Amplitudes in $\mathcal{N} = 4$ supergravity

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based on
[arXiv:1202.3692] and [arXiv:1208.1255]
with Piotr Tourkine

A little tribute to this triangular meeting and Pietro Fre’

И Паниковский от правого конца прямой повел вверх волнистый перпендикуляр. [...] Тут Паниковский соединил обе линии третьей, так что на песке появилось нечто похожее на треугольник, и закончил: [...] Балаганов **с уважением посмотрел на треугольник** [...]
– Поезжайте в **Дубну!** - сказал он неожиданно. - И тогда вы поймете, что я прав. Обязательно поезжайте в **Дубну**!.

Freely adapted from Илф и Петров, “Золотой теленок”

Motivations

$\mathcal{N} = 4$ and $\mathcal{N} = 8$ supergravity arises as the low-energy limit of string

String theory provides a consistent ultraviolet finite theory of quantum gravity. One could wonder if one can remove the string massive modes and address the question of ultraviolet behaviour of *pure supergravity*

In this talk we will discuss

- the role of supersymmetry in perturbative computation
- the role of non-perturbative duality symmetries in string theory

Behavior of supergravity amplitudes

Gravity has a dimensional coupling constant

$$[1/\kappa^2_{(D)}] = \text{mass}^{D-2}$$

An L-loop n-point gravity amplitude in D-dimensions has the dimension

$$[\mathfrak{M}_L^{(D)}] = \text{mass}^{(D-2)L+2}$$

4-graviton amplitudes factorize an $\mathcal{R}^4$ term and possibly higher derivatives

$$[\mathfrak{M}_L^{(D)}] = \text{mass}^{(D-2)L-6-2\beta_L^N}\, \partial^{2\beta_L^N}\mathcal{R}^4$$

- Critical dimension for UV divergences is

$$D \geqslant D_c = 2 + \frac{6 + 2\beta_L^N}{L}$$

Supersymmetry and UV behaviour

- Critical dimension for UV divergence is

$$D \geqslant D_c = 2 + \frac{6 + 2\beta_L^N}{L}$$

- Depending on the various implementations of supersymmetry

$$6 \leqslant 6 + 2\beta_L^N \leqslant 18$$

- With a first possible divergence in $D = 4$ at
  - $L \geqslant 3$: $\beta_L^N = 0$ [Howe, Lindstrom, Stelle '81]
  - $L \geqslant 5$: $\beta_L^8 = 2$ [Howe, Stelle '06; Bossard, Howe, Stelle '09]
  - $L \geqslant 8$: $\beta_L^8 = 5$ [Kallosh '81]
  - $L \geqslant 7$: $\beta_L^8 = 4$ [Vanhove '10; Green, Bjornsson '10]
  - $L \geqslant 9$: $\beta_L^8 = 6$ [Green,Russo,Vanhove '06]
  - $L = \infty$: $\beta_L^8 = L$ [Green,Russo,Vanhove '06]

Supersymmetry and UV behaviour

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$$D \geqslant D_c = 2 + \frac{6 + 2\beta_L^N}{L}$$

- Depending on the various implementations of supersymmetry

$$6 \leqslant 6 + 2\beta_L^N \leqslant 18$$

- With a first possible divergence in $D = 4$ for $\mathcal{N} = 4$ supergravity
  - $\beta_3^4 > 0$ [Bern, Davies, Dennen, Huang, '12]
  - $L \geqslant 2$: $\beta_L^4 = 2$ [Tourkine, Vanhove, '12]

Non-renormalisation theorems

Supersymmetry implies various non-renormalisation theorems for higher dimension operators.

- Heterotic compactification ($\mathcal{N} = 4$ models)
  - $\mathcal{R}^4$ is a $\frac{1}{2}$-BPS coupling 1-loop exact in perturbation [Bachas, Kiritsis; Bachas, Fabre, Kiritsis, Obers, Vanhove; Tourkine, Vanhove]
- Type II compactifications on a torus ($\mathcal{N} = 8$ models) [Green, Russo, Vanhove; Berkovits]
  - $\mathcal{R}^4$ is $\frac{1}{2}$-BPS : 1-loop exact
  - $\partial^4\mathcal{R}^4$ is $\frac{1}{4}$-BPS : 2-loop exact
  - $\partial^6\mathcal{R}^4$ is $\frac{1}{8}$-BPS : 3-loop exact

These operators are potential UV divergences counter-term to supergravity in various dimensions

How these stringy results allow to conclude about the ultraviolet behaviour of supergravity?

Part I

$\mathcal{N} = 8$ supergravity

The ultraviolet behaviour of $\mathcal{N}=8$ supergravity

- $\mathcal{N}=8$ non-renormalisation theorems imply that [Green, Russo, Vanhove] imply that up to and including 4 loops the rule $\beta_L^8 = L$ is satisfied and $\mathcal{N}=8$ SUGRA as the same UV behaviour as $\mathcal{N}=4$ SYM

$$[\mathfrak{M}_{4;L}^{(D)}] \sim \Lambda^{(D-4)L-6}\, \partial^{2L}\, \mathcal{R}^4 \qquad 2 \leqslant L \leqslant 4$$

- The critical dimension for UV divergences is

$$D_c = 4 + \frac{6}{L}$$

Same critical UV behaviour for $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA at $1 \leqslant L \leqslant 4$ loops

The ultraviolet behaviour of $\mathcal{N} = 8$ supergravity

Should we expect a deviation from the $\beta_L^8 = L$ rule?
At which order $\mathcal{N} = 8$ SUGRA can have a worse UV behaviour of $\mathcal{N} = 4$ SYM?

An equivalent question: In $\mathcal{N} = 8$ is the $\partial^8 \mathcal{R}^4$ is protected ?

- After 4-loop it is expected a worse UV behaviour than for $\mathcal{N} = 4$ SYM

[Green, Russo, Vanhove], [Vanhove], [Green, Bjornsson]

$$[\mathfrak{M}_{4;L}^{(D)}] \sim \Lambda^{(D-2)L-14}\, \partial^8\, \mathcal{R}^4 \qquad \beta_L^8 = 4 \text{ for } L \geqslant 4$$

- At five-loop order the 4-point amplitude in
  - $\mathcal{N} = 4$ SYM divergences for $5 < 26/5 \leqslant D$
  - $\mathcal{N} = 8$ SUGRA divergences for $24/5 \leqslant D$

The ultraviolet behaviour of $\mathcal{N}=8$ supergravity

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$$[\mathfrak{M}_{4;L}^{(D)}] \sim \Lambda^{(D-2)L-14}\,\partial^8\,\mathcal{R}^4 \qquad \beta_L^8 = 4 \text{ for } L \geqslant 4$$

- This behaviour indicates a *seven-loop* divergence in $D=4$ with counter-term $\partial^8\mathcal{R}^4$

Candidate counter-term in Harmonic superspace

- Using harmonic superspace we constructed candidate counter-term to UV divergence in $D = 4$ for $\mathcal{N} = 8$ and $\mathcal{N} = 4$
  [Bossard, Howe, Stelle, Vanhove]

- The $\partial^8 \mathcal{R}^4$ term for $\mathcal{N} = 8$ supersymmetric and $E_{7(7)}$ invariant

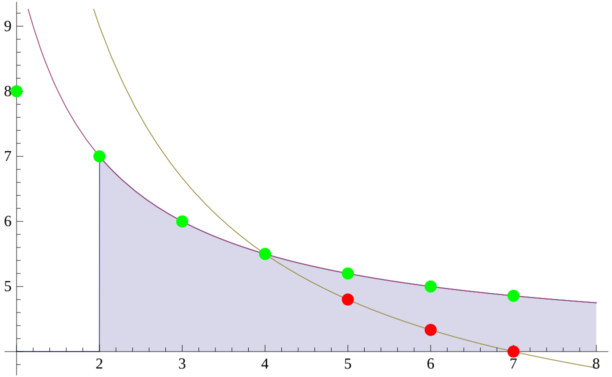
$$\int d\mu_{(8,1,1)}\, \bar{\chi}^{1mn} \chi_{8mn} \bar{\chi}^{1pq} \chi_{8pq} \sim \int d^4x e \,(\partial^8 \,\mathcal{R}^4 + \cdots)$$

- The $\mathcal{R}^4$ term for $\mathcal{N} = 4$ supersymmetric and $SU(1,1)$ invariant expression

$$\int d\mu_{(4,1,1)}\, \bar{\chi}^{1mn} \chi_{4mn} \bar{\chi}^{1pq} \chi_{4pq} \sim \int d^4x e \,(\mathcal{R}^4 + \cdots)$$

- Since the volume of superspace vanishes for $\mathcal{N} \leqslant 8$
  [Bossard, Howe, Stelle, Vanhove] are these terms F-terms or D-terms?

The ultraviolet behaviour of $\mathcal{N} = 8$ supergravity



- red dots the predicted UV behaviour [Green, Russo, Vanhove]
- (green dots) confirmed behaviour using various field theory supersymmetry [Bossard, Stelle, Howe], and direct loop computation [Bern, Carrasco, Dixon, Johansson, Roiban], and continuous E_7 arguments [Elvang, Keirmaier, Freedman et al.]

Part II

$\mathcal{N} = 4$ supergravity

Constructing $\mathcal{N} = 4$ supergravity from string theory

- String theory constructions lead to models that has pure $\mathcal{N} = 4$ SUGRA coupled to $0 \leqslant n_v$ vector multiplet in four dimensions

- The string theory moduli space is (with $\Gamma \subset SL(2, \mathbb{Z})$)

$$\Gamma \backslash SU(1,1,\mathbb{R})/U(1) \times SO(6, n_v, \mathbb{Z}) \backslash SO(6, n_v, \mathbb{R})/SO(6) \times SO(n_v)$$

To get pure $\mathcal{N} = 4$ supergravity we want to set $n_v = 0$ and decouple the string modes.

The string theory effective action is given by

$$\mathcal{S} = \frac{1}{\ell_4^2} \int d^4x\, e \, \left(\mathcal{R} + \ell_4^2 f(S, \bar{S}) \mathcal{R}^2 + \ell_4^6 \, g(S, \bar{S}) \, \mathcal{R}^4\right)$$

Constructing $\mathcal{N} = 4$ supergravity from string theory

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To get pure $\mathcal{N} = 4$ supergravity we want to set $n_v = 0$ and decouple the string modes.

- For $(4,0)$ models the complex scalar S of the supergravity multiplet is in the $SU(1,1,\mathbb{R})/U(1)$ factor
- For $(2,2)$ models the scalar S parametrizes a $SU(1,1,\mathbb{R})/U(1) \subset SO(6,n_v,\mathbb{R})/SO(6) \times SO(n_v)$
- these models are *non-perturbatively dual* in string theory

The one-loop amplitudes in $\mathcal{N} = 4$ supergravity

- We [Tourkine, Vanhove] have checked that for the one-loop four-graviton amplitudes in *all* the $(4, 0)$ and $(2, 2)$ we can decouple the vector multiplet and obtain the *same* result as the computation in the supergravity done by [Dunbar et al.; Bern et al.]
- In a recent work [Carrasco, Chiodaroli, Gunaydin, Roiban] have shown that all the four-point amplitudes (including external vector) are the *same* in all the models.

The $\mathcal{N} = 4$ supergravity at higher-loops I

One can construct $(4,0)$ heterotic string models with $6 \leqslant n_v \leqslant 22$ vector multiplets using the CHL asymmetric orbifold construction

In this construction one considers a compactification on $T^5 \times S^1$ and take an orbifold of the current algebra and right moving modes of the string on torus together with an order $1/N$ shift along the S^1 direction

Importantly this construction does not affect the supersymmetric sector of the heterotic string and fermionic zero mode saturation is the same for *all* these models and identical to the torus compactification

The $\mathcal{N} = 4$ supergravity at higher-loops II

- 4 graviton amplitude computations gives that [Tourkine, Vanhove]

$$M_4^{1-loop} \sim \mathcal{R}^4 I_{box}[\ell^4] \quad : \qquad \beta_1^4 = 0$$
$$M_4^{2-loop} \sim \partial^2 \mathcal{R}^4 I_{double-box}[\ell^4] \quad : \qquad \beta_2^4 = 1$$

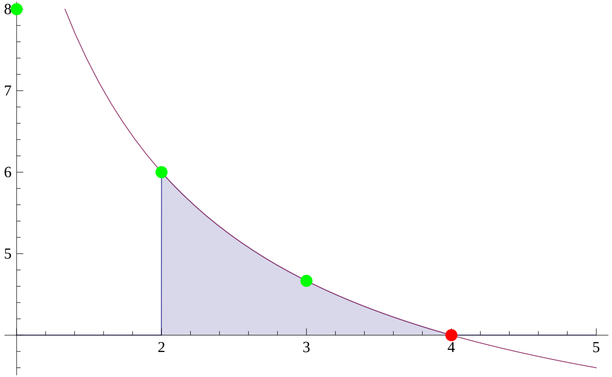
- 1-loop non-renormalisation of $\mathcal{R}^4$: $\beta_L^4 \geqslant 1$ for $L \geqslant 2$

First UV divergence in 4D: $\qquad L \geqslant 3 + \beta_L^4 \geqslant 4$ loops

$\mathcal{N} = 4$ non-renormalisation theorems for $\mathcal{R}^4$ term $\beta_L^4 = 1$ for $L \geqslant 2$

$$[\mathfrak{M}_L^{(D)}] = \text{mass}^{(D-2)L-8}\, \partial^2 \mathcal{R}^4 \qquad \text{for } L \geqslant 2$$

The $\mathcal{N} = 4$ supergravity at higher-loops III



- red dots predicted behaviour [Tourkine, Vanhove]
- green dots Absence of three loop divergence in $D = 4$ at $L = 3$ and $D = 5$ at $L = 2$ was obtained from direct field theory computation [Bern, Davies, Dennen, Huang] in agreement with the general formula above

The special case of $\mathcal{N} = 4$ supergravity

- We are facing a puzzle because a seemingly valid $L = 3$ in $D = 4$ counter-term was constructed using harmonic superspace

$$\int d^4xe\,(\mathcal{R}^4 + \text{susy}) = \int d\mu_{(4,1,1)}\,(\chi\bar{\chi})^2$$

- What is wrong with this operator?

- [Tourkine, Vanhove] suggested that the off-shell version of $\mathcal{N} = 4$ sugra coupled to vector multiplets could still be active in the pure supergravity case. This 'fake' F-term would be a true F-term

The special case of $\mathcal{N} = 4$ supergravity

$\mathcal{N} = 4$ supergravity is special because of the $U(1)$ R-symmetry anomaly [Marcus].

Therefore the $SU(1,1)$ duality symmetry is broken in perturbation and full superspace integrals of functions of the axion-dilaton $\mathcal{S} \in SU(1,1)/U(1)$ are allowed

$$\int d^{16}\theta\, E(x,\theta)\, G(\mathcal{S},\bar{\mathcal{S}}) = f(S,\bar{S})\, \mathcal{R}^4 + \text{susy completion}$$

Only for $f = 1$ this is a 3-loop UV divergence counter-term in the 4 graviton amplitude.

But $f = 1$ would violate the $\mathcal{R}^4$ 1-loop non renormalisation theorems derived from string theory

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Because of the anomaly canceling term $\Re\mathrm{e} \int d^4x\, h(S,\bar{S})\, \mathrm{tr}(R - i * R)^2$ it is tempting to conclude that the $\mathcal{R}^4$ will also have a non-trivial dependence on the scalar field S

$$\kappa_{(4)}^4 \int d^4x f(S,\bar{S})\, \mathcal{R}^4$$

This would be compatible with the string theory non-renormalisation theorems

The special case of $\mathcal{N} = 4$ supergravity

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- Recently [Bossard, Howe, Stelle] have shown that *if* there is an off-shell version for the pure $\mathcal{N} = 4$ supergravity, the 3-loop counter-term would be ruled out by the dualities invariance (although anomalous)

- [Kallosh, Ferrara, van Proyen] have argued for a superconformal invariance that makes pure $\mathcal{N} = 4$ supergravity finite to all orders

Outlook

- Using string theory we have put constraints on the possible counter-terms for UV divergences of four gravitons amplitudes in $\mathcal{N}=4$ and $\mathcal{N}=8$ supergravity
- Using harmonic superspace we have constructed *supersymmetric duality invariant* candidate counter-terms for first possible UV divergence in $D=4$
- We showed that the $\mathcal{R}^4$ term satisfies a *non-renormalisation* theorem in $\mathcal{N}=4$

$$\kappa_{(4)}^4 \int d^4x f(S)\,\mathcal{R}^4$$

- Where $f(S) = tree + 1 - loop$ and no constant contribution
- Since the 3-loop $\mathcal{N}=4$ candidate is not associated to a divergence. Should we expect the same for $\mathcal{N}=8$ given as well by an harmonic superspace 'fake' F-term type of integral? Not really because $\mathcal{N}=8$ duality symmetry is *not* anomalous and there no real expectation that $\mathcal{N}=8$ has an hidden *off-shell* formalism