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On geometrical structures and properties of solutions to Hamiltonian systems of PDEs

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Plan:

- Hamiltonian PDEs depending on small parameter. Dispersionless limit.

- Gradient catastrophe; phase transitions from regular to oscillatory behavior.

- Universality conjecture. Some special functions (Painlevé transcendents).

Class of systems of PDEs depending on a small parameter ϵ

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + \epsilon\, A_2(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}) + \epsilon^2 A_3(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}, \mathbf{u}_{xxx}) + \dots$$

$$\mathbf{u} = (u^1, \dots, u^n)$$

Terms of order ϵ^k are differential polynomials

of degree $k+1$

$$\deg u^{(m)} = m, \quad m = 1, 2, \dots$$

This class is invariant with respect to the group of transformations of the form

$$\mathbf{u} \mapsto \tilde{\mathbf{u}} = F_0(\mathbf{u}) + \epsilon\, F_1(\mathbf{u}, \mathbf{u}_x) + \epsilon^2 F_2(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}) + \dots$$

$$\deg F_k(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k)}) = k$$

$$\det\left(\frac{DF_0(\mathbf{u})}{D\mathbf{u}}\right) \neq 0$$

The Hamiltonian structure is given by a flat metric

$$ds^2 = \eta_{ij} du^i du^j$$

$$\eta_{ji} = \eta_{ij} = \text{const}, \quad \det(\eta_{ij}) \neq 0$$

(B.D., S.Novikov, 1983)

$$u^i_t = \eta^{ij} \frac{\partial}{\partial x} \frac{\delta H}{\delta u^j(x)}, \quad i = 1, \dots, n$$

Local Hamiltonians

$$H = \int \left[h_0(u) + \epsilon h_1(u, u_x) + \epsilon^2 h_2(u, u_x, u_{xx}) + \dots \right] dx$$

Triviality of Poisson cohomology: Getzler 2001

Examples

1) KdV $\quad u_t + u\,u_x + \frac{\epsilon^2}{12} u_{xxx} = 0 \qquad (n=1)$

In the zero dispersion limit $\epsilon = 0 \quad \Rightarrow$ Hopf equation

$$u_t + u\,u_x = 0$$

2) Toda lattice

$$\left.\begin{aligned} \epsilon\,u_t &= \quad v(x) - v(x-\epsilon) \\ \epsilon\,v_t &= \quad e^{u(x+\epsilon)} - e^{u(x)} \end{aligned}\right\} \qquad (n=2)$$

Long wave limit

$$\left.\begin{aligned} u_t &= \quad v_x \\ v_t &= \quad e^u u_x \end{aligned}\right\}$$

More general class of systems of the Fermi-Pasta-Ulam type

$$H = \sum_{n=1}^{N} \frac{p_n^2}{2} + V(q_n - q_{n-1})$$

For large N the equations of motion can be replaced by

$$\left.\begin{array}{rcl} u_t &=& \frac{1}{\epsilon}[v(x) - v(x-\epsilon)] \\ v_t &=& \frac{1}{\epsilon}\left[V'(u(x+\epsilon)) - V'(u(x))\right] \end{array}\right\}$$

$$p_n = v(n\epsilon), \quad q_n - q_{n-1} = u(n\epsilon), \quad \epsilon = \frac{1}{N}$$

In the leading term one obtains an integrable PDE

$$\left.\begin{array}{rcccl} u_t &=& \partial_x \frac{\delta H}{\delta v(x)} &=& v_x \\ v_t &=& \partial_x \frac{\delta H}{\delta u(x)} &=& V''(u)u_x \end{array}\right\} \qquad H = \int \left[\frac{1}{2}v^2(x) + V\left(u(x)\right)\right] dx$$

3) Nonlinear Schrödinger equation

$$i\epsilon\,\psi_t + \frac{\epsilon^2}{2}\psi_{xx} + |\psi|^2\psi = 0$$

In the real variables

$$u = |\psi|^2, \quad v = \frac{\epsilon}{2i}\left(\frac{\psi_x}{\psi} - \frac{\bar{\psi}_x}{\bar{\psi}}\right)$$

can be recast into the form

$$u_t + (u\,v)_x = 0$$

$$v_t + v\,v_x - u_x + \frac{\epsilon^2}{4}\left(\frac{1}{2}\frac{u_x^2}{u^2} - \frac{u_{xx}}{u}\right)_x = 0$$

The Hamiltonian formulation

$$u_t + \frac{\partial}{\partial x}\frac{\delta H}{\delta v(x)} = 0$$

$$v_t + \frac{\partial}{\partial x}\frac{\delta H}{\delta u(x)} = 0$$

$$H = \int \left[\frac{1}{2}\left(u\,v^2 - u^2\right) + \frac{\epsilon^2}{8u}u_x^2\right]\,dx$$

The metric $$(\eta_{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The main goal: to compare the properties of solutions to the perturbed system

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + \epsilon\, A_2(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}) + \epsilon^2 A_3(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}, \mathbf{u}_{xxx}) + \dots$$

with solutions to the "dispersionless limit $\epsilon \to 0$

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x$$

- Hamiltonian
- completely integrable
- finite life span (nonlinearity!)

For the dispersionless system $\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x$

a *gradient catastrophe* takes place: the solution exists

for $t < t_0$, there exists the limit $\lim_{t\to t_0} \mathbf{u}(x,t)$

but, for some x_0

$\mathbf{u}_x(x,t), \quad \mathbf{u}_t(x,t) \quad \to \quad \infty$ for $(x,t) \to (x_0, t_0)$

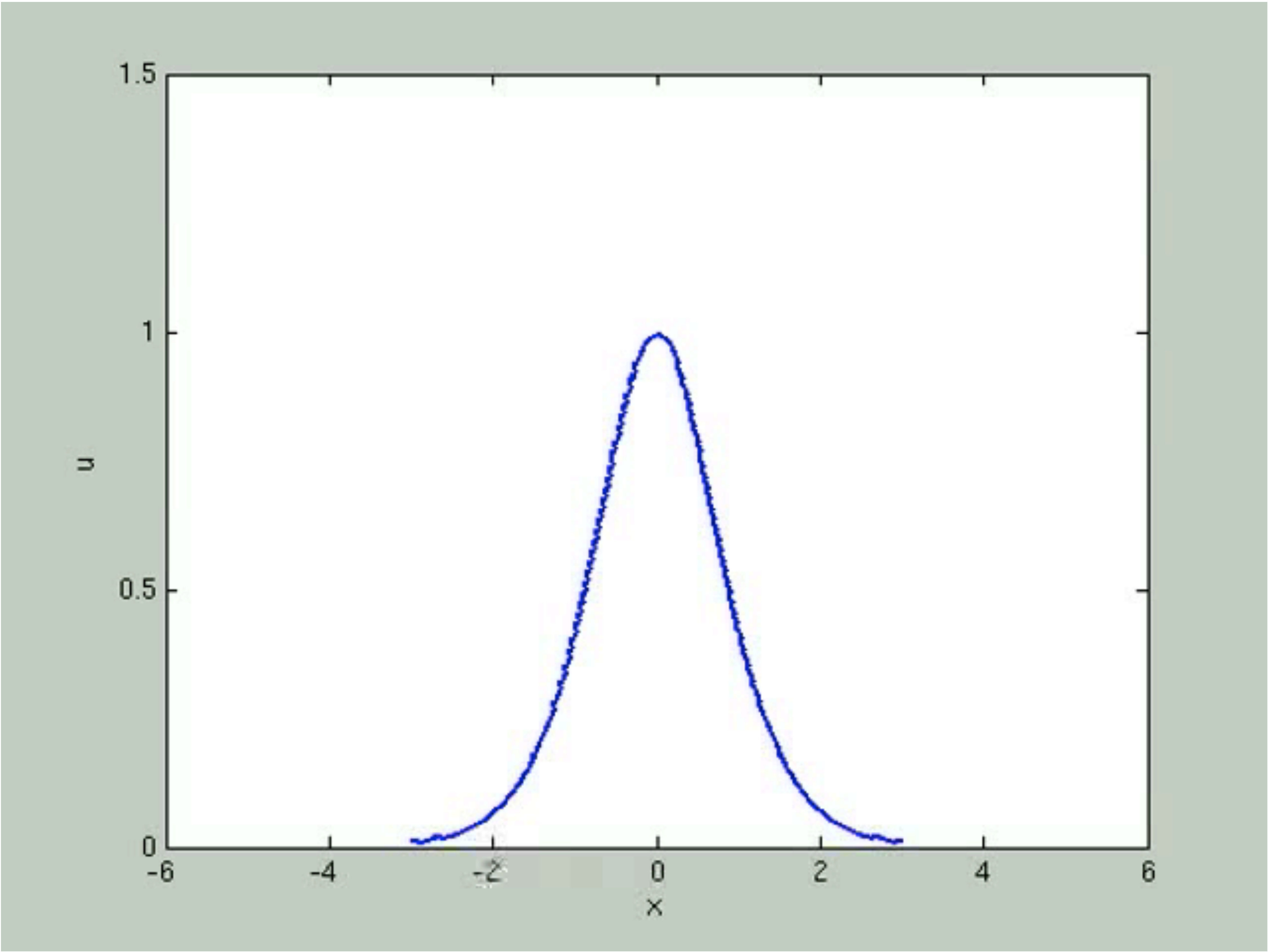
The problem: to describe the asymptotic behaviour of the generic solution $\mathbf{u}(x,t;\epsilon), \quad \mathbf{u}(x,0;\epsilon) = \mathbf{u}_0(x)$

to the perturbed system for $\epsilon \to 0$

in a neighborhood of the point of catastrophe (x_0, t_0)

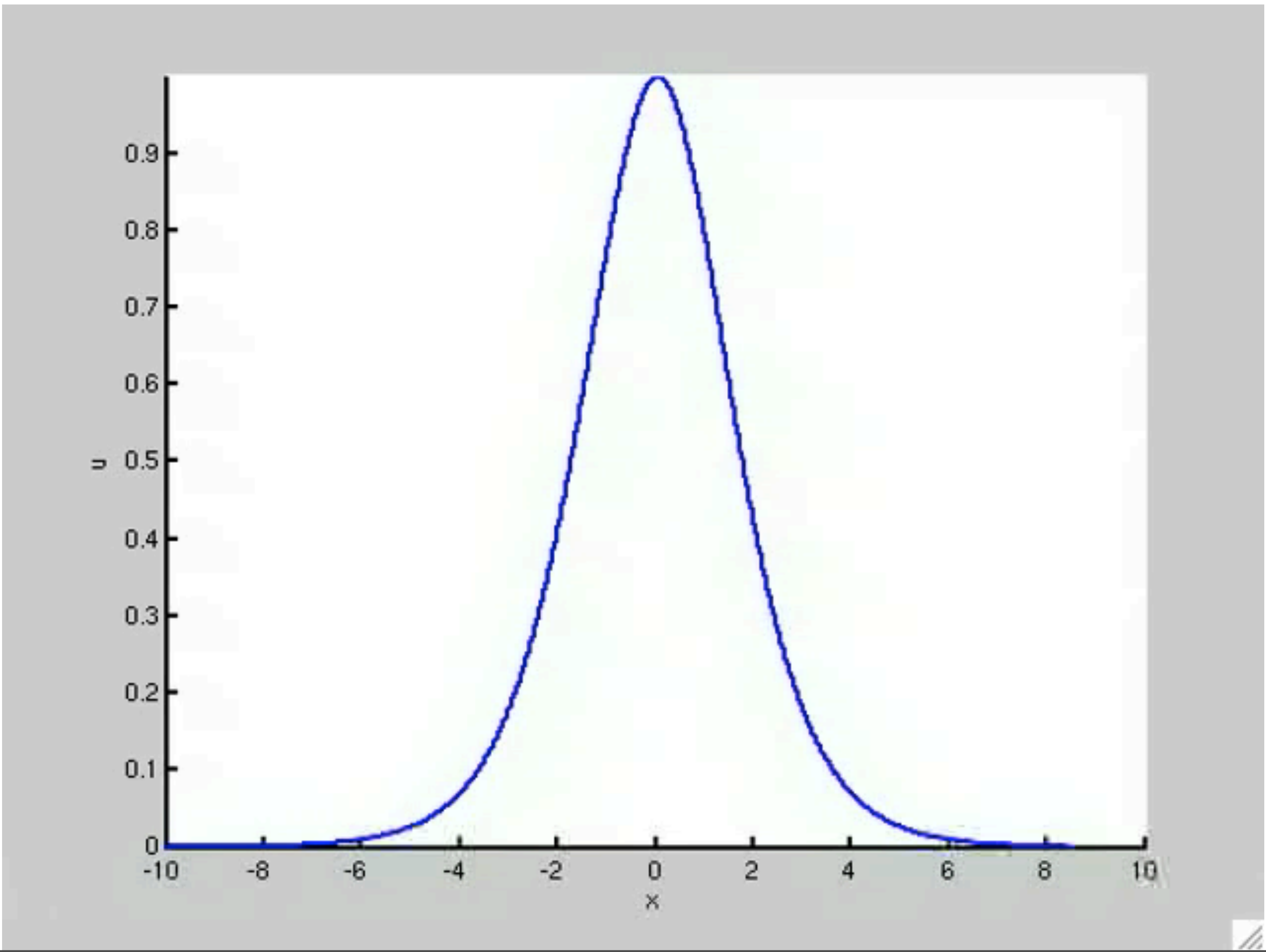
Gradient catastrophe for Hopf equation

$$u_t + u\,u_x = 0$$



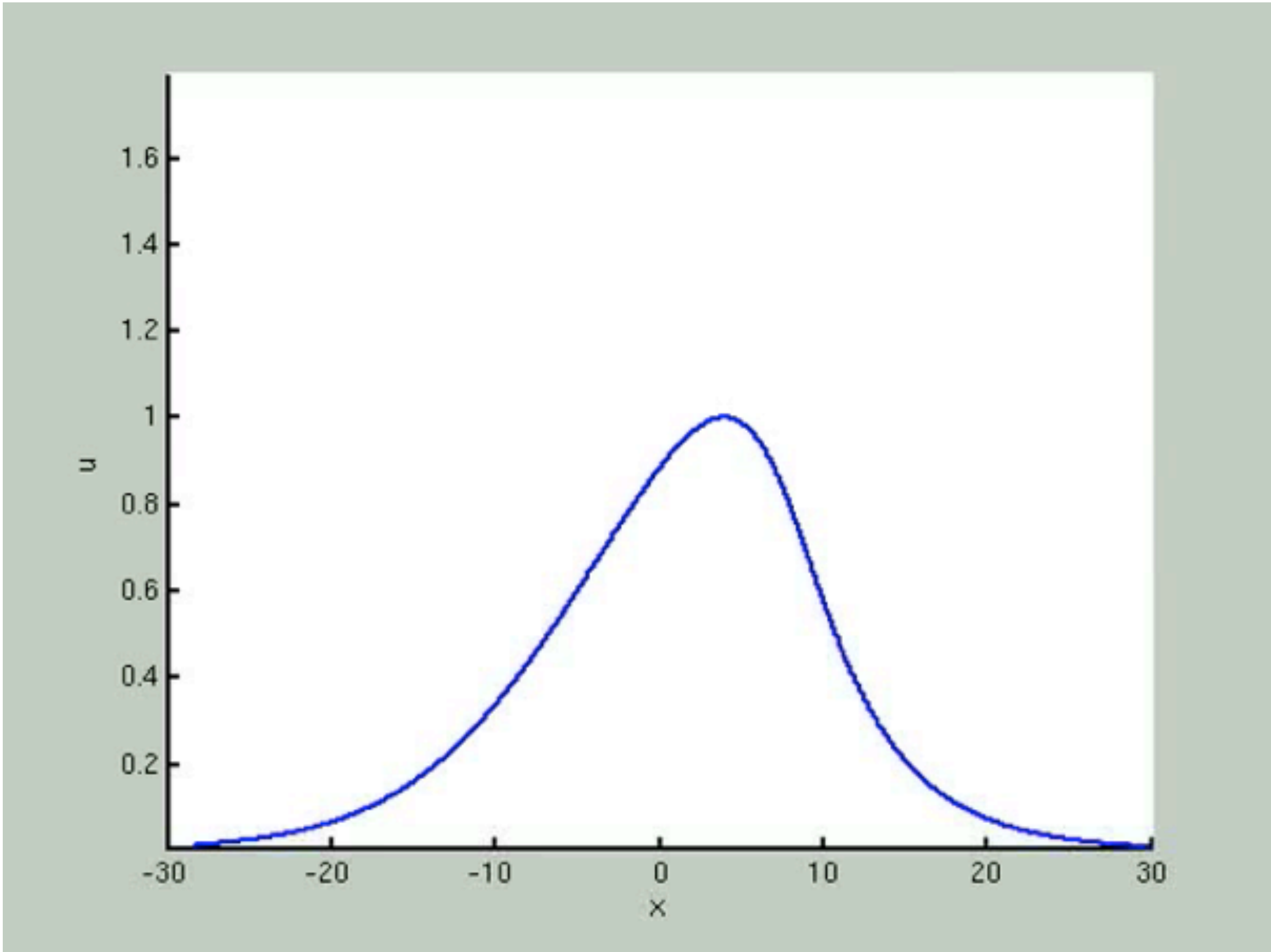
Perturbation: Burgers equation $u_t + u\, u_x = \epsilon\, u_{xx}$

(dissipative case)



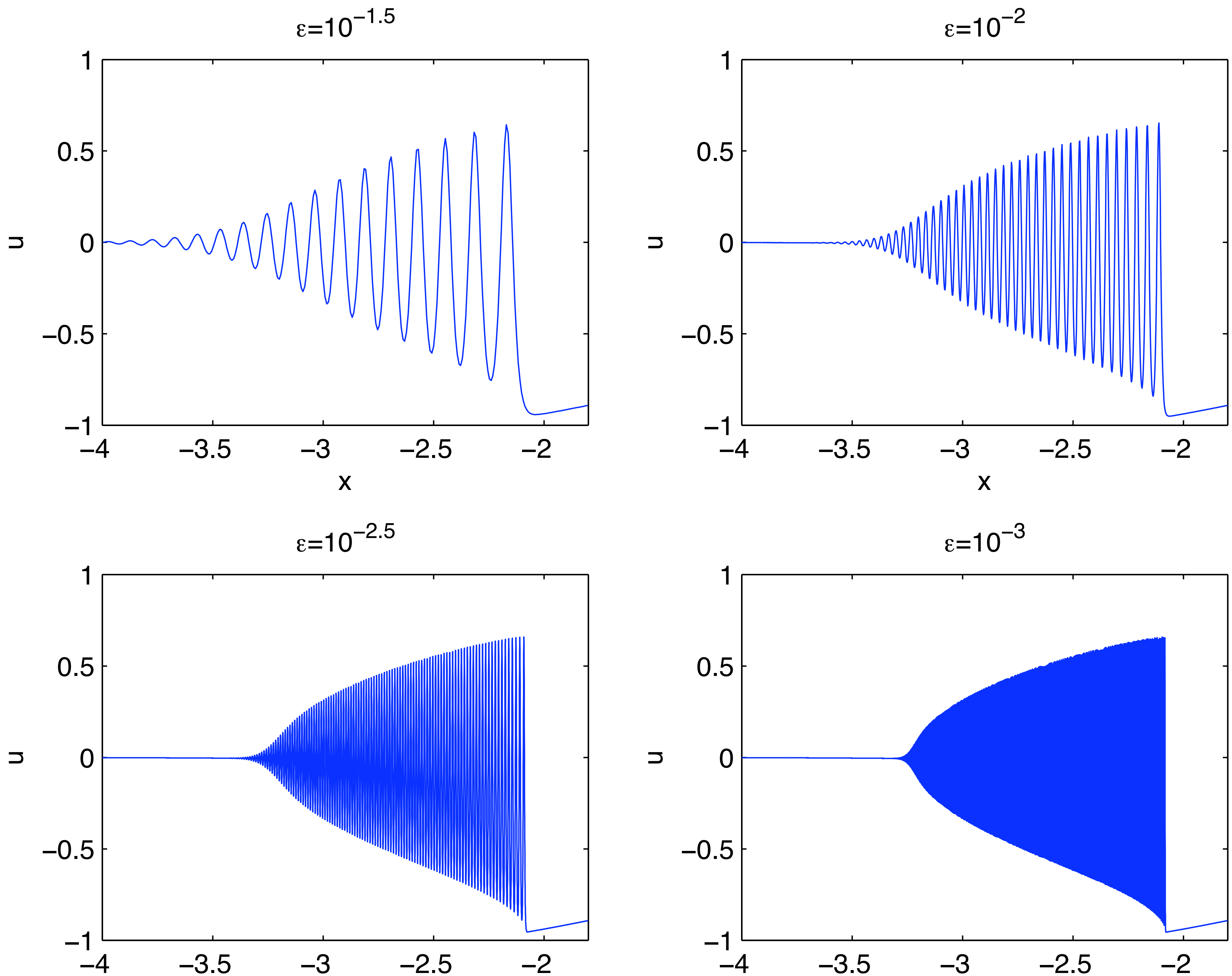
Perturbation: KdV equation $u_t + u\,u_x + \epsilon^2 u_{xxx} = 0$

(Hamiltonian case)



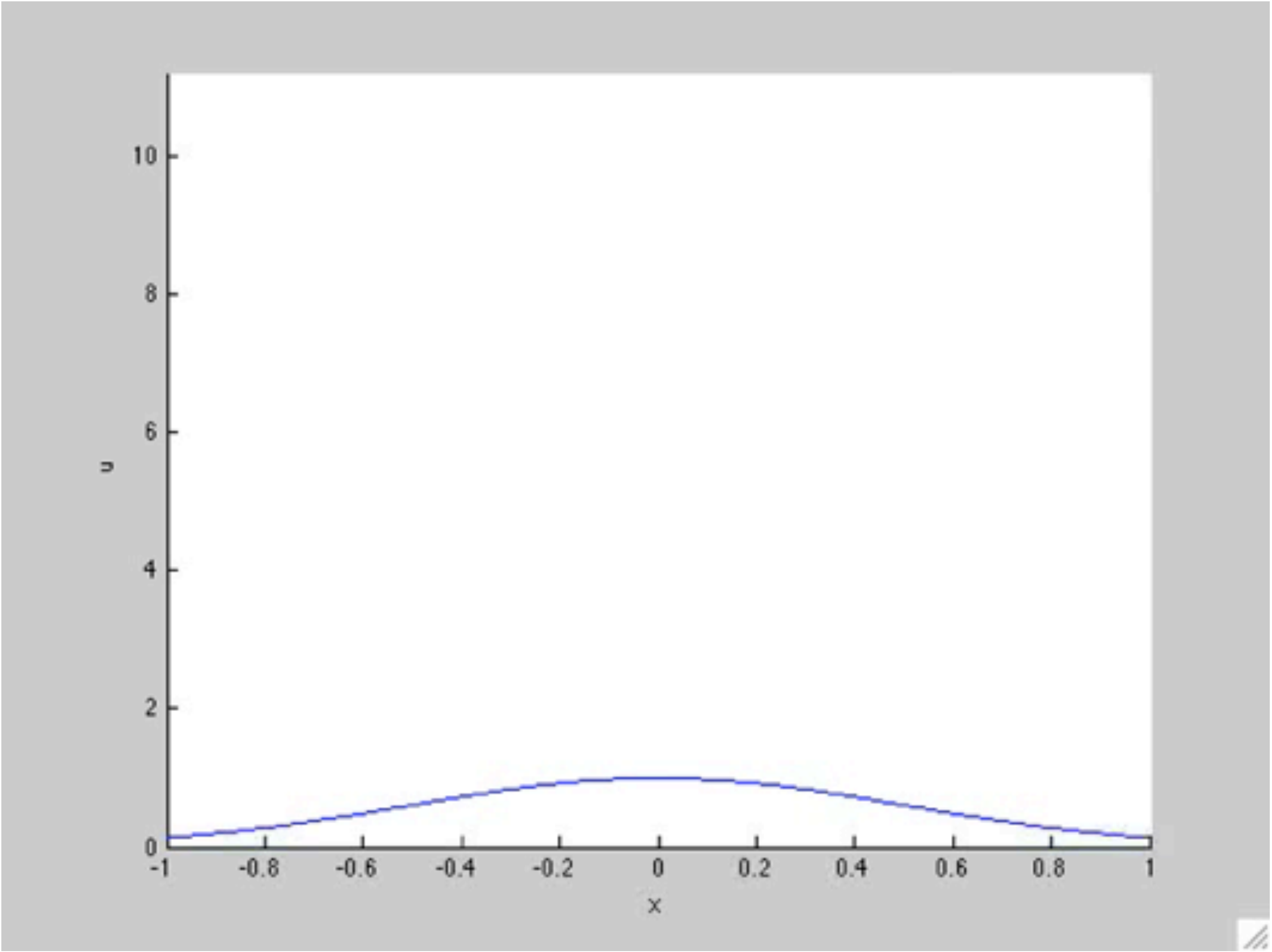
Phase transition from regular to oscillatory behavior (N.Zabusky, M.Kruskal, 1965)

The smaller is ϵ, the faster are the oscillations



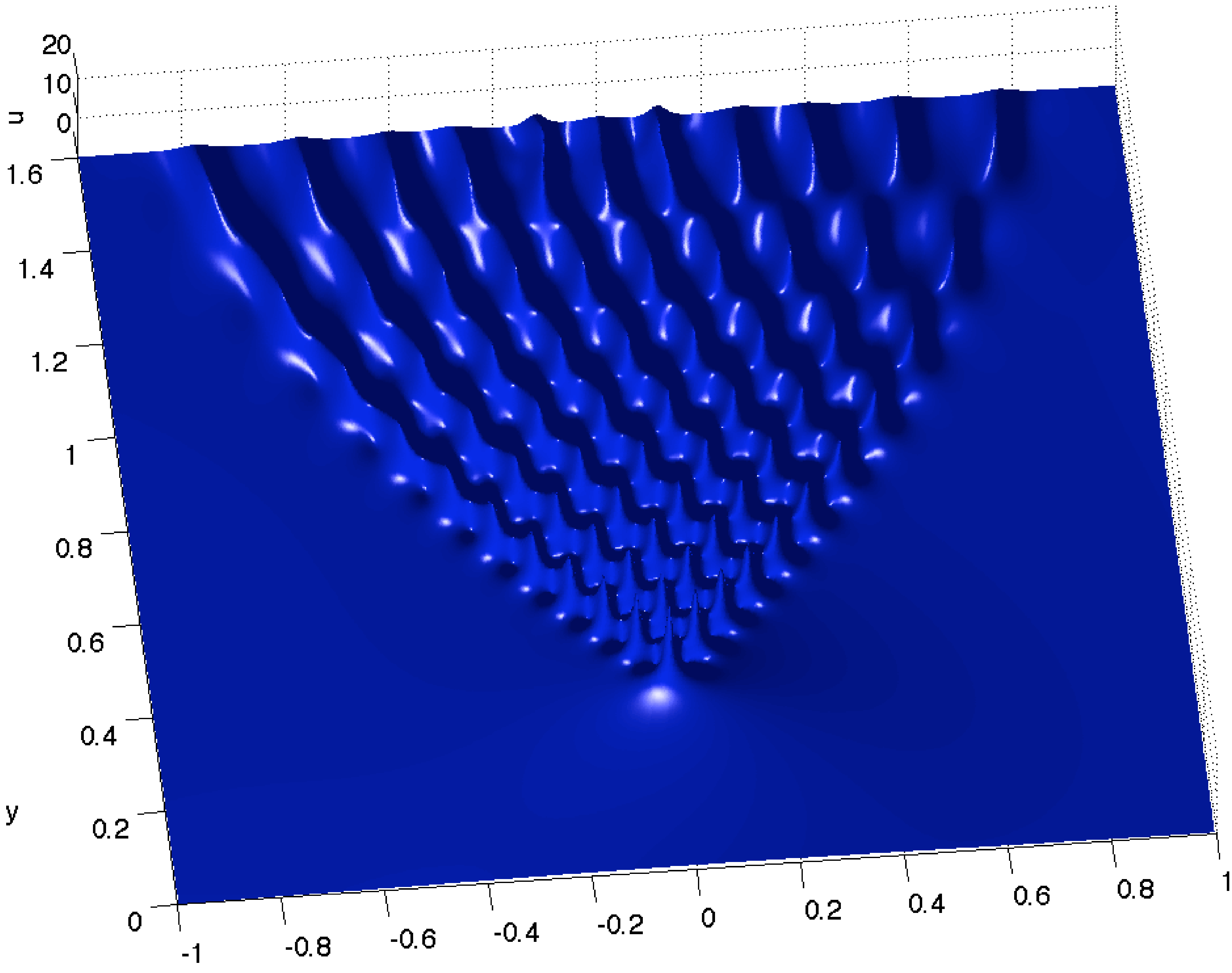
Nonlinear Schrödinger equation (the focusing case)

$$i\epsilon\,\psi_t + \frac{\epsilon^2}{2}\psi_{xx} + |\psi|^2\psi = 0$$



NB: the dispersionless limit is a PDE of *elliptic type*

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} + \begin{pmatrix} v & u \\ -1 & v \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} = 0$$, eigenvalues $\lambda = v \pm i\sqrt{u}$


20
10
0
u
1.6
1.4
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1
0.8
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y
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-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
x

Main Conjecture (B.D., 2005): a finite list of types of the critical behaviour

(Universality)

KdV asymptotics

$$u_t + u\,u_x + \frac{\epsilon^2}{12} u_{xxx} = 0$$

$$u(x, t = 0, \epsilon) = \varphi(x)$$

$$u(x,t,\epsilon) \simeq U_{\rm ell}\left(\frac{S(x,t)}{\epsilon}; r_1(x,t), r_2(x,t), r_3(x,t)\right)$$

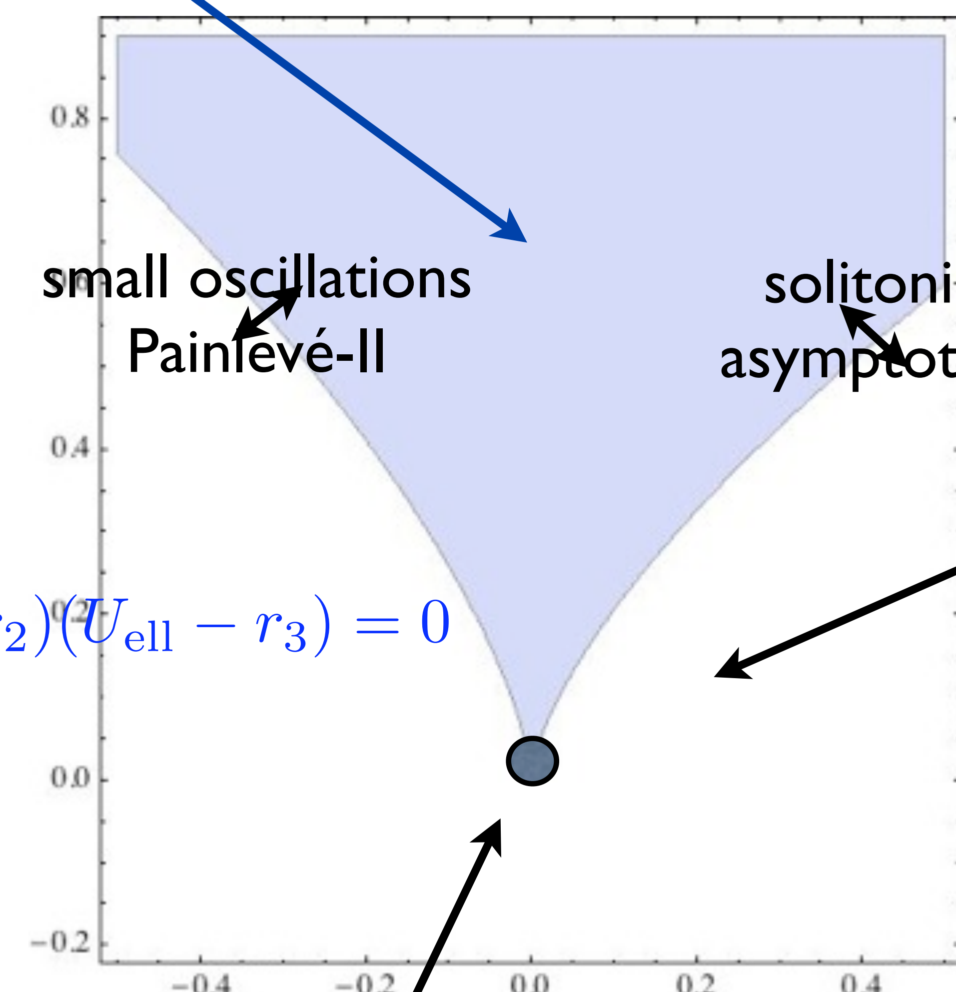
A.Gurevich, L.Pitayevski '73

P.Lax, D.Levermore '83

$$U_{\rm ell}(z; r_1, r_2, r_3)$$

Weierstrass elliptic function

$$(U_{\rm ell}')^2 + 4(U_{\rm ell} - r_1)(U_{\rm ell} - r_2)(U_{\rm ell} - r_3) = 0$$



$$u(x,t,\epsilon) = v(x,t) + \mathcal{O}(\epsilon)$$

where

$$v_t + v\,v_x = 0$$

$$v(x,0) = \varphi(x)$$

point of gradient catastrophe for the Hopf solution $v(x,t)$

Universality, the case of scalar Hamiltonian PDEs

at the point of phase transition $x = x_0, t = t_0, u = u_0$

the generic solution has the following asymptotics

$$u(x,t) = v_0 + \alpha\, \epsilon^{2/7} U\left(\beta \frac{x - a_0(t-t_0) - x_0}{\epsilon^{6/7}}, \gamma \frac{t-t_0}{\epsilon^{4/7}}\right) + O\left(\epsilon^{4/7}\right)$$

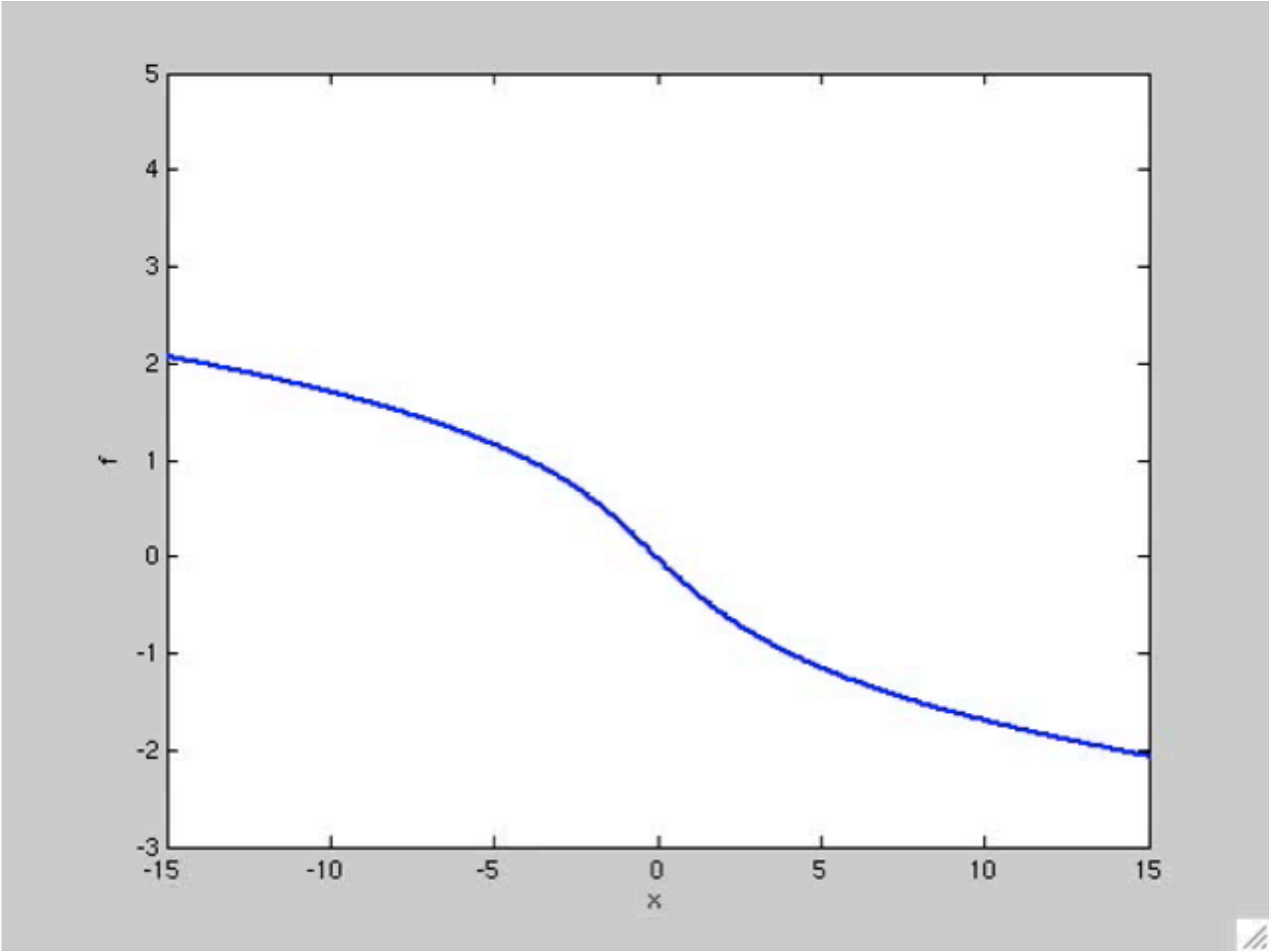
where $U(X,T)$ is a particular solution to the ODE P_I^2

$$X = T\,U - \left[\frac{1}{6}U^3 + \frac{1}{24}\left(U'^2 + 2U\,U''\right) + \frac{1}{240}U^{IV}\right]$$

(a differential equation in *X* depending on the parameter *T*)

The smooth solution $U(X,T)$ to P_I^2

$$X = T\,U - \left[\frac{1}{6}U^3 + \frac{1}{24}\left(U'^2 + 2U\,U''\right) + \frac{1}{240}U^{IV}\right]$$



Proof of existence: T.Claeys, M.Vanlessen, 2007

The case of NLS $i\epsilon\,\psi_t + \frac{\epsilon^2}{2}\psi_{xx} + |\psi|^2\psi = 0$

A complex combination of the real variables

$$u = |\psi|^2, \quad v = \frac{\epsilon}{2i}\left(\frac{\psi_x}{\psi} - \frac{\bar{\psi}_x}{\bar{\psi}}\right)$$

$$w(x,t) = u(x,t) + i\sqrt{u_0}\,v(x,t) =$$

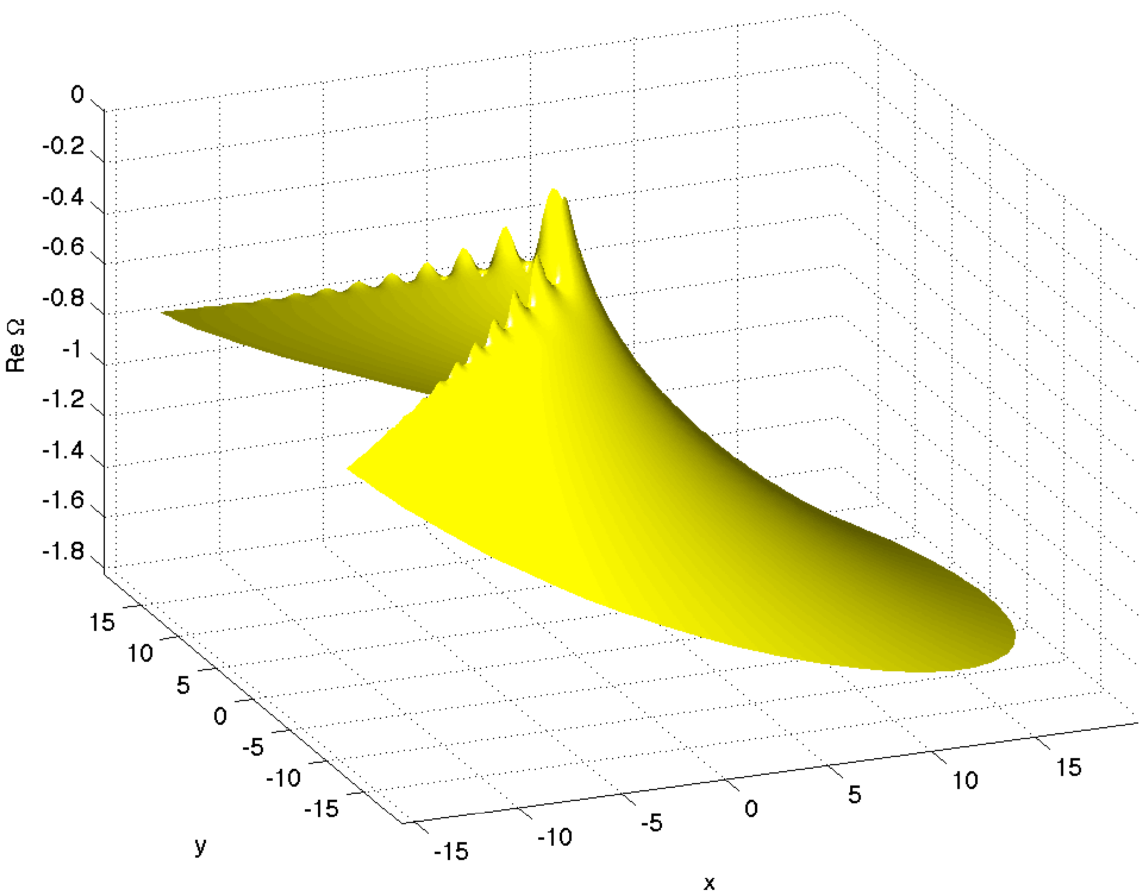
$$= w^0 + \gamma\,\epsilon^{2/5}W(\epsilon^{-4/5}Z) + \mathcal{O}(\epsilon^{4/5}),$$

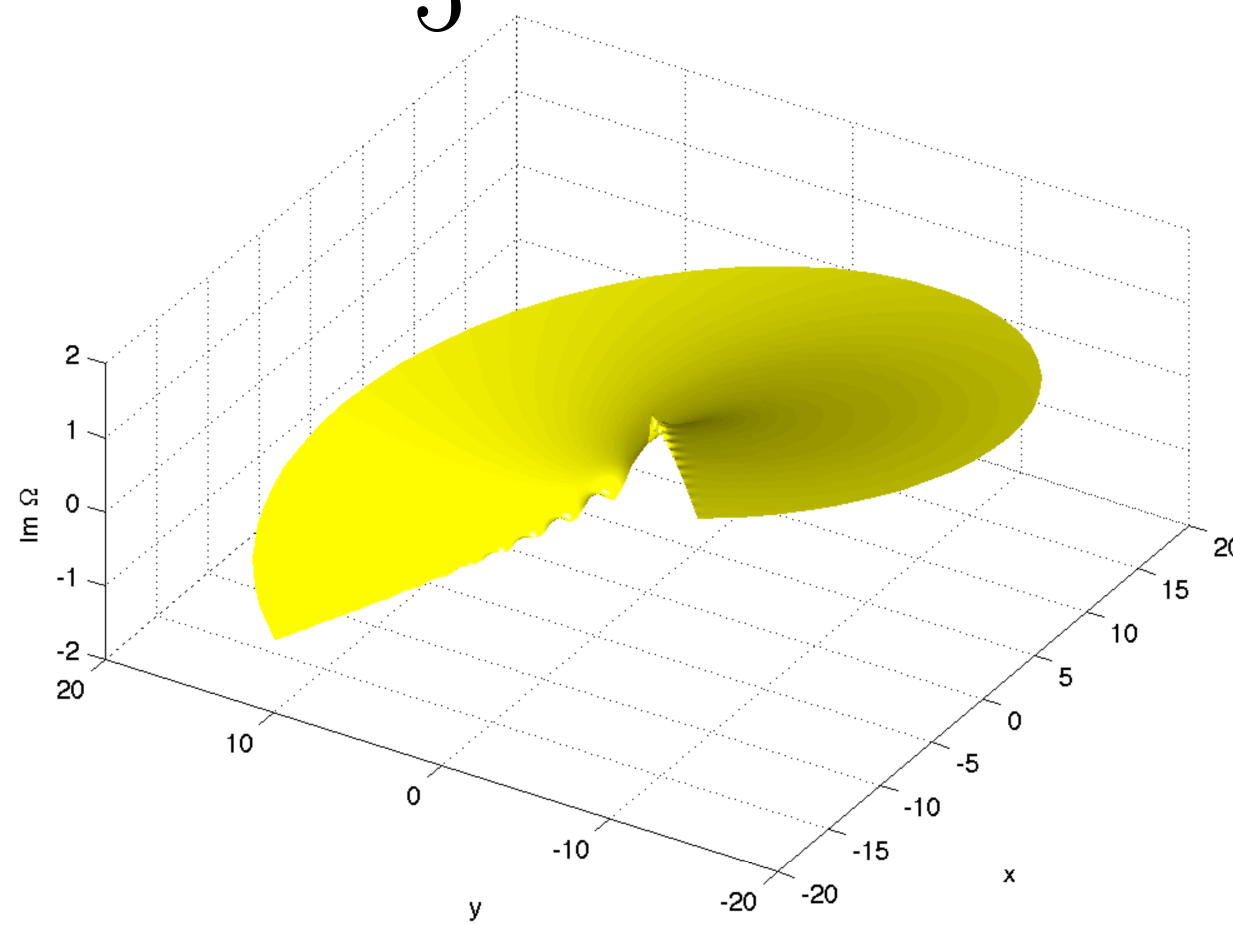
$$Z = \alpha\,x + \beta\,t + Z_0, \quad \alpha, \beta, \gamma, Z_0 \in \mathbf{C}$$

Here $W(Z)$ is the *tritronquée* solution to the Painlevé–I eq.

$$W'' = 6W^2 - Z$$

Conjecture (B.D., T.Grava, C.Klein 2009): analyticity of the solution in the sector $|\arg Z| < \frac{4\pi}{5}$





Proof (September 2012): O.Costin *et al.*

Motivations (for a scalar Hamiltonian equation)

$$u_t + a(u)u_x + \epsilon^2 \left[c(u)u_{xxx} + b(u)u_{xx}u_x + d(u)u_x^3\right] + \mathcal{O}\left(\epsilon^3\right) = 0$$

The solution to a Cauchy problem for the leading term

$$v_t + a(v)\, v_x = 0$$
$$v(x,0) = u_0(x)$$

can be written in the implicit form

$$x = a(v)\, t + f(v), \quad f\left(u_0(x)\right) \equiv x$$

Point of gradient catastrophe (x_0, t_0, v_0) such that

$$x_0 = a(v_0)\, t_0 + f(v_0)$$
$$0 = a'(v_0)\, t_0 + f'(v_0)$$
$$0 = a''(v_0)\, t_0 + f''(v_0)$$

Step 1: near the generic point of gradient catastrophe the solution to the dispersionless equation can be approximated by

$$\bar{x} + a_0' \bar{v}\, \bar{t} \simeq \frac{1}{6} f_0''' \bar{v}^3$$

where

$$\begin{aligned} \bar{x} &= x - x_0 - a_0(t - t_0) \\ \bar{t} &= t - t_0 \\ \bar{v} &= v - v_0 \end{aligned}$$

$$a_0 = a(v_0), \quad a_0' = a'(v_0) \quad \text{etc.}$$

After rescaling

$$\begin{aligned} \bar{x} &\mapsto \lambda\, \bar{x} \\ \bar{t} &\mapsto \lambda^{2/3} \bar{t} \\ \bar{v} &\mapsto \lambda^{1/3} \bar{v} \end{aligned}$$

one obtains the above cubic equation modulo

$$\mathcal{O}\left(\lambda^{1/3}\right), \quad \lambda \to 0$$

Step 2: replacing the original dispersive equation by KdV near the point (x_0, t_0, v_0)

$$u_t + a(u)u_x + \epsilon^2 \left[c(u)u_{xxx} + b(u)u_{xx}u_x + d(u)u_x^3\right] + \mathcal{O}\left(\epsilon^3\right) = 0$$

Rescaling

$$\begin{array}{rclcr} \bar{x} & = & x - x_0 - a_0(t - t_0) & \mapsto & \lambda\,\bar{x} \\ \bar{t} & = & t - t_0 & \mapsto & \lambda^{2/3}\bar{t} \\ \bar{u} & = & u - v_0 & \mapsto & \lambda^{1/3}\bar{u} \\ \epsilon & & & \mapsto & \lambda^{7/6}\epsilon \end{array}$$

yields

$$\bar{u}_{\bar{t}} + a_0'\bar{u}\,\bar{u}_{\bar{x}} + \epsilon^2 c_0 \bar{u}_{\bar{x}\bar{x}\bar{x}} \simeq 0, \quad c_0 = c(v_0)$$

modulo $\mathcal{O}\left(\lambda^{1/3}\right), \quad \lambda \to 0$

Step 3: choosing a particular solution to KdV

The trick: to replace PDE (the KdV) + initial condition by an ODE ("string equation")

$$u_t + u\, u_x + \frac{\epsilon^2}{12} u_{xxx} = 0$$

$$u(x, 0; \epsilon) = u_0(x)$$

First, rewrite the solution $x = v\,t + f(v), \quad f\,(u_0(x)) \equiv x$

to Hopf equation $v_t + v\, v_x = 0$

in the form

$$\frac{\partial}{\partial v}\left[F(v) + t\,\frac{v^2}{2} - x\,v\right] = 0, \quad F'(v) = f(v)$$

The idea: to determine the solution to the KdV equation

$$u_t + u\,u_x + \frac{\epsilon^2}{12} u_{xxx} = 0$$
$$u(x,0;\epsilon) = u_0(x)$$

with the **same**, modulo $\mathcal{O}\left(\epsilon^2\right)$
initial data from the Euler-Lagrange equation

$$\frac{\delta}{\delta u(x)} \left\{ H_F[u;\epsilon] + \int \left(t\,\frac{u^2}{2} - x\,u \right)\, dx \right\} = 0$$

("string equation")

Construction of the functional $H_F[u;\epsilon]$

uses the theory of deformations of the conservation laws

For the Hopf equation $v_t + v\, v_x = 0$ the functional

$$H_F^0 = \int F(v)\, dx$$

is a conservation law for an arbitrary function F

Theorem. For any function F there exists

a deformed functional

$$H_F = \int \left[F(u) - \frac{\epsilon^2}{24} F'''(u) u_x^2 + \epsilon^4 \left(\frac{1}{480} F^{(4)} u_{xx}^2 - \frac{1}{3456} F^{(6)} u_x^4 \right) + \dots \right] dx$$

being a conservation law for the KdV equation

$$u_t + u\, u_x + \frac{\epsilon^2}{12} u_{xxx} = 0$$

An explicit formula in terms of Lax operator

$$L = \frac{\epsilon^2}{2}\frac{d^2}{dx^2} + u(x)$$

Then

$$H_F = \int h_F \, dx$$

where

$$h_F = \mathrm{res}\, F^{(1/2)}\,(L)$$

So, one arrives at studying solutions to the Euler-Lagrange equation

$$\frac{\delta}{\delta u(x)}\left\{H_F[u;\epsilon]+\int\left(t\,\frac{u^2}{2}-x\,u\right)\,dx\right\}=0$$

where

$$H_F=\int\left[F(u)-\frac{\epsilon^2}{24}F'''(u)u_x^2+\epsilon^4\left(\frac{1}{480}F^{(4)}u_{xx}^2-\frac{1}{3456}F^{(6)}u_x^4\right)+\dots\right]\,dx$$

Explicitly

$$x=u\,t+f(u)+\frac{\epsilon^2}{24}\left[2f''(u)u_{xx}+f'''(u)u_x^2\right]+\frac{\epsilon^4}{240}f'''(u)u_{xxxx}+\dots$$

The last step: to apply to the "string equation"

$$x = u\,t + f(u) + \frac{\epsilon^2}{24}\left[2f''(u)u_{xx} + f'''(u)u_x^2\right] + \frac{\epsilon^4}{240}f'''(u)u_{xxxx} + \ldots$$

a rescaling near the point of phase transition

$$\begin{array}{rclcr} \bar{x} & = & x - x_0 - v_0(t - t_0) & \mapsto & \lambda\,\bar{x} \\ \bar{t} & = & t - t_0 & \mapsto & \lambda^{2/3}\bar{t} \\ \bar{u} & = & u - v_0 & \mapsto & \lambda^{1/3}\bar{u} \\ \epsilon & & & \mapsto & \lambda^{7/6}\epsilon \end{array}$$

to arrive at

$$\bar{x} = \bar{u}\,\bar{t} + \frac{1}{6}f_0'''\left[\bar{u}^3 + \frac{\epsilon^2}{4}\left(2\bar{u}\,\bar{u}_{\bar{x}\bar{x}} + \bar{u}_{\bar{x}}^2\right) + \frac{\epsilon^4}{40}\bar{u}_{\bar{x}\bar{x}\bar{x}\bar{x}}\right] + \mathcal{O}\left(\lambda^{1/3}\right)$$

Choosing $\lambda = \epsilon^{6/7}$ one obtains P_I^2

Solutions to P_I^2

$$X = T\,U - \left[\frac{1}{6}U^3 + \frac{1}{24}\left(U'^2 + 2U\,U''\right) + \frac{1}{240}U^{IV}\right]$$

satisfy KdV

$$U_T = U\,U_X + \frac{1}{12}U_{XXX}$$

Matching condition: for large $|X|$

$U(X,T) \simeq$ (unique) root of cubic equation $X = U\,T - \frac{1}{6}U^3$

Matching + smoothness $\Rightarrow$ choice of a particular solution

Proof of the Universality Conjecture for analytic solutions to KdV in T.Claeys, T.Grava, 2008

Uses:

- Riemann-Hilbert formulation of inverse scattering
- asymptotics of the scattering data of Lax operator

$$L = \frac{\epsilon^2}{2}\frac{d^2}{dx^2} + u$$

- Deift-Zhou asymptotic analysis of the Riemann-Hilbert problem

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Thank you!