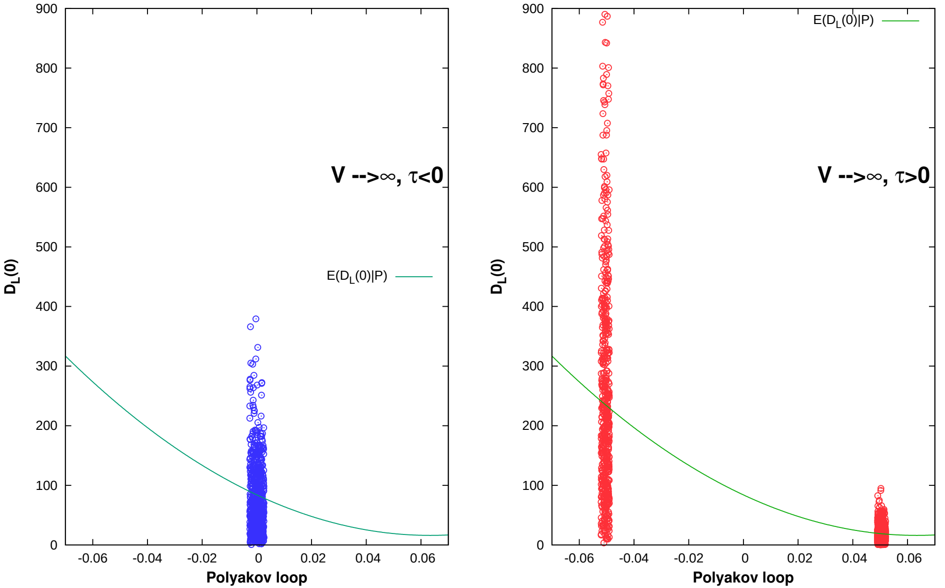
whereas, by definition,

$$\mathcal{D} \simeq \mathcal{D}\tau^{\beta_{\mathcal{D}}}. \qquad (14)$$

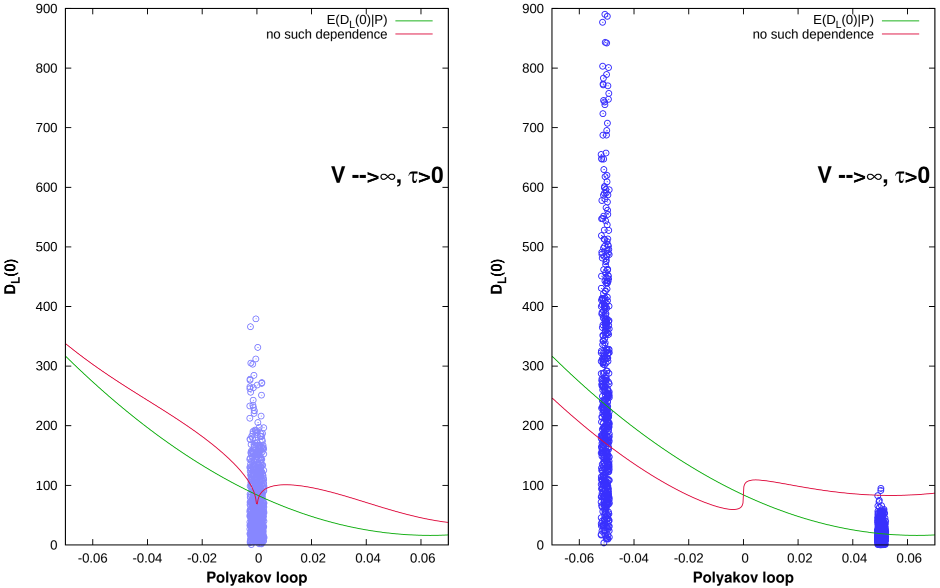
Therefore,

$$\beta_{\mathcal{D}} = \beta = 0.326419(3),$$
$$B_{\mathcal{D}} = D_1 B = -330(80)\ \text{GeV}^{-2}$$

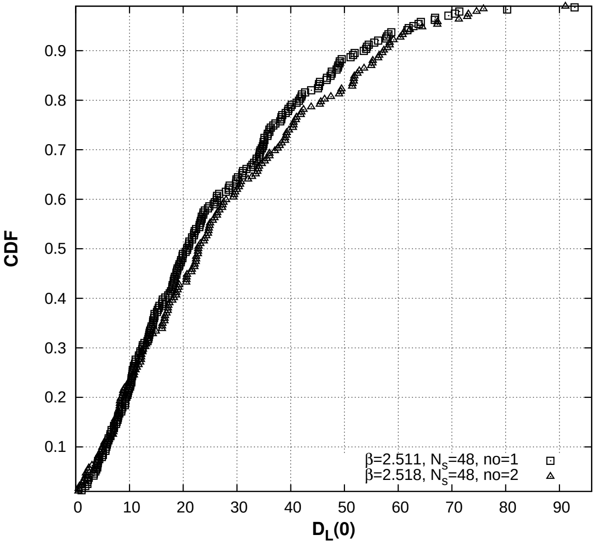
$a^{-1} \sim 2.5$ GeV



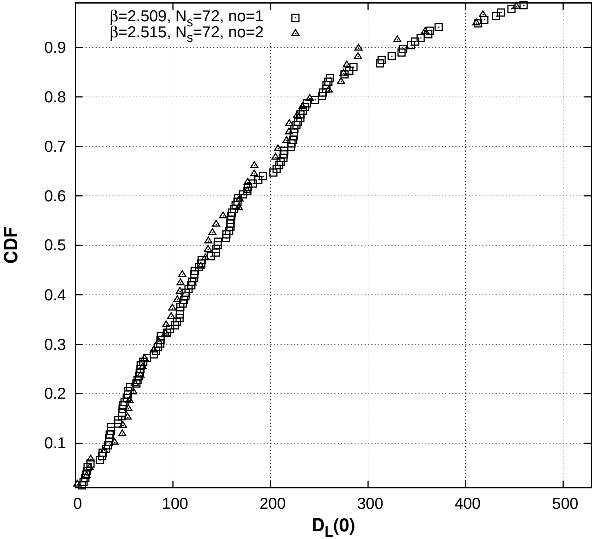
V -->∞, τ<0
E(D_L(0)|P)
D_L(0)
Polyakov loop
V -->∞, τ>0
E(D_L(0)|P)
D_L(0)
Polyakov loop



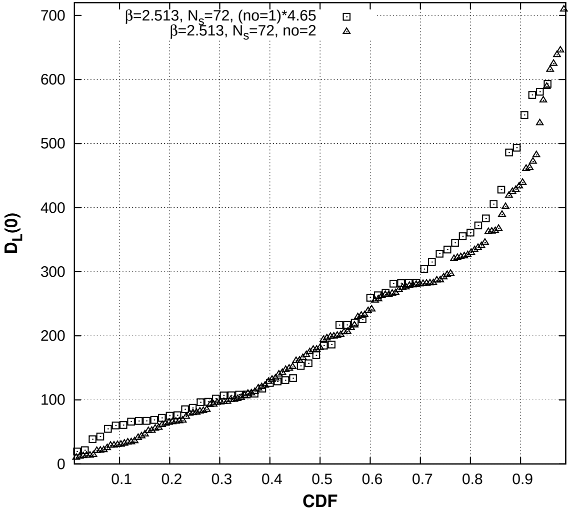
E(D_L(0)|P)
no such dependence
V -->∞, τ>0
D_L(0)
Polyakov loop
E(D_L(0)|P)
no such dependence
V -->∞, τ>0
D_L(0)
Polyakov loop



$0.035 < \mathcal{P} < 0.040$; $L = 4$ fm; $1 \rightarrow \tau = 0.002$; $2 \rightarrow \tau = 0.025$



$-0.030 < \mathcal{P} < -0.025$; $L = 6$ fm; *1* $\rightarrow \tau = -0.0045$; *2* $\rightarrow \tau = 0.0148$



$L = 6$ fm;
$\tau = 0.0048$;

$1 \rightarrow 0.035 < \mathcal{P} < 0.040$; $2 \rightarrow -0.030 < \mathcal{P} < -0.025$;

Conclusions

- Both the asymmetry and the longitudinal propagator have a significant correlation with the Polyakov loop.
- Regression analysis reveals the dependence of each of these quantities on the Polyakov loop $\mathcal{P}$ as follows:

$$D \simeq D_0 + D_1\mathcal{P} + D_2\mathcal{P}^2$$

- Such dependence implies that in the infinite-volume limit both Δ_{A^2} and $D_L(0)$

$$\beta_{\mathcal{A}} = \beta_{\mathcal{D}} = \beta = 0.326419(3)$$

- $B_{\mathcal{A}} = -9.6(2.3), \quad B_{\mathcal{D}} = -330(80)\ \mathrm{GeV}^{-2}$ (bare quantities)
- Scaling in the conditional distribution of $D_L(0)$ is observed