Statistical models of hadron production

simple models for complicated processes

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Theoretical description of particle production

$$\mathcal{P}_n(i \to f) =$$

$$\int d^4 p'_1 \dots d^4 p'_n \delta(p'_1 + \dots + p'_n - P_i) \prod_{j=1}^{n} \delta(p'^2_j - m_j^2) |\langle p'_1, \dots p'_n | \mathcal{S} | i \rangle|^2$$

The dynamical part

$$\langle p'_1, \dots p'_n | \mathcal{S} | i \rangle$$

The kinematical part

$$\delta(p'_1 + \dots + p'_n - P_i) \prod_{j=1}^{n} \delta(p'^2_j - m_j^2)$$

Place for statistical physics

More particles (degrees of freedom) in the process: kinematics tends to dominate the behavior of the system

- measurable quantities are much less detailed than $\langle p'_1, \ldots p'_n | \mathcal{S} | i \rangle$
- with the integration over a large region of the phase space the dynamical details are averaged and only a few parameters remains
- restricted knowledge of $\langle p'_1, \ldots p'_n | \mathcal{S} | i \rangle$ is not needed

$$P_n = \bar{S}_n \, \mathcal{R}_n$$

$$\mathcal{R}_n = \int d^4 p'_1 \ldots d^4 p'_n \delta(p'_1 + \cdots + p'_n - P_i) \prod_{j=1}^{n} \delta(p'^2_j - m^2_j)$$

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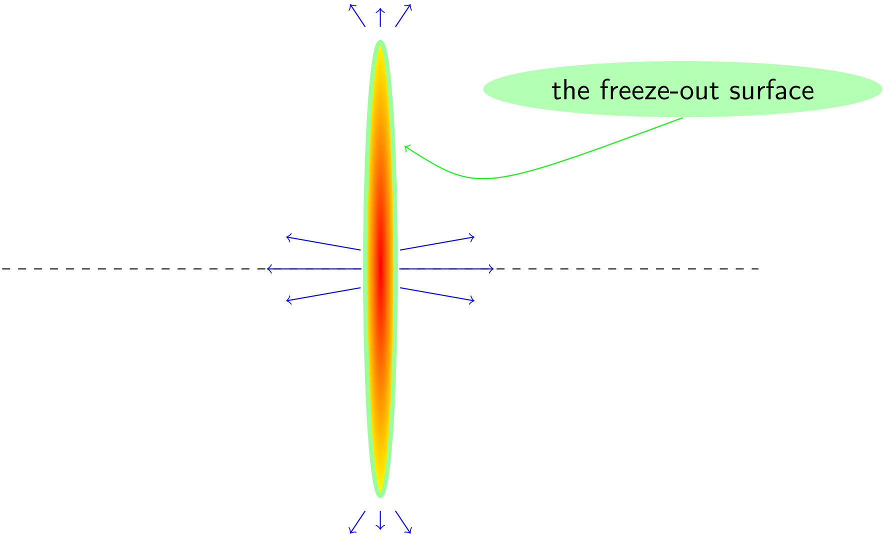
Arguments work if the thermodynamic equilibrium is reached

HIC collision – graphics



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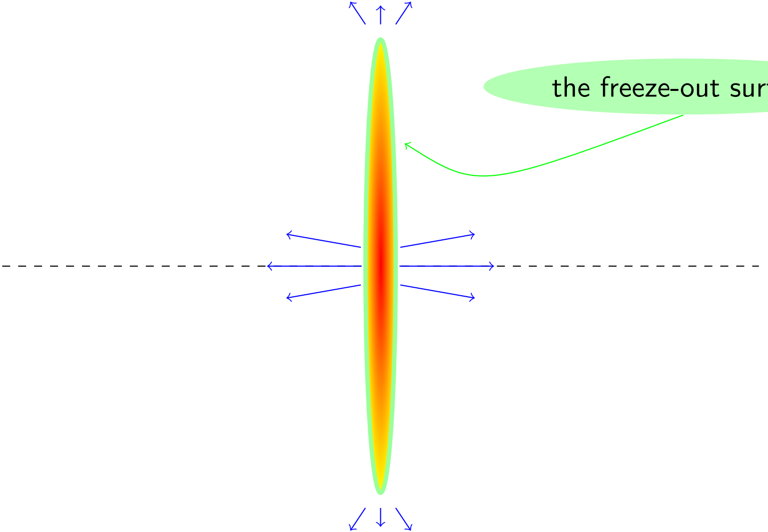
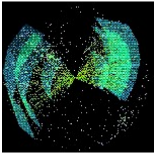
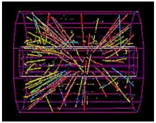






The aim of statistical models

To derive the equilibrium properties of a macroscopic system from the measured yields of the constituent particles

but

Not to describe how a system approaches equilibrium.

The chemical freeze-out

The stage where hadrons have been created and the net numbers of stable particles of each type no longer change in further system evolution.

Almost reality

The time-space evolution of the system after collision is given by kinetic equations

$$\left(p_\mu \partial^\mu + F_\mu^{(j)} \partial^\mu(p)\right) f_j(x,p) = C_j(x,p)$$

long distance forces

interaction with other particles

phase space distribution function

Fewer degrees of freedom: small number of particles or low temperature: dynamics more and more important

Local thermodynamic equilibrium

In the adiabatic processes

$$\partial_\mu s^\mu = 0$$

and

$$f_j^{eq}(x,p) = \frac{1}{e^{(u^\mu(x)p_\mu - b_j\mu_b(x) - s_j\mu_s(x))/T(x)} \pm 1}$$

Fireball parameters

The mean free path

$$\lambda_j = \frac{1}{\sum_k \langle \sigma_{jk} \rangle}$$

Between scattering time

$$\tau^{(j)}_{scatt} = \frac{\lambda_j}{\langle v_j \rangle} \sim \frac{1}{\sum_k \langle v_{jk} \sigma_{jk} \rangle}$$

Escape time

$$\tau^{(j)}_{esc} = \frac{R}{\langle v_j \rangle} \sim \frac{R}{c}$$

Expansion time

$$\tau_{exp} = \frac{1}{\partial_\mu u^\mu(x)}$$

Freeze-out condition

$$\tau_{scatt}^{(j)} \gtrsim \min(\tau_{esc}^{(j)}, \tau_{exp})$$

gives

freeze-out hypersurface $\Sigma_f^{(j)}(x)$

Statistical model calculations – in principle

$$\epsilon = \frac{1}{2\pi^2}\sum_{j=1}(2S_j+1)\int_0^\infty \frac{dp\, p^2 E_j}{\exp\left\{\frac{E_j-\mu_j}{T}\right\}+g_j}\ ,$$

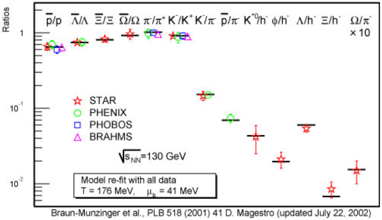
$$n_b = \frac{1}{2\pi^2}\sum_{j=1}(2S_j+1)b_j\int_0^\infty \frac{dp\, p^2}{\exp\left\{\frac{E_j-\mu_j}{T}\right\}+g_j}\ ,$$

$$n_s = \frac{1}{2\pi^2}\sum_{j=1}(2S_j+1)s_j\int_0^\infty \frac{dp\, p^2}{\exp\left\{\frac{E_j-\mu_j}{T}\right\}+g_j}\ ,$$

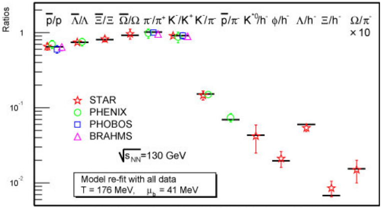
where

$$\mu_j = b_j\mu_b + s_j\mu_s$$

It works


Ratios
p̄/p Λ̄/Λ Ξ̄/Ξ Ω̄/Ω π⁻/π⁺ K⁻/K⁺ K⁻/π⁻ p̄/π⁻ K*0/h⁻ φ/h⁻ Λ/h⁻ Ξ/h⁻ Ω/π⁻ × 10
STAR
PHENIX
PHOBOS
BRAHMS
√sNN=130 GeV
Model re-fit with all data
T = 176 MeV, μb = 41 MeV
Braun-Munzinger et al., PLB 518 (2001) 41 D. Magestro (updated July 22, 2002)

It works



Braun-Munzinger et al., PLB 518 (2001) 41 D. Magestro (updated July 22, 2002)

although still there are discussions is this "the real" temperature or "a fake" temperature due to the phase space dominance effect.

Puzzle

Multiproduction at high energy e^+e^- and pp is still well described within statistical model.

Finite volume effects

- The quark - gluon - hadrons systems are created in a finite volume
- In the thermodynamic limit different statistical ensembles are equivalent,

Finite volume effects

- The quark - gluon - hadrons systems are created in a finite volume
- In the thermodynamic limit different statistical ensembles are equivalent,but what does it mean?
- selection of the appropriate scale parameter

The scale parameter:

$$VT^3\,; \quad 1\,\mathrm{fm}\cdot 1\,\mathrm{MeV} \approx 200\hbar c$$

The thermodynamic limit is not sufficient

There are physical quantities which are finite in the thermodynamic limit

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The scaled variance

$$\omega_N = \frac{\Delta N^2}{\langle N \rangle} = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle}\,.$$

has different thermodynamic limits in different statistical ensembles:

$$\omega_N = \begin{cases} 1 & \text{in the grand canonical ensemble at } \langle Q \rangle = 0\,, \\ \frac{1}{2} & \text{in the canonical ensemble at } Q = 0\,. \end{cases}$$

See e.g.: V. V. Begun, M. Gazdzicki, M. I. Gorenstein and O. S. Zozulya, Phys. Rev. C **70**,034901 (2004); V. V. Begun, M. I. Gorenstein, A. P. Kostyuk and O. S. Zozulya, Phys. Rev. C **71**,054904 (2005); V. V. Begun, M. I. Gorenstein and O. S. Zozulya, Phys. Rev. C **72**,014902 (2005); A. Keränen, F. Becattini, V.V. Begun, M.I. Gorenstein, O.S. xZozulya, J. Phys. G **31**, S1095 (2005)

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Look at densities:

$$\langle N^2 \rangle = V^2 \langle n^2 \rangle$$

$$\omega_N = \frac{\Delta N^2}{\langle N \rangle} = \frac{V^2 \Delta n^2}{V \langle n \rangle} \equiv V \omega_n \,.$$

ω_N finite in the thermodynamic limit

$$\Downarrow$$

$$\omega_n = \frac{1}{V} \mathcal{R} + \mathcal{O}(V^{-2}) \,.$$

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Corollary

Look at $\mathcal{O}(V^{-1})$ terms.
They are important also in the thermodynamic limit.

More:

J. Cleymans, K. Redlich and L.T.:
Phys. Rev. C **71** 047902 (2005);
J. Phys. G **31** 1421 (2005)

Canonical and grand canonical ensembles

Non-HEP physics

Number of particles is important

- The canonical ensemble: fixed number of particles N.
- The grand canonical ensemble: fixed average number of particles $\langle N \rangle$.

Statistical ensembles $\Longrightarrow$ probabilities

- $\mathcal{P}_N^C(E, V)$ for the canonical distribution,

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$$\mathcal{P}_{\langle N \rangle}^{GC}(N, E, V) = \mathcal{P}_{\langle N \rangle}^{GC}(N, V)\mathcal{P}_N^C(E, V)$$

The thermodynamic limit

The thermodynamic limit

The canonical distribution

$$V \to \infty,\ N \to \infty,\ \frac{N}{V} = n$$

The grand canonical distribution

$$V \to \infty,\ \langle N \rangle \to \infty,\ \frac{\langle N \rangle}{V} = \langle n \rangle$$

Probability distributions

The canonical distribution

$$\mathcal{P}_n^{C\infty}(\epsilon) = \lim_{V\to\infty} V\mathcal{P}_{Vn}^{C}(V\epsilon, V)$$

The grand canonical distribution

$$\mathcal{P}_{\langle n \rangle}^{GC\infty}(\epsilon) = \lim_{V\to\infty} V\mathcal{P}_{V\langle n \rangle}^{GC}(V\epsilon, V)$$

Equivalence

$$\mathcal{P}_{\langle n \rangle}^{GC\infty}(\epsilon) = \mathcal{P}_n^{C\infty}(\epsilon)$$

The grand canonical distribution

For particle number

$$P_{\langle N\rangle}(N) = \frac{\langle N\rangle^N}{N!}\,\mathrm{e}^{-\langle N\rangle}\ .$$

For the density

$$\mathbf{P}_{\langle n\rangle}(n) = VP_{V\langle n\rangle}(Vn) = V\frac{(V\langle n\rangle)^{Vn}}{\Gamma(Vn+1)}\,\mathrm{e}^{-V\langle n\rangle}\ .$$

T-limit: $V \to \infty$

$$\mathbf{P}_{\langle n\rangle}(n) \sim V^{1/2}\frac{1}{\sqrt{2\pi n}}\left(\frac{\langle n\rangle}{n}\right)^{Vn}\mathrm{e}^{V(n-\langle n\rangle)}\left\{1-\frac{1}{12Vn}+\mathcal{O}(V^{-2})\right\}$$

Generalized function limit

$$\langle G\rangle = \int dnG(n)\mathbf{P}_{\langle n\rangle}(n)$$

$+$ saddle point method $\Downarrow$

Generalized function limit

$$\langle G \rangle = \int dn G(n) \mathbf{P}_{\langle n \rangle}(n)$$

$+$ saddle point method $\Downarrow$

$$\langle G \rangle = G(\langle n \rangle) + \frac{\langle n \rangle}{2V} G''(\langle n \rangle) + \mathcal{O}(V^{-2}),$$

$\Downarrow\Downarrow$

$$\mathbf{P}_{\langle n \rangle}(n) \sim \delta(n - \langle n \rangle) + \frac{\langle n \rangle}{2V} \delta''(n - \langle n \rangle) + \mathcal{O}(V^{-2}).$$

Moments

$$\langle n^2 \rangle = \langle n \rangle^2 + \frac{\langle n \rangle}{V}\,.$$

$$\Delta n^2 = \frac{\langle n \rangle}{V} \to 0\,.$$

Expressed by particle number (extensive variable)

$$\langle N^2 \rangle = \langle N \rangle^2 + \langle N \rangle \Rightarrow \frac{\Delta N^2}{\langle N \rangle} = 1\,.$$

For the canonical ensemble

$$\Delta N^2 = 0$$

by definition.

This does not contradict the GC and C ensemble equivalence because in the T-limit

$$\mathbf{P}^{GC}_{\langle n \rangle}(n) = \delta(n - \langle n \rangle) = \mathbf{P}^{C}_{n}(n)$$

Part II

High Energy Statistical Physics

Crucial points

All people are equal

- any statistical ensemble is connected with the corresponding probability distribution (for densities),
- in finite volume those probability distributions are different for different ensembles,
- statistical ensembles are equivalent in the thermodynamic limit i.e. corresponding probability distribution have the same T-limit,

Crucial points

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But not the same

- semi-intensive variables remain finite in the thermodynamic limit,
- they remain different for different ensembles even in the thermodynamic limit.

Variables

In the thermodynamical limit the relevant probabilities distributions are those related to densities. These distributions are given by moments calculated for densities – not for particles. In the practice, however, we measure particles – not densities. By taking corresponding ratios volumes are canceled.

The density variance $\Delta^2 n$

$$\Delta^2 n = \langle n^2 \rangle - \langle n \rangle^2 = \frac{\langle N^2 \rangle - \langle N \rangle^2}{V^2} .$$

By taking the relative variance

$$\frac{\Delta^2 n}{\langle n \rangle^2} = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle^2} ,$$

volume is eliminated

Variables

A special care should be taken for calculations of ratios of particles momenta. These momenta are extensive variables but their ratios can be finite in the thermodynamic limit. A behavior of such quantities depend on higher terms in the saddle point procedure.

The scaled particle variance

$$\frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle} = V\frac{\langle n^2\rangle - \langle n\rangle^2}{\langle n\rangle}\,.$$

The term

$$\frac{\langle n^2\rangle - \langle n\rangle^2}{\langle n\rangle}\,.$$

tends to zero in the thermodynamic limit. A behavior of the scaled variance depends on the $\mathcal{O}(V^{-1})$ term in the scaled density variance

To get a proper answer one should use high V behavior of momenta $\langle n^k\rangle_q$

Canonical and grand canonical distribution

High energy physics

HEP vocabulary

- particle number is not conserved
- charge is conserved
- canonical and grand canonical with respect to the charge

Conserved charges are important

- Canonical ensemble: charge Q is fixed.
- Grand canonical ensemble: The average charge $\langle Q \rangle$ is fixed.

Ensembles $\Longrightarrow$ probability distributions

- $\mathcal{P}^{C}_{Q}(N, V)$ for the canonical distribution,
- $\mathcal{P}^{GC}_{\langle Q \rangle}(N, Q, V), \mathcal{P}^{GC}_{\langle Q \rangle}(N, V)$ for the grand canonical distribution.

Canonical and grand canonical distribution

High energy physics. The thermodynamic limit

The thermodynamic limit: $V \to \infty$

The canonical ensemble

$$Q, N \to \infty; \; \frac{Q}{V} = q \, , \; \frac{N}{V} = n$$

The grand canonical ensemble

$$\langle Q \rangle, N \to \infty; \; \frac{\langle Q \rangle}{V} = \langle q \rangle \, , \; \frac{N}{V} = n$$

N means here and it will mean till the end of that lecture: number of negative charged particles.

Ideal gas with conserved charge Q

$$\mathcal{P}_Q^{C}(N,V) = \frac{z^{2N+Q}}{N!(N+Q)!}\frac{1}{I_Q(2z)}\,.$$

$$\mathcal{P}_{\langle Q\rangle}^{GC}(Q,V) = I_Q(2z)\left[\frac{\langle Q\rangle + \sqrt{\langle Q\rangle^2 + 4z^2}}{2z}\right]^Q \mathrm{e}^{-\sqrt{\langle Q\rangle^2+4z^2}}$$

$$\mathcal{P}_{\langle Q\rangle}^{GC}(N,V) = \frac{1}{N!}\left[\frac{\sqrt{\langle Q\rangle^2 + 4z^2} - \langle Q\rangle}{2}\right]^N \exp\left[-\frac{\sqrt{\langle Q\rangle^2 + 4z^2} - \langle Q\rangle}{2}\right].$$

$$z(T) = \frac{V}{(2\pi)^3}\sum_i g_i \int d^3p\ \mathrm{e}^{-\beta\sqrt{p^2+m_i^2}} \equiv Vz_0(T)\,,$$

$$
\begin{aligned}
\mathbf{P}_q^C(n, V) &:= V\mathcal{P}_{Vq}^C(Vn, V)\,, \\
\mathbf{P}_{\langle q\rangle}^{GC}(n, q, V) &:= V^2\mathcal{P}_{V\langle q\rangle}^{GC}(Vn, Vq, V)\,, \\
\mathbf{P}_{\langle q\rangle}^{GC}(q, V) &:= V\mathcal{P}_{V\langle q\rangle}^{GC}(Vq, , V)\,.
\end{aligned}
$$

In the thermodynamic limit $V \to \infty$

$$
\mathbf{P}_q^C(n, V) = \mathcal{P}_q^\infty(n) + \frac{1}{V}R_q^C(n) + \mathcal{O}(V^{-2})
$$

$$
\begin{aligned}
\mathbf{P}_{\langle q\rangle}^{GC}(n, q, V) &= \mathcal{P}_{\langle q\rangle}^\infty(n, q) + \frac{1}{V}R_{\langle q\rangle}^{GC}(n, q) + \mathcal{O}(V^{-2})\,, \\
\mathbf{P}_{\langle q\rangle}^{GC}(q, V) &= \mathcal{P}_{\langle q\rangle}^\infty(q) + \frac{1}{V}S_{\langle q\rangle}^{GC}(q) + \mathcal{O}(V^{-2})\,.
\end{aligned}
$$

Some technique

Grand canonical distribution

$$\mathcal{Z}^{GC}(V,T) = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{Q})} \ .$$

So GC leads to

$$\mathcal{Z}^{GC}(V,T,\mu) = \sum_{N_+}\sum_{N_-} \langle N_+|\, e^{-\beta\hat{H}}\, |N_+\rangle\, e^{\beta\mu N_+} \langle N_-|\, e^{-\beta\hat{H}}\, |N_-\rangle\, e^{-\beta\mu N_-} = e^{(\lambda_+ + \lambda_-)z} \ ,$$

where

$$\lambda_\pm = e^{\pm\beta\mu} \ .$$

$\lambda_\pm$ can be used later as formal fugacities to calculate average numbers of positive and negative particles in the system.

$$\langle N_\pm\rangle = \lambda_\pm \frac{\partial}{\partial\lambda_\pm} \ln \mathcal{Z}^{GC} = e^{\pm\beta\mu}\, z \, .$$

Some technique

The average charge is given as

$$\langle Q \rangle = T \frac{\partial}{\partial \mu} \ln \mathcal{Z}^{GC} \, .$$

The charge dispersion is

$$\Delta Q^2 = \langle Q^2 \rangle - \langle Q \rangle^2 = T^2 \frac{\partial^2}{\partial \mu^2} \ln \mathcal{Z}^{GC} \, .$$

This gives

$$\langle Q \rangle = \left(\mathrm{e}^{\beta\mu} - \mathrm{e}^{-\beta\mu} \right) z = \langle N_+ \rangle - \langle N_- \rangle \, ,$$

$$\Delta Q^2 = \left(\mathrm{e}^{\beta\mu} + \mathrm{e}^{-\beta\mu} \right) z = \langle N_+ \rangle + \langle N_- \rangle \, .$$

The chemical potential is untangled as

$$\frac{\mu}{T} = \frac{1}{2} \ln \frac{\langle N_+ \rangle}{\langle N_- \rangle} \, .$$

Some technique

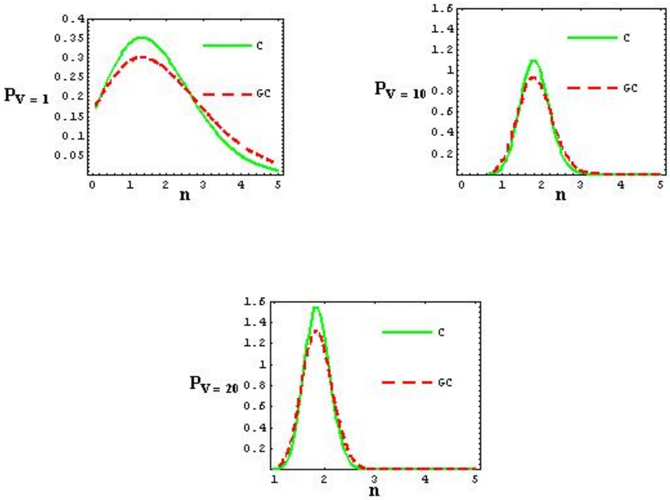
For fugacities

$$\lambda_+ = \frac{\langle Q\rangle + \sqrt{\langle Q\rangle^2 + 4z^2}}{2z}\,; \quad \lambda_- = \frac{2z}{\langle Q\rangle + \sqrt{\langle Q\rangle^2 + 4z^2}}\,;$$

The probability distribution $\mathcal{P}^{GC}_{\langle Q\rangle}(N,V)$ to have the charge Q, $N+Q$ positive (and N negative) particles at the given average charge $\langle Q\rangle$ has a form

$$\mathcal{P}^{GC}_{\langle Q\rangle}(N,V) = \frac{z^{2N+Q}}{N!(N+Q)!}\left[\frac{\langle Q\rangle + \sqrt{\langle Q\rangle^2 + 4z^2}}{2z}\right]^Q e^{-\sqrt{\langle Q\rangle^2 + 4z^2}}\,.$$

Volume dependence



Moment with NLO corrections

$$\langle n^k \rangle_q^C \simeq \int dn\, n^k\, \mathcal{P}_q^\infty(n) + \frac{1}{V} \int dn\, n^k\, R_q^C(n)\,,$$

$$\langle n^k \rangle_{\langle q \rangle}^{GC} \simeq \int dn\, n^k \int dq\, \mathcal{P}_{\langle q \rangle}^\infty(n,q) + \frac{1}{V} \int dn\, n^k \int dq\, R_{\langle q \rangle}^{GC}(n,q)\,.$$

Average density of charged particles

$$\langle n_\pm \rangle_\infty = \frac{\sqrt{q^2 + 4z_0^2} \pm q}{2}$$

The same T-limit in the grand canonical and canonical ensemble

$$\mathcal{P}_{\langle q \rangle}^\infty(n,q) = \mathcal{P}_q^\infty(n) \cdot \delta(q - \langle q \rangle).$$

The canonical ensemble

Probability distribution up to NLO terms

$$\mathbf{P}_q^C(n;V) \simeq \delta\left(n - \langle n\rangle_\infty\right) + \\ + \frac{1}{V}\frac{z_0^2}{q^2+4z_0^2}\,\delta'\left(n - \langle n\rangle_\infty\right) + \frac{1}{V}\frac{z_0^2}{2\sqrt{q^2+4z_0^2}}\,\delta''\left(n - \langle n\rangle_\infty\right)\,.$$

The grand canonical ensemble

Probability distributions up to NLO terms

$$\mathbf{P}^{GC}_{\langle q\rangle}(q,n;V) \simeq \delta\left(n-\langle n\rangle_\infty\right)\delta(q-\langle q\rangle)+\frac{\langle n\rangle_\infty}{2V}\delta''\left(n-\langle n\rangle_\infty\right)\delta(q-\langle q\rangle)\,,$$

$$\mathbf{P}^{GC}_{\langle q\rangle}(q,V) \simeq \delta\left(q-\langle q\rangle\right)+\frac{\sqrt{\langle q\rangle^2+4z_0^2}}{2V}\,\delta''\left(q-\langle q\rangle\right),$$

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Moments up to NLO terms

The canonical ensemble

$$\langle n_{\pm}^{k}\rangle^{C} \simeq \langle n_{\pm}\rangle_{\infty}^{k} - \frac{k}{V}\frac{z_0^2}{q^2+4z_0^2}\langle n_{\pm}\rangle_{\infty}^{k-1} + \frac{k(k-1)}{2V}\frac{z_0^2}{\sqrt{q^2+4z_0^2}}\langle n_{\pm}\rangle_{\infty}^{k-2}\,.$$

The grand canonical ensemble

$$\langle n_{\pm}^{k}\rangle^{GC} \simeq \langle n_{\pm}\rangle_{\infty}^{k} + \frac{k(k-1)}{2V}\langle n_{\pm}\rangle_{\infty}^{k-1}\,.$$

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The grand canonical ensemble

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$\frac{q}{z_0}$ is an observable:

$$\frac{q^2}{z_0^2} = \frac{\langle N_{+}\rangle_{\infty}}{\langle N_{-}\rangle_{\infty}} + \frac{\langle N_{-}\rangle_{\infty}}{\langle N_{+}\rangle_{\infty}} - 2$$

Approaching the thermodynamic limit

$$\Delta_k = \frac{\langle N^k \rangle - \langle N \rangle_\infty^k}{\langle N \rangle_\infty^k} \approx \frac{\langle N^k \rangle - \langle N \rangle^k}{\langle N \rangle^k}$$

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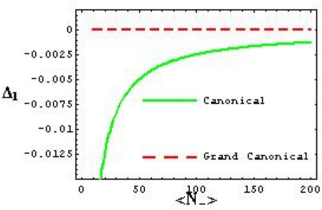
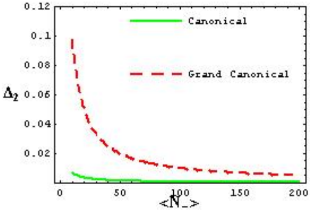
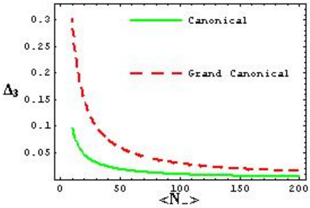
$$\Delta_k = \frac{\langle N^k \rangle - \langle N \rangle_\infty^k}{\langle N \rangle_\infty^k} \approx \frac{\langle N^k \rangle - \langle N \rangle^k}{\langle N \rangle^k}$$







N^{2}LO terms

From

$$\widetilde{\mathbf{P}}^{GC}_{\langle q\rangle}(q,V) \sim V^{1/2}\frac{e^{-V\left(\sqrt{\langle q\rangle^2+4z_0^2}-\sqrt{q^2+4z_0^2}\right)}}{\sqrt{2\pi}(q^2+4z_0^2)^{1/4}}\left[\frac{\langle q\rangle+\sqrt{\langle q\rangle^2+4z_0^2}}{q+\sqrt{q^2+4z_0^2}}\right]^{Vq}\times$$
$$\left\{1+\frac{6z_0^2-q^2}{12V(q^2+4z_0^2)^{3/2}}+\frac{324z_0^4-300z_0^2q^2+q^4}{288V^2(q^2+4z_0^2)^3}\right\}.$$

one gets

$$\widetilde{\mathbf{P}}^{GC}_{\langle q\rangle}(q,V) = \delta\left(q-\langle q\rangle\right)+\frac{\sqrt{\langle q\rangle^2+4z_0^2}}{2V}\,\delta''\left(q-\langle q\rangle\right)+$$
$$\frac{1}{V^2}\left(-\frac{\langle q\rangle}{6}\,\delta^{(3)}(q-\langle q\rangle)+\frac{\langle q\rangle^2+4\,z_0^2}{8}\,\delta^{(4)}(q-\langle q\rangle)\right)+\mathcal{O}(V^{-3}).$$

Example 1: Moments

$$\mathcal{S}_k = \frac{\langle N_\pm^k \rangle - \langle N_\pm \rangle^k}{\langle N_\pm \rangle^{k-1}}$$

For the canonical distribution

$$\text{T-}\lim \mathcal{S}_k^C = \frac{k(k-1)}{4} \frac{\sqrt{q^2 + 4z_0^2} \mp q}{\sqrt{q^2 + 4z_0^2}},$$

For the grand canonical distribution

$$\text{T-}\lim \mathcal{S}_k^{GC} = \frac{k(k-1)}{2}.$$

Example 2: Susceptibilities

κ_p: $p-$th order susceptibility

$$\kappa_p = \frac{\partial^p \ln \mathcal{Z}}{\partial \mu^p}$$

$$\mathcal{K}_{p;r} = \frac{\kappa_p}{\kappa_r}\,,$$

is a semi–intensive quantity.

Conclusions

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