Quantum black hole models and Schroedinger-like equations in finite differences

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Classical Schwarzschild black hole

$$ds^2 = \left(1 - \frac{2Gm}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2Gm}{r}} - r^2 d\sigma^2.$$

G - Newton's constant, m - total mass (energy), r - radius $(4\pi r^2 = \text{area})(c = 1)$

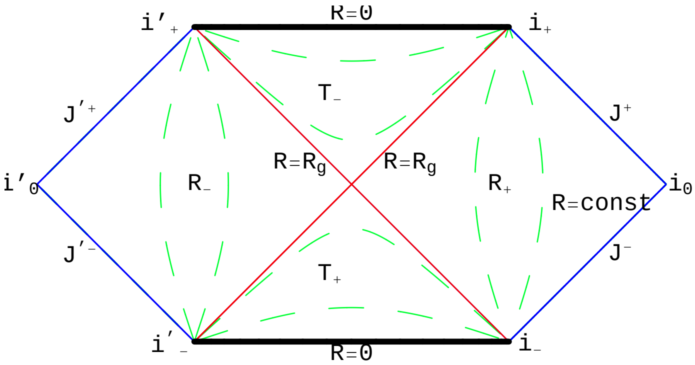

Рис.: Carter-Penrose diagram for geodesically complete Schwarzshcild space-time.

- Wormhole geometry. Rather complex boundary
- No dynamical degrees of freedom!

- Constraint equation

$$\sigma_{in}\sqrt{\dot{\rho}^2+1-\frac{2Gm_{in}}{\rho}}-\sigma_{out}\sqrt{\dot{\rho}^2+1-\frac{2Gm_{out}}{\rho}}=\frac{GM}{\rho}\,.$$

$M = const > 0$ - bare mass. Black holes $\rightarrow m_{in} > 0$, $m_{out} > 0$.
$\sigma = \pm 1$

- Bound motion.Two types of shells

- Black hole case: $\Delta m > \frac{1}{2}(\sqrt{m_{in}^2 + M^2} - m_{in})$

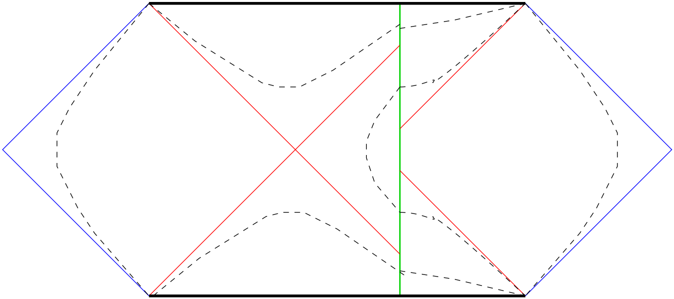

Рис.: Carter-Penrose diagram for bound motion with turning point at $\rho = \rho_0$ and $\Delta m > 0 \Longrightarrow \sigma_{in}(\rho_0) = +1,\ \sigma_{out}(\rho_0) = +1$.

- Wormhole case: $\Delta m < \frac{1}{2}(\sqrt{m_{in}^2 + M^2} - m_{in})$

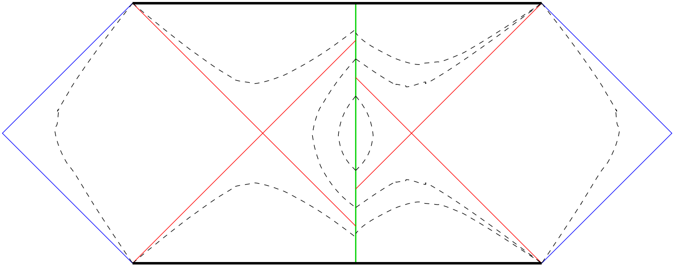

Рис.: Carter-Penrose diagram for bound motion with turning point at$\rho = \rho_0$ and $\Delta m > 0 \Longrightarrow \sigma_{in}(\rho_0) = +1, \ \sigma_{out}(\rho_0) = -1.$

- Quantum shells - ?

[V.A.B., N.G.Kozimirov, V.A.Kuzmin, I.I.Tkachev (1988)]

- $\sigma_{in} = +1$, $m_{in} = 0$, $m_{out} = m$)

$$\sqrt{\dot{\rho}^2 + 1} - \sigma_{out} \sqrt{\dot{\rho}^2 + 1 - \frac{2Gm}{\rho}} = \frac{GM}{\rho} \, .$$

- Pre-Hamiltonian

$$m = M \sqrt{\dot{\rho}^2 + 1} - \frac{GM^2}{2\rho} \, .$$

- Restoration scheme

$$p = \frac{\partial L}{\partial \dot{\rho}},$$

$$H = p\dot{\rho} - L = \dot{\rho}\frac{\partial L}{\partial \dot{\rho}} - L,$$

$$L = \dot{\rho}\int H\frac{d\dot{\rho}}{\dot{\rho}^2} = \dot{\rho}\int \frac{\partial H}{\partial \dot{\rho}}\frac{d\dot{\rho}}{\dot{\rho}} - H$$

$$p = \int \frac{\partial H}{\partial \dot{\rho}}\frac{d\dot{\rho}}{\dot{\rho}}.$$

- Hamiltonian

$$p = M \log(\dot{\rho} + \sqrt{\dot{\rho}^2 + 1}) + F(\rho),$$

$$L = M(\dot{\rho} \log(\dot{\rho} + \sqrt{\dot{\rho}^2 + 1}) - \sqrt{\dot{\rho}^2 + 1}) + \dot{\rho} F(\rho),$$

$F(\rho) = 0.$

$$\dot{\rho} = \sinh \frac{p}{M},$$

$$H = M \cosh \frac{p}{M} - \frac{GM^2}{2\rho}.$$

$$x = M\rho, \qquad \Pi = \frac{1}{M}\rho,$$

$$H = M\left(\cosh \Pi - \frac{GM^2}{2x}\right).$$

- Quantization

$$[\Pi, x] = -i.$$

Coordinate representation: $\Pi = -i\partial/\partial x$

$$e^{-i\frac{\partial}{\partial x}}\Psi(x) = \Psi(x - i)$$

$$H\Psi(x) = E\Psi(x),$$

$\epsilon = m/M$, $\alpha = GM^2$

$$\Psi(x + i) + \Psi(x - i) = \left(2\epsilon + \frac{\alpha}{x}\right)\Psi(x).$$

- Non-relativistic limit

$$x \pm i \to M\left(\rho \pm \frac{1}{M} i\right) \to \rho \pm \frac{\hbar}{Mc} i,$$

$\rho \gg \hbar/Mc$. Up to second order in $(\hbar/Mc)$:

$$-\frac{1}{2M}\frac{d^2\Psi}{d\rho^2} - \frac{GM^2 - e^2}{2\rho}\Psi = (E - M)\Psi$$

$$(E - M)_n = -\frac{M(GM^2 - e^2)^2}{8n^2}, \qquad n = 1, 2, \ldots$$

- Rydberg formula

- Trancated equation. Matrix method

$$\Psi^{(2N)} = -\sum_{K=1}^{N-1}(-1)^{N-K}\frac{(2N)!}{(2K)!}\Psi^{(2K)}+(-1)^{N}(2N)!\left(\epsilon-1+\frac{\alpha}{2x}\right)\Psi(x),$$

- Asymptotics at zero radius

$$\Psi = x^{n} \quad x \to 0,$$
$$n = 0, 1, 2, ...$$

- Asymptotics at infinity

$$\Psi = e^{\lambda x} e^{-\frac{\alpha}{2\sin\lambda}\log x},$$
$$\cos\lambda = \epsilon.$$

- Disappearing term

$(1 + x^{2N} \log x)$

- Analyticity

Shift in the argument is along imaginary axis - solution must be analytical functions on some Riemannian surface.
At singular points of equations the asymptotical solutions have, in general, branching, and the number of leaves of the Riemannian surface should be the same at the ends of every cut between two branching points.
Ranks of such branchig points could differ only by some integer number n.

- Discrete spectrum

$$\frac{GM^2}{2\sin\lambda} = n$$
$$m = M\sqrt{1 - \frac{G^2M^4}{4n^2}}.$$

- Solution

$$\Psi_\beta(x) = \left(-4\pi\beta e^{-i\lambda}\sin\lambda\right) x e^{-\lambda x} F\left(1 - ix, 1-\beta; 2; 1 - e^{-2i\lambda}\right).$$

$F(a, b; c; z)$ - Gauss's hyper-geometric function.

$$a = 1 - ix, \qquad b = 1 - \beta; \qquad c = 2; \qquad z = 1 - e^{-2i\lambda},$$

- Asymptotics $\Rightarrow$ quantization $\Rightarrow \beta = n$

[V.A.B., A.M.Boyarsky, A.Yu.Neronov]

- ADM formalism in the presence of the shell
- Hamiltonian constraint

$$C = F_{out} + F_{in} - \sqrt{F_{out}}\sqrt{F_{in}}\left(\exp\frac{G\dot{P}_R}{R} + \exp-\frac{G\dot{P}_R}{R}\right) - \frac{M^2G^2}{R^2}$$

- Idea

$$\sqrt{F} \to F^{1/2}$$

$$\begin{aligned} F^{1/2} &= |F|\, e^{i\phi} \\ \phi = 0 &\quad \text{в } R_+\text{-region} \\ \phi = \pi/2 &\quad \text{в } T_-\text{-region} \\ \phi = \pi &\quad \text{в } R_-\text{-region} \\ \phi = -\pi/2 &\quad \text{в } T_+\text{-region} \end{aligned}$$

for the black hole case, and the reverse bypass in the wormhole case.

- Equation for the wave function

$$\mu = m_{in}/m_{out},\ m_{out} = m,\ \zeta = \frac{1}{2}\left(\frac{m_{pl}}{m}\right)^2, S = \frac{R^2}{R_g^2}$$

$$\Psi(m, \mu, S + i\zeta) + \Psi(m, \mu, S - i\zeta) =$$
$$(F_{in}F_{out})^{-1/2}\left(F_{in} + F_{out} - M^2/4m^2S\right)\Psi(m, \mu, S)$$

$$m_{in} = 0,\ F_{in} = 1,\ F_{out} = F$$

$$\Psi(m, S+i\zeta)+\Psi(m, S-i\zeta) = F^{-1/2}\left(1 + F - M^2/4m^2S\right)\Psi(m, \mu, S)$$

- Asymptotics in R_+-region. $\zeta \ll 1$ $y \gg \zeta$

$S = \infty$, $s = (1+y)^2$

$$\Psi \sim y^{\frac{1}{2} - \frac{\frac{M^2}{m^2} - 2}{4\mu\zeta^2}} \exp(-\mu y), \quad \mu = \frac{1}{\zeta}\sqrt{\frac{M^2}{m^2} - 1},$$

$S \to 1 + 0$, $s = (1+z^2)^2, y = \sqrt{z}, s \gg \zeta, y \gg \zeta$

$$\Psi \sim 1 - \frac{8}{3\zeta^2}\left(1 - \frac{M^2}{4m^2}\right) y^{3/2}$$

Compare the ranks of branching points

$$2 - \frac{M^2}{m^2} = 4\zeta\sqrt{\frac{M^2}{m^2} - 1n}, \quad n = integer$$

This is the first quantum condition.

- Asymptotics in R_--region

$s \to \infty$. Because of the "minus"sign in front of $F^{1/2}$, the wave equation in R_--region differs considerably from that in R_+-region:

$$\Psi \sim y^{\frac{\frac{M^2}{m^2} - 1}{8\zeta}} \exp\left(-\frac{2}{\zeta} y^2\right)$$

Compare ranks of turning points at $s \to 1+0$ and $s \to \infty$

$$\frac{\frac{M^2}{m^2} - 1}{8\zeta} = \frac{1}{2} + p, \qquad p = positive \quad integer$$

The appearance of the second quantum number is rather surprising and needs some explanation.

- Massless shell

$$\Psi(S+i\zeta)+\Psi(S-i\zeta)=\frac{F_{in}+F_{out}}{\sqrt{F_{in}F_{out}}}\Psi.$$

$$\Psi=\exp\left(\frac{i\Omega(S)}{\zeta}\right)(\phi_0+\zeta\phi_1+\dots).$$

Zero order in ζ - Hamilton-Jacobi equation for $P_S=d\Omega/dS$:

$$\cosh\{P_S\}=\frac{F_{in}+F_{out}}{2\sqrt{F_{in}F_{out}}}$$

$$P_S=\pm\ln\left(\frac{F_{out}}{F_{in}}\right)^{1/2}=\pm\ln\left(\frac{\sqrt{S}-1}{\sqrt{S}-\mu}\right)^{1/2}.$$

In the region $S\in(\mu^2,1)$ between horizons the momentum P_S acquires an imaginary part, therefore this is an analog of the "classically forbidden"region

$$\Psi = A\exp\left\{\int_S P_{\tilde S} d\tilde S\right\} \to A\exp\left\{\int_S P_{\tilde S} d\tilde S\right\}\exp\left(\frac{\pi}{\zeta}\frac{1-\mu^2}{2}\right).$$

$$\Psi = A\exp\left\{-\int_S P_{\tilde S} d\tilde S\right\} \to A\exp\left\{-\int_S P_{\tilde S} d\tilde S\right\}\exp\left(-\frac{\pi}{\zeta}\frac{1-\mu^2}{2}\right).$$

- Hawking radiation

The ratio of the squared amplitudes Z_{in} и Z_{out}:

$$P = \frac{Z_{in}^2}{Z_{out}^2} = \exp\left\{-\frac{2\pi}{\zeta}(1-\mu^2)\right\} = \exp\left\{-\frac{4\pi}{m_{Pl}^2}\frac{\delta R_g^2}{4}\right\}.$$

A - horizon area

$$P = \exp\left\{-\frac{1}{4}\frac{\delta A}{m_{Pl}^2}\right\}.$$

J.Hartle and S.Hawking: distribution of probabilities as the Gibbs distribution

$$P = \exp\{-\delta m/T\} \implies \delta m = T\,\delta A/4.$$

- Equation

$$\Psi(m,\mu,S+i\zeta)+\Psi(m,\mu,S-i\zeta)=$$
$$(F_{in}F_{out})^{-1/2}\left(F_{in}+F_{out}-M^2/4m^2S\right)\Psi(m,\mu,S)$$

- Matrix method - degenerate principal matrix
- Matrices of infinite order $\Longrightarrow$ infinite system of finite order matrices
- Discrete spectrum

$$\frac{2(1-\mu)^2-M^2\big/m^2}{\zeta\sqrt{2}\sqrt{(1-\mu)^2-M^2\big/m^2}}=\mathbf{n}$$
$$\frac{1}{\zeta 2\sqrt{2}}\left((1-\mu)^2-M^2\Big/m^2\right)=\mathbf{p}$$

where n and p - integers.

- Quantum gravitational collapse
- Discrete spectrum

$$\frac{2(\Delta m)^2 - M^2}{\sqrt{M^2 - (\Delta m)^2}} = \frac{2m_{Pl}^2}{\Delta m + m_{in}} n ,$$
$$M^2 - (\Delta m)^2 = 2(1 + 2p) m_{Pl}^2 ,$$

$\Delta m = m_{out} - m_{in}$, n и $p \geq 0$ - целые числа. Two quantum number (n, p) - for two parameters $(\Delta m, M)$ with fixed m_{in} $\rightarrow$ shell does not collapse.
After switching on the radiation - gravitational collapse starts with (necessary) creation of new particles (shells) $\rightarrow$ m_{in} increases. Such a process can procees in different ways - the origin of entropy.
How a quantum collapse can be stopped?
The natural final point - crossing of the Einstein-Rosen bridge because the construction of the semiclosed world requires inclusion of the infinite volume - zero probability.
And this happens exactly when $n = 0$!

- "No-memory"state

$n = 0$ - special point in the spectrum.
Being in this quantum state the shell "does not fell"what is going both outside and inside $\rightarrow$ it "feels"only itself.
This resembles the "no hair"feature of the classical black holes.
Eventually, when all the shells (both the initial one and newly born) become in the corresponding states with $n_i = 0$, the whole system will not "feel"its own history.
It is this "no-memory"state that we call "the quantum black hole".

Thank you very much!