Pseudotoric structures and exotic lagrangian tori

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Let (X, ω) — be a symplectic manifold of real dimension $2n$.
We understand it as *the phase space of a classical mechanical system*
We are interested in the case of compact phase space
The main problem we have in mind — **Quantization** of such systems
The main approach — lagrangian quantization:
· for $\mathbb{R}^{2n}, \omega = \sum dp \wedge dq$ —**V.Maslov**, *semiclassical aproximation*;
· for $T^*S, d\alpha$ — **S. Dobrokhotov, A. Shafarevich**,
· for general compact (X, ω) — **N.T.** (*algebraic lagrangian geometry*)
basic geometrical idea — lagrangian submanifolds in X look and behave like points (Darboux - Weinstein theorem) of an infinite dimensional variety, and any classical Hamiltonian function on X generates the corresponding dynamics on this variety.

Lagrangian geometry — questions about *lagrangian submanifolds* of X:

1) which homology classes from $H_n(X, \mathbb{Z})$ can be realized by smooth lagrangian submanifolds;

2) what are the topological types of these lagrangian submanifolds;

3) classification up to lagrangian deformations of lagrangian submanifolds of the same topological type and homology class;

4) classification up to Hamiltonian isotopy of lagrangian submanifolds of the same deformation type.

5) unification of all lagrangian submanifolds in an appropriate category

Recall that $S \subset X$ is lagrangian if

$$\omega|_S \equiv 0 \quad \text{and} \quad \dim S = n$$

Thus at least $[S]$ *is perpendicular to* $[\omega]$.

Two lagrangian submanifolds $S_0, S_1 \subset X$ are of the same deformation type if there is a lagrangian film

$$\begin{array}{ccccccc} \mathcal{S} & \subset & X & \times & \mathbb{C}, & & \omega \oplus dz \wedge d\bar{z} \\ & & p \swarrow & & \searrow q & & \\ & X & & & & \mathbb{C} & \end{array}$$

such that $p(\mathcal{S} \cap X \times \{i\}) = S_i, \quad i = 0, 1.$

Thus at least $[S_0] = [S_1]$ *and* $S_0 \simeq S_1$

Hamiltonian isotopy of lagrangian submanifold $S_0 \subset X$ is given by a time dependent Hamiltonian function $H(x, t) : X \times \mathbb{R} \to \mathbb{R}$ which generates the flow ϕ_H^t, and $S_t = \phi_H^t(S_0)$ is the corresponding isotopy.

Toy example: dim $= 2$. Let Σ be a Riemann surface equipped with a symplectic form.

Then since every loop is lagrangian (dimensional reason):

1) every primitive homology class from $H_1(\Sigma, \mathbb{Z})$ is realizable by a smooth lagrangian submanifold;

2) every smooth lagrangian submanifold is isomorphic to S^1;

3) two loops from the same homology class are deformation equivalent;

4) two loops are Hamiltonian isotopic if the symplectic area of the oriented film bounded by the loops is zero;

5) the Fukaya category for a curve of any genus exists

thus for this case the problem is completely solved

Example: $\mathbb{CP}^2$**.** The projective plane is the simplest compact symplectic manifold in dimension 4:

1) since $H^2(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}$, any lagrangian submanifold must present trivial homology class;

2) vanishing results for 2- spheres (M. Gromov), riemann surfaces of genus > 1 (M. Audin), Klein bottle (S. Nemirovskiy, V. Shevchishin) — they are not realizable as lagrangian submanifolds;

3) — 4) it was believed that well known Clifford tori are unique examples of lagrangian tori in $\mathbb{CP}^2$ since in 1996 Yu. Chekanov proposed a construction of lagrangian torus which is not Hamiltonian isotopic to a Clifford torus — and nobody knows are there other types of lagrangian tori;

5) nevertheless certain constructions of appropriate categories exist (Fukaya - Seidel).

thus even for this basic case in dimension 4 the problem is not solved yet

Why we are interested in lagrangian geometry?

If we would like to proceed in the **lagrangian approach to Geometric Quantization** —

there lagrangian submanifolds represent quantum states

— it is necessury to know all these states = all types of lagrangian submanifolds.

F.e. in **ALAG** the Chekanov result ensures that the moduli space of half weighted Bohr - Sommerfeld lagrangian cycles of level 3, $\mathcal{B}_{S,3}^{hw,r}$, has **at least two disjoint components**

and may be there is a tunneling between these components?

As well for **Homological Mirror Symmetry** — one should try to describe all objects in the Fukaya category, so all types of nonisotopic lagrangian tori.

Well known Clifford tori in $\mathbb{CP}^2$ comes from the **toric geometry**:

there are two real Morse functions f_1, f_2 in involution:

$$f_1 = \frac{|z_1|^2 - |z_2|^2}{\sum_{i=0}^2 |z_i|^2}, f_2 = \frac{|z_0|^2 - |z_1|^2}{\sum_{i=0}^2 |z_i|^2}, \{f_1, f_2\}_\omega = 0$$

in homogeneous coordinates $[z_0 : z_1 : z_2]$;

the degeneration set

$$\Delta(f_1, f_2) = \{df_1 \wedge df_2 = 0\} \subset \mathbb{CP}^2$$

is formed by three lines $l_i, l_i = \{z_i = 0\}$;

the action map $F = (f_1, f_2) : \mathbb{CP}^2 \to P_{\mathbb{CP}^2} \subset \mathbb{R}^2$ sends $\Delta(f_1, f_2)$ to the boundary component $\partial P_{\mathbb{CP}^2}$, and the preimage of any inner point $p \in P_{\mathbb{CP}^2}$ is a smooth lagrangian torus, labeled by values of f_1, f_2.

It is the standard picture for a toric manifold

Exotic Chekanov tori — the first version for $\mathbb{R}^4$:

fix a complex structure, so we have $\mathbb{C}^2$ with a coordinate system (z_1, z_2);

choose a smooth contractible loop $\gamma \subset \mathbb{C}^*$, which lies in a half plane so $\mathrm{Re}\gamma > 0$;

consider two - dimensional subset given in the coordinates by $(z_1, z_2) = (e^{i\phi}\gamma, e^{-i\phi}\gamma)$ — it is a lagrangian torus;

Remark. *If γ is not contractible, we get a standard torus.*

since $\mathbb{CP}^2 \backslash l$ is symplectomorphic to an open ball in $\mathbb{R}^4$ one implements the construction to the projective plane;

and the last step:

using **Hofer's capacity technique**, Chekanov proved that this torus is not equivalent to the standard one.

This torus is called **the Chekanov torus**; the forthcoming paper by Yu. Chekanov and F. Schlenk contains the details how to construct these nonstandard tori in $\mathbb{CP}^n$ for certain n, the products $S^1 \times ... \times S^1$, and some other cases.

An alternative description of the Chekanov tori based on the notion of **pseudotoric structure:**

· again we take $\mathbb{C}^2$ and consider pencil $\{Q_w\}$,
$Q_w = \{z_1 z_2 = w\} \subset \mathbb{C}^2$ of quadrics;

· take real Morse function $F = |z_1|^2 - |z_2|^2$;

· note that the Hamiltonian vector field X_F of this function F preserves each quadric Q_w from the pencil;

· take a smooth contractible loop $\gamma' \subset \mathbb{C}^*_w$ where $\mathbb{C}_w$ parameterizes our pencil $\{Q_w\}$;

· on each quadric $Q_w, w \in \gamma'$, mark the level set
$S_w = \{F = 0\} \cap Q_w$ which is a smooth loop;

· collect these loops along γ':
$T(\gamma') = \bigcup_{w \in \gamma'} S_w$, getting a torus

— *it is not hard to see, that we again get the Chekanov torus from the previous slide*, if we put $\gamma = \sqrt{\gamma'}$.

Let us repeat the construction for for the projective plane:

· consider pencil of quadrics $\{Q_p\}$, $p \mapsto [\alpha : \beta] \subset \mathbb{CP}^1_{\alpha,\beta}$
$Q_p = \{\alpha z_1 z_2 = \beta z_0^2\} \subset \mathbb{CP}^2$;

· consider real Morse function $F = \frac{|z_1|^2 - |z_2|^2}{\sum_{i=0}^2 |z_i|^2}$;

· note that its Hamiltonian vector field X_F preserves each element of the pencil;

· choose a smooth contractible loop $\gamma \subset \mathbb{CP}^1_{\alpha,\beta} \backslash \{[1 : 0], [0 : 1]\}$;

· on each quadric $Q_p, p \in \gamma$ take the level set
$S_p = \{F = 0\} \cap Q_p$ which is a smooth loop;

· collect the level sets S_p along the loop γ
$T(\gamma) = \bigcup_{p \in \gamma} S_p$ getting again a lagrangian torus.

The resulting torus is exactly the Chekanov torus, given by the identification of symplectic ball in $\mathbb{R}^4$ *and* $\mathbb{CP}^2 \backslash$ *line*.

Another remark:*if* $\gamma \subset \mathbb{CP}^1_{\alpha,\beta}$ *is non contractible, then the resulting torus is equivalent to a Clifford torus*.

Thus **equivalence classes** $\Rightarrow \pi_1(\mathbb{CP}^1_{\alpha,\beta} \backslash \{[0 : 1], [1 : 0]\})$.

What is the difference between toric and pseudo toric considerations?

$\mathbb{R}$	$\mathbb{C}$
real Morse function f	Lefschetz pencil $\{Q_p\}$
$\Vert$	$\Vert$
$f : X \to \mathbb{R}$	$\psi : X \backslash B \to \mathbb{CP}^1, Q_p = \overline{\psi^{-1}(p)}$
$\downarrow$	$\downarrow$
toric case	pseudtoric case
(f_1, f_2) on $\mathbb{CP}^2$	$(f, \{Q_p\})$ on $\mathbb{CP}^2$
such that $\{f_1, f_2\}_\omega = 0$	such that $X_f \parallel Q_p$
$\uparrow$	$\uparrow$
standard commutation rel.	new commutation rel.

New commutation relation: pencil $\{Q_p\}$ commutes with real function f if the Hamiltonian vector field X_f is parallel to each element Q_p of the pencil at each point.

In other words, *pseudotoric structure* (of rank one) is a combination of

· real data $(f_1, \ldots, f_{n-1})$ — first integrals in involution

· complex data $\{Q_p\}$ — a pencil of symplectic divisors, covering whole X s.t.

$$\psi : X \backslash B \to \mathbb{CP}^1$$

has generically smooth symplectic fibers

$$Q_p = \overline{\psi^{-1}(p)} = \psi^{-1}(p) \cup B$$

and H_{f_i} is parallel to Q_p at each point (for all i, p)
Distinguished points $p_1, \ldots, p_k \in \mathbb{CP}^1$ - singular fibers - form $D_{\mathrm{Sing}} \subset \mathbb{CP}^1$

· $B \subset X$ is the base set of pencil $\{Q_p\}$

· $Q_p, (f_i|_{Q_p})$ — toric manifold with the same convex polytop.

Now we have

Theorem (S. Belyov, N.T.) *Let $(f_1, ..., f_{n-1}, \psi)$ be a regular pseudotoric structure of rank one on a compact symplectic manifold X. Let $S \subset \mathbb{CP}^1$ be a smooth lagrangian torus which doesn't pass through p_i. Then the choice of non critical values $(c_1, ...c_{n-1})$ of $f_1, ..., f_{n-1}$ defines a smooth lagrangian torus $T(S, c_1, ..., c_{n-1}) \subset X$.*

· Thus we get a correspondence

$H_1((\mathbb{CP}^1 \backslash D_{\mathrm{Sing}}), \mathbb{Z}) \longrightarrow$ different types of lagrangian tori

For example, coming back to $\mathbb{CP}^2$, Clifford and Chekanov tori:

$$H_1(\mathbb{CP}^1 \backslash ([1:0],[0:1]), \mathbb{Z}) \begin{array}{l} \nearrow \ \text{Clifford type} = \text{primitive elem.} \\ \searrow \ \text{Chekanov type} = \text{trivial elem.} \end{array}$$

This hints how to construct non standard lagrangian tori in toric symplectic manifolds in view of the following

Theorem (S. Belyov, N.T.):

· *Any smooth compact toric symplectic manifold admits regular pseudotoric structure* $(f_1, ..., f_{n-1}, \psi, \mathbb{CP}^1)$ *of rank one.*

· *For this structure the singular divisor* $D_{sing} \subset \mathbb{CP}^1$ *consists of exactly two distinct points,* $p_N, p_S \subset \mathbb{CP}^1$*.*

· *The primitive and the trivial elements of* $H_1(\mathbb{CP}^1 \backslash (p_N \cup p_S), \mathbb{Z})$ *generates lagrangian tori of the standard type and of the Chekanov type respectively.*

Suppose additionaly that our given toric (X, ω_X) is monotone, so $K_X = k[\omega_X] \subset H^2(X, \mathbb{Z})$ — f.e. Fano varieties in AG — then

· *if there is a standard monotone lagrangian torus then there exists a monotone lagrangian torus of the Chekanov type.*

Main conjecture: these monotone tori are not Hamiltonian isotopic.

Outline of the proof:

· take for a given toric X the set of commuting Morse moment maps $(f_1, ...f_n)$, which give the action map by "action coordinates" $F = (f_1, ..., f_n) : X \to P_X$ to convex moment polytop $P_X \subset \mathbb{R}^n$;

· for the components D_i of the boundary divisor $D = F^{-1}(\partial P_X)$ find an integer combination $\sum \lambda_i D_i$ equals to zero;

· rearrange this to the form $\sum_{\lambda_i > 0} \lambda_i D_i = \sum_{\lambda_j < 0} |\lambda_j| D_j$, $D_i \neq D_j$, thus we have two divisors from the same linear system

$$D_+ = \sum_{\lambda_i > 0} \lambda_i D_i, \quad D_- = \sum_{\lambda_j < 0} |\lambda_j| D_j \in |\sum_{\lambda_i > 0} \lambda_i D_i|;$$

· take the pencil $< D_+, D_- >$ with the base set $B = D_+ \cap D_-$ — it is our pencil ψ, and for generic point $p \in \mathbb{CP}^1$, $p \neq [1:0] (\mapsto D_+)$, $[0:1] (\mapsto D_-)$, the divisor $\overline{\psi^{-1}(p)} \subset X$ is smooth outside the base set B;

· the same linear combination $\sum \lambda_i D_i$ after substitution of linear forms l_i which correspond to D_i in $\mathbb{R}^n$ gives a linear relation on x_i — and this relation derive our real data $f'_1, ...f'_{n-1}$ from $f_1, ..., f_n$.

Example: $\mathbb{CP}^2_3$ — **del Pezzo surface of degree 6** can be realized in the direct product $\mathbb{CP}^2_x \times \mathbb{CP}^2_y \supset \mathcal{U} = \{x_0 y_0 = x_1 y_1 = x_2 y_2\}$ with the projection $p_x : \mathcal{U} \to \mathbb{CP}^2_x, \quad p_x(x_i, y_j) = [x_0 : x_1 : x_2]$.

$p_x^0 : \mathcal{U} \backslash \text{three lines} \simeq \mathbb{CP}^2_x \backslash \text{three points}$,

but $(p_x^0)^{-1}(T_{\mathrm{Ch}}) \subset \mathcal{U}$ is **not lagrangian** — we cann't lift the Chekanov torus, but we can lift the corresponding pseudotoric structure!

· take the pencil $\{Q_{\alpha,\beta}\} = \{\alpha x_0 x_1 y_2^2 = \beta x_2^2 y_0 y_1\} \subset \mathbb{CP}^2_x \times \mathbb{CP}^2_y$, and the intersections $Q_{\alpha,\beta} \cap \mathcal{U}$ gives the Lefschetz pencil ψ on $\mathcal{U}$;

· the real Morse function $\quad F = \frac{|x_0|^2 - |x_1|^2}{\sum_{i=0}^2 |x_i|^2} + \frac{|y_1|^2 - |y_0|^2}{\sum_{i=0}^2 |y_i|^2}$

preserves by the Hamiltonian action $\mathcal{U}$ and each element $Q_{\alpha,\beta}$ of the pencil, and the restriction $f = F|_{\mathcal{U}}$ gives the real data;

· the choice of a smooth loop $\gamma \subset \mathbb{CP}^1 \backslash ([1:0],[0:1])$ gives a lagrangian torus $T(0,\gamma) = \bigcup_{p \in \gamma} \{f|_{\overline{\psi^{-1}(p)}} = 0\}$, and if γ is contrctible, we get a Chekanov torus in $\mathbb{CP}^2_3$.

Another usage of pseudotoric structure — in construction of **special lagrangian fibrations** on Fano varieties.
D. Auroux, an approach to Mirror Symmetry conjecture:
(X, I, ω, g) — Kahler manifold, $|K_X^{-1}| \supset D$
$D \mapsto \Theta_D$ — holomorphic form with pole along D.
Lagrangian fibration $\pi : X \backslash D \to B$ is said to be *special* if the proportionality coefficient ρ from

$$\Theta_D|_{\pi^{-1}(p)} = \rho \mathrm{Vol}(g|_{\pi^{-1}(p)})$$

has the same phase: $\mathrm{Arg}\rho =$ const for each $p \in B$.
Example: standard toric fibration.
X with collection of Morse commuting moment maps $(f_1, ..., f_n)$ with the degeneration locus $\Delta(f_1, ..., f_n) = D \in |K_D^{-1}|$
The corresponding form Θ_D is preserved by the moment maps, so

$$\Theta_D(X_{f_1} \wedge ... X_{f_n}) = \text{const on } X \backslash D$$

but essentially this constant is our ρ.
Question: what about another elements from $|K_X^{-1}|$?
Auroux's conjecture for $\mathbb{CP}^2$: each $D \in |3H|$ is realizable.

Example: the flag variety. Take F^3 — full flag in $\mathbb{C}^3$, realize it as $\mathcal{U} \subset \mathbb{CP}^2_x \times \mathbb{CP}^2_y$, given by the equation $\sum_{i=0}^3 x_i y_i = 0$.
Pseudotoric structure on $\mathcal{U}$: two real Morse functions

$$f_1 = \frac{|x_0|^2 - |x_1|^2}{\sum |x_i|^2} + \frac{|y_1|^2 - |y_0|^2}{\sum |y_i|^2}, \quad f_2 = \frac{|x_1|^2 - |x_2|^2}{\sum |x_i|^2} + \frac{|y_2|^2 - |y_1|^2}{\sum |y_i|^2};$$

Lefschetz pencil $\psi : \mathcal{U} \to \mathbb{CP}^1$ given by

$$\psi([x_0 : x_1 : x_2] \times [y_0 : y_1 : y_2]) = [x_0 y_0 : x_1 y_1 : x_2 y_2], \sum_{i=0}^3 x_i y_i = 0.$$

The base set $B \subset \mathcal{U}$ is a hexagon, general element of the pencil is toric del Pezzo surface $\mathbb{CP}^2_3$; three singular elements correspond to points $[1 : -1 : 0], [1 : 0 : -1], [0 : 1 : -1] \in \mathbb{CP}^1$ have the form $\mathbb{CP}^2_2 \cup \mathbb{CP}^2_2$ with intersection along a diagonal of hexagon B.
Now take a Morse function h on $\mathbb{CP}^1$ which preserve the Kahler structure by the Hamiltonian action and which has critical points at $p_1 = [1 : -1 : 0]$ and $p_2 = [1 : 0 : -1]$.

Then

· we get a lagrangian fibration on $\mathcal{U} \backslash D_1 \cup D_2$ where $D_i = \overline{\psi^{-1}(p_i)}$ has the type $\mathbb{CP}^2_2 \cup \mathbb{CP}^2_2$;

· in the fibration there is a 1- dimensional subfamily of singular lagrangian tori while generic fiber is smooth;

· the bundary divisor $D_1 \cup D_2$ lies in the anticanonical system $|K^{-1}_{\mathcal{U}}|$;

· and this fibration is **special**.

In contrast with the previous examples: F^3 is not toric, but it admits pseudotorci structure. It is natural to call such a manifold **pseudotoric** since it carries a lagrangian fibration which looks similiar to the standard toric lagrangian fibrations. Another examples of pseudotoric manifolds are complex quadrics and certain complete intersections in $\mathbb{CP}^n$; it is reasonable to ask: **which symplectic manifolds are pseudotoric?**