Renormalization Group Approach

towards the QCD Phase Diagram

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I. Introduction to the Renormalization Group

Helmholtz International Summer School

Dense Matter In Heavy Ion Collisions and Astrophysics

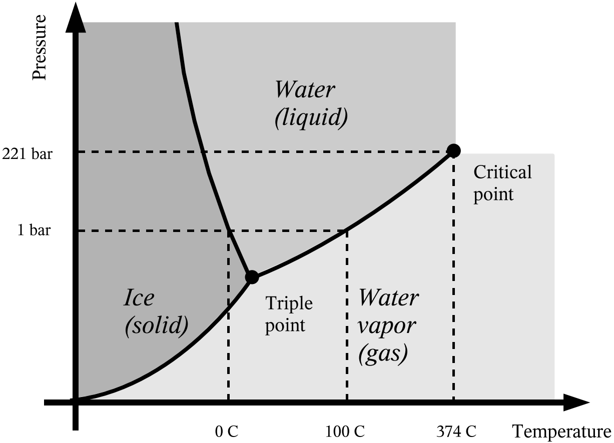
21st Aug. - 1st Sept, 2006

Dubna, Russia

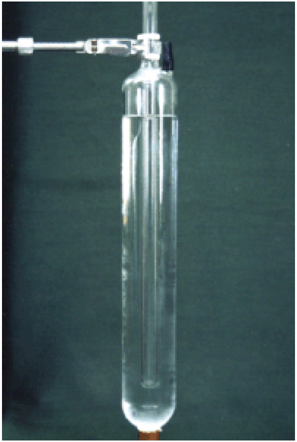
Outline Part I

1. Motivation
   - boiling water
   - why should we use non-perturbative RG?
   - what is the idea of the functional RG?
   - some historical comments on RG
2. RG equation for a $0+0$ dim. field theory
   - derivation of the flow equation
   - a comparison: RG versus perturbation theory
3. Exact RG
   - properties of the ERG
   - truncation schemes
4. Summary

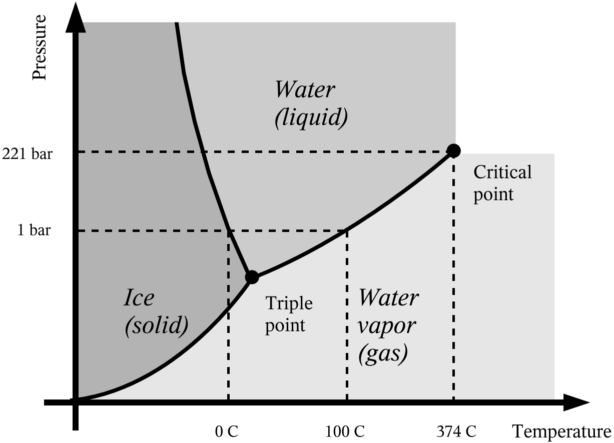
Phase Diagram of Water



Triple point



Phase Diagram of Water



- water-steam transition (first-order transition with latent heat) ends in critical point (second-order)

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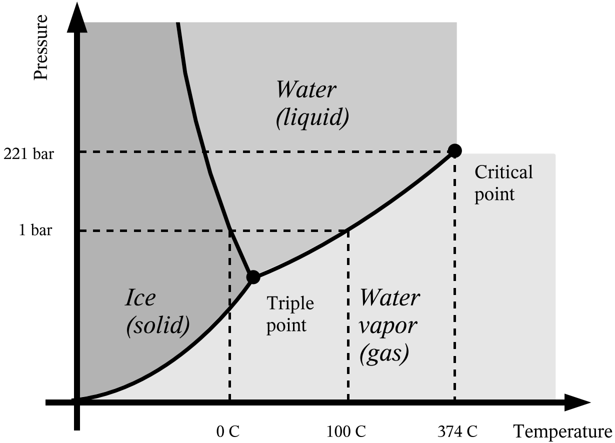

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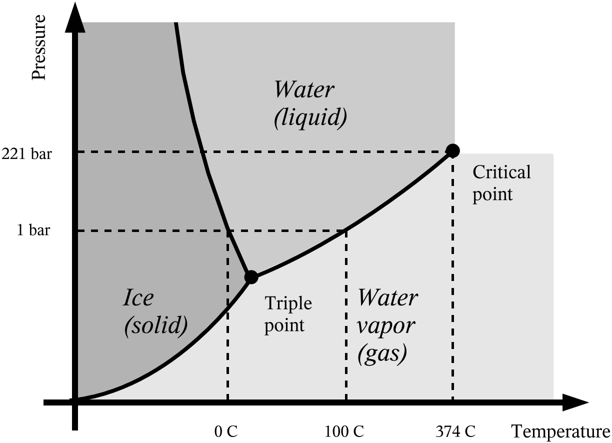
Phase Diagram of Water



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- vicinity of crit. point: density fluctuations on many different scales

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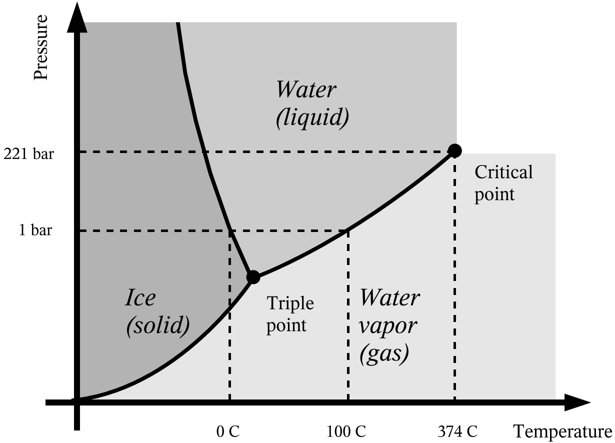

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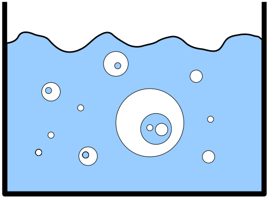
- ▷ vicinity of crit. point: density fluctuations on many different scales
- ▷ it is measurable! $\rightarrow$ critical opalescence

Water in the Vicinity of the Critical Point



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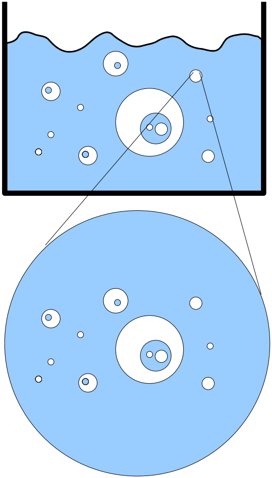
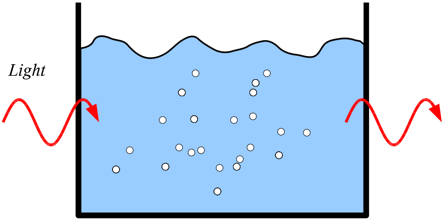


▷ fluctuations on all length scales

Critical Opalescence



below the critical point

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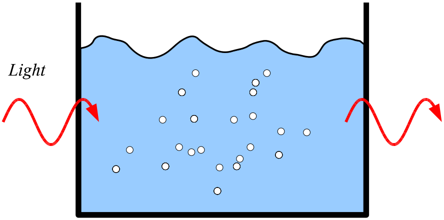

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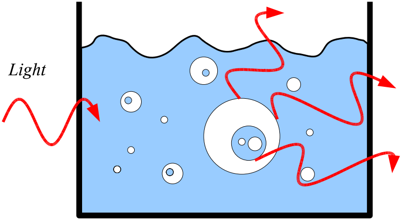

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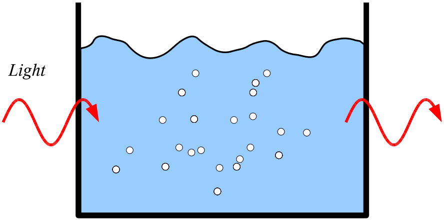

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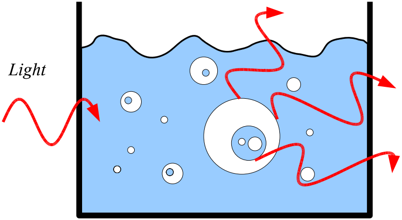

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▷ in the vicinity of the critical point: all length scales are equally important.

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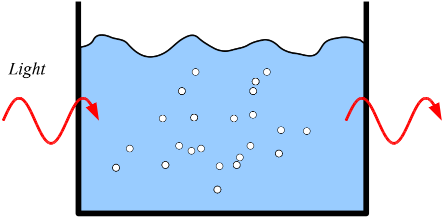

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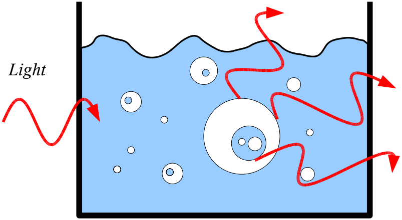

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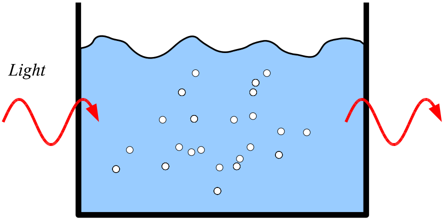
- in the vicinity of the critical point: all length scales are equally important.
- fluctuation scales become arbitrarily large $\rightarrow$ measurable.

Critical Opalescence



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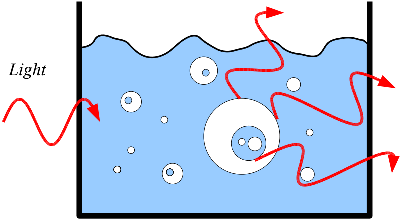

at the critical point

- in the vicinity of the critical point:
  all length scales are equally important.
- fluctuation scales become arbitrarily large → measurable.
- at the critical point:
  light is strongly scattered.

Critical Phenomena

There are much more similar phenomena, where one has to take many (length or time) scales equally into account:

$\longrightarrow$ critical phenomena!

$\Rightarrow$ critical behavior (phase boundaries)

best described by renormalization group methods

Why non-perturbative Renormalization Group?

- allows to describe physics across different length scales

  2nd-order phase transition $\rightarrow$ long-wavelength fluctuations $(\xi \rightarrow \infty)$

  dissimilar systems exhibit same critical exponents $\rightarrow$ universality
  assign each system to a universality class

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- chiral fermions, implementation of quarks w/ & w/o quark masses

- with standard perturbation theory

  $\rightarrow$ not possible to describe spontaneous symmetry breaking

Idea of the Renormalization Group

- Quantum field theory: generating functional

$$\mathcal{Z}[J] = \frac{1}{\mathcal{N}} \int \mathcal{D}\phi e^{-S[\phi,J]} \quad ; \qquad \text{“ill-defined”}$$

- path integral $\Longleftrightarrow$ functional diffential equation (FDE)
- FDE well-defined since original divergences are relegated to the boundary values of its solution

(Wilsonian) RG's

describe very efficiently universal and non-universal aspects of phase diagrams

Wilsonian Renormalization Group

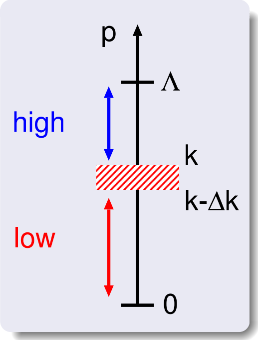
- split field: $\phi(p) = \underset{0 \le |p| < k}{\phi_{\text{low}}(p)} + \underset{k \le |p| \le \Lambda}{\phi_{\text{high}}(p)}$
- integrate high modes:

$$\mathcal{Z}[J] \sim \int \mathcal{D}\phi \, e^{-S[\phi,J]} = \int \mathcal{D}\phi_{\text{low}} \underbrace{\mathcal{D}\phi_{\text{high}} \, e^{-S[\phi_{\text{low}},\phi_{\text{high}},J]}}$$

$$= \int \mathcal{D}\phi_{\text{low}} \qquad e^{-S_{k,\text{eff}}[\phi_{\text{low}},J]}$$

$$\Rightarrow \mathcal{Z}_{k,\text{low}}[J] = e^{-S_{k,\text{eff}}[\phi_{\text{low}},J]}$$

coarse graining cf. block spin transformation



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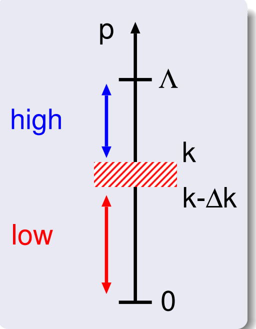
coarse graining   cf. block spin transformation

- integrate from *micro* $\longrightarrow$ *macro*

$$\Rightarrow \lim_{k \to 0} \mathcal{Z}_{k,\text{low}}[J] = \mathcal{Z}[J]$$

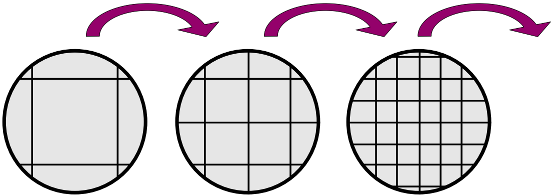
direction unique



▷ procedure: step-by-step magnification of the smallest scale up to larger scales.

$$\texttt{microscopical} \longrightarrow \texttt{macroscopical}$$

▷ look at physics with a microscope with varying resolution



On the History of Renormalization Group

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  $\rightarrow$ qualitative & quantitative

- Name historically, nowadays:
  scale dependence of physics
  strategy to solve problems with many scales

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- ▷ pioneered 1953 by A. Petermann & E.C.G. Stückelberg
- ▷ Gell-Mann & Low (1954) $\rightarrow$ asymptotic behavior of Green's functions in QED
- ▷ Bogoliubov & Shirkov (1959)

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- ▷ Kadanoff (1966)
- ▷ K.G. Wilson (1970)
- ▷ C.G. Callan and K. Symanzik (1970)
- ▷ F. Wegner and A. Houghton (1973)

Kenneth Geddes Wilson

- born June 8th, 1936 in Waltham, Massachusetts
- Ph.D., California Institute of Technology, 1961
- long time at Cornell University, NY
- since August 1988 at Ohio State University (Columbus,OH)

Nobel prize 1982

theory for critical phenomena in connection with phase transitions

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Derivation of the Flow Equation

QFT in $d = 1 + 3$: generating functional $(m^2 = 1)$

$$Z[j] = \int \mathcal{D}\phi(x)\ \exp\left[-\int d^d x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - j\,\phi\right)\right]$$

QFT in $d = 0 + 0$: generating function

$$Z[j] = \int dx\ \exp\left[-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + j\,x\right]$$

Derivation of the Flow Equation

generating function

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Derivation of the Flow Equation

generating function with 'cutoff' R

$$Z[j;R] = \int dx \exp\left[-\frac{1}{2}(1+R)x^2 - \frac{\lambda}{4!}x^4 + j\,x\right]$$

▷ introduce 'cutoff' R

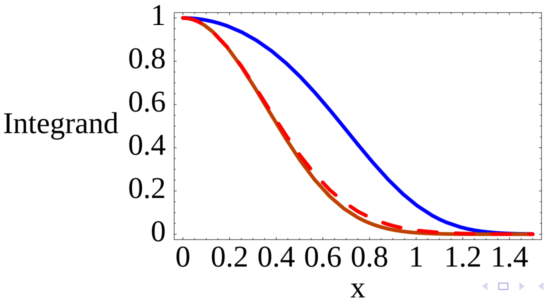
Derivation of the Flow Equation

generating function with 'cutoff' R

$$Z[j;R] = \int dx \exp\left[-\frac{1}{2}(1+R)x^2 - \frac{\lambda}{4!}x^4 + j\,x\right]$$

▷ properties of R

- $R \to \infty$: $Z[j;R\to\infty] \to \int dx \exp\left[-\frac{1}{2}(1+R)x^2 + j\,x\right]$



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- $R \to 0$: $Z[j;R\to 0] \to Z[j]$
- Normalization: $j = 0$:

$$Z[0;R] = \frac{2}{\sqrt{1+R}} e^{\frac{3(R+1)^2}{4\lambda}} \sqrt{\frac{3(R+1)^2}{4\lambda}} K_{\frac{1}{4}}\left(\frac{3(R+1)^2}{4\lambda}\right)$$

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aim: find flow equation

. . . for $\ln Z[j; R]$

$$\partial_R \ln Z[j; R]$$

. . . for effective action $\Gamma[x; R]$

$$\partial_R \Gamma[x; R]$$

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Expectation values

$$\partial_j \ln Z[j] = \langle x \rangle_j \qquad ; \qquad \partial_j^2 \ln Z[j] = -\langle x \rangle_j^2 + \langle x^2 \rangle_j$$

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$\Longrightarrow$

$$\partial_R \ln Z[j;R] = -\frac{1}{2}\left[\partial_j^2 \ln Z[j] + \left(\partial_j \ln Z[j]\right)^2\right]$$

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Legendre transform

$$\frac{\partial \Gamma[x;R]}{\partial j} = 0 \qquad \longrightarrow \qquad \partial_j \ln Z[j] = \langle x \rangle_j = x$$

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Legendre transform

$$x = \partial_j \ln Z[j]$$

generating function with 'cutoff' R

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$$1 = \partial_x \partial_j \ln Z[j]$$

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$$\Gamma[x;R] = jx - \ln Z[j;R] - \frac{1}{2}Rx^2 \;\rightarrow\; \partial_x \Gamma[x;R] = j - Rx \;\rightarrow\; \partial_x^2 \Gamma[x;R] = \partial_x j - R$$

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$$\partial_R \Gamma[x;R] = \frac{1}{2}\partial_j^2 \ln Z[j]$$

$$1 = \partial_j^2 \ln Z[j] \,\left(\partial_x^2 \Gamma[x;R] + R\right)$$

Derivation of the Flow Equation

generating function with 'cutoff' R

$$Z[j;R] = \int dx \exp\left[-\frac{1}{2}(1+R)x^2 - \frac{\lambda}{4!}x^4 + j\,x\right]$$

aim: find flow equation

. . . for $\ln Z[j;R]$

$$\partial_R \ln Z[j;R] = -\langle \frac{1}{2}x^2 \rangle_j$$

. . . for effective action $\Gamma[x;R]$

$$\partial_R \Gamma[x;R] = \frac{1}{2}\partial_j^2 \ln Z[j]$$

flow equation for effective action $\Gamma[x;R]$

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Solution of the Flow Equation

flow equation for effective action $\Gamma[x;R]$

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- initial condition:

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  $R \to \infty$

- boundary condition:

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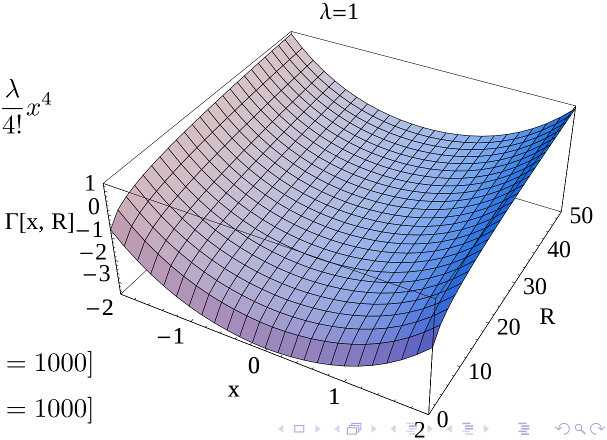

Comparison: RG and 'Perturbation Theory'

$$\ln Z[j] = \ln \int dx \, \exp\left[-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + j\,x \right]$$

calculate $\ln Z[j]$ with

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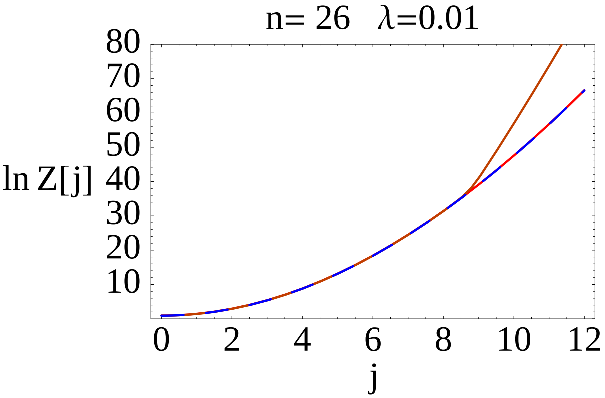

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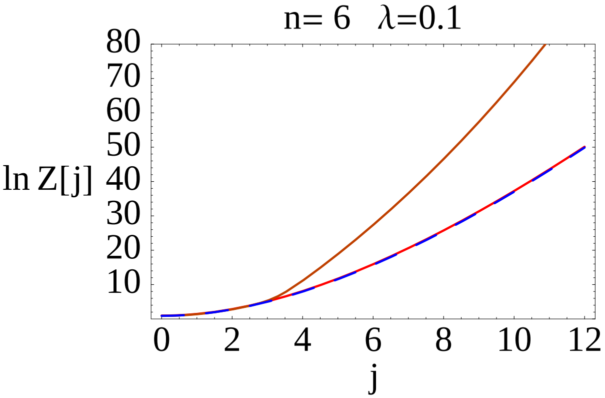

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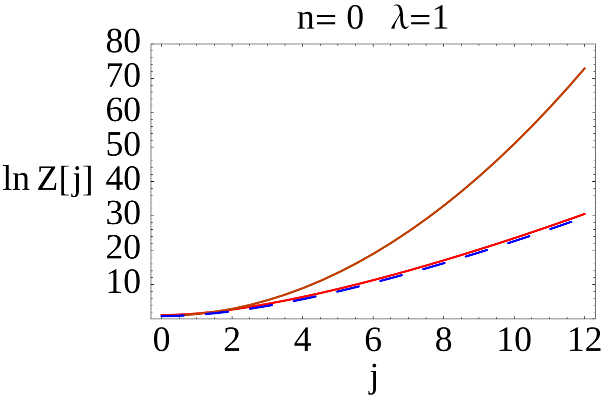

Outline

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   - boiling water
   - why should we use non-perturbative RG?
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some properties of average effective action Γ_k:

- Γ_k contains only fluctuations with $q^2 \geq k^2$

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Γ_k interpolates between S_{class} & Γ

$$\Gamma_\Lambda = S_{\text{class}} \qquad ; \qquad \lim_{k\to 0} \Gamma_k = \Gamma$$

$\Rightarrow$ ability to follow $k \to 0$ evolution $\equiv$ ability to solve the theory

Exact RG

QFT in $d = 1 + 3$: generating functional

$$Z[j] = \int \mathcal{D}\phi \, \exp\left[-S[\phi] + \int j\,\phi\right]$$

Exact RG

addition of an IR cutoff term

$$Z_k[j] = \int \mathcal{D}\phi \, \exp\left[-S[\phi] - \Delta S_k[\phi] + \int j\,\phi\right]$$

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with:

$\lim_{q^2/k^2 \to \infty} R_k(q) = 0$: remove cutoff for $k \to 0$ and UV not suppressed

no modes are integrated out

$\lim_{k \to \infty(\Lambda)} R_k(q) \to \infty$: $\to$ acts like a functional $\delta(\phi)$

(see exercises)

Exact RG

addition of an IR cutoff term

$$Z_k[j] = \int \mathcal{D}\phi \, \exp\left[-S[\phi] - \Delta S_k[\phi] + \int j\,\phi\right]$$

- modified Legendre transform (details in seminar)

$$\begin{aligned}\Gamma_k[\phi] &= -\ln Z_k[j] + j\phi - \Delta S_k[\phi] \\ &= -\ln \int \mathcal{D}\tilde{\chi} e^{-S[\tilde{\chi}+\phi] - \Delta S_k[\tilde{\chi}] + \frac{\delta \Gamma_k[\phi]}{\delta\phi}\tilde{\chi}}\end{aligned}$$

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- 1st term: $S[\phi]$ classical contribution
- 2nd term: $\tilde{\chi}$ fluctuations with background field ϕ

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$$\lim_{k\to\Lambda} \Delta S_k[\phi] \to \infty \quad : \quad \Gamma_\Lambda[\phi] = S[\phi]$$

Exact RG

addition of an IR cutoff term

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(details in seminar)

flow equation for average effective action [Wetterich]

$$k\partial_k \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr}\, k\partial_k R_k \left(\frac{1}{\Gamma_k^{(2)} + R_k} \right) \quad ; \quad \Gamma_k^{(2)}[\phi] = \frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi}$$

RG Approaches

$\Gamma_k[\phi]$ scale dependent effective action ; $t = \ln(k/\Lambda) \quad \rightarrow \quad k\partial_k \equiv \partial_t$

1. Exact RG

ERG (average effective action) [Wetterich]

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr}\, \partial_t R_k \left(\frac{1}{\Gamma_k^{(2)} + R_k} \right) \quad ; \quad \Gamma_k^{(2)} = \frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi}$$

2. Proper-time RG (more in 2nd lecture)

PTRG [Liao]

$$\partial_t \Gamma_k[\phi] = -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \left[\partial_t f_k(\Lambda^2 \tau)\right] \mathrm{Tr} \exp\left(-\tau \Gamma_k^{(2)}\right)$$

3. other approximations

Truncations

exact RG impossible to solve $\rightarrow$ systematic approximations needed

$\Rightarrow$ projection onto *sub-theory* space

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- derivative expansion

$$\Gamma_k[\phi] = \int d^d x \left\{ U_k(\phi) + \frac{1}{2} Z_k \partial_\mu \phi \partial_\mu \phi + \ldots + \mathcal{O}(\partial^4) \right\}$$

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- expansion in powers of the fields

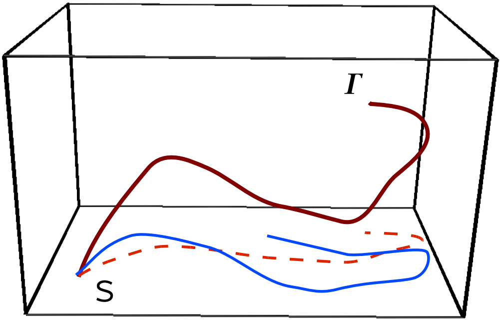
$$\Gamma_k[\phi] = \sum_n \frac{1}{n!} \int \left(\prod_i^n d^d x_i \phi(x_i) \right) \Gamma_k^{(n)}(x_1, \ldots, x_n)$$

- ... (some more expansion schemes)

Truncations

consider a 3-dim. subset of operators

here: RG flow in a 3-dim. space of all action functionals:



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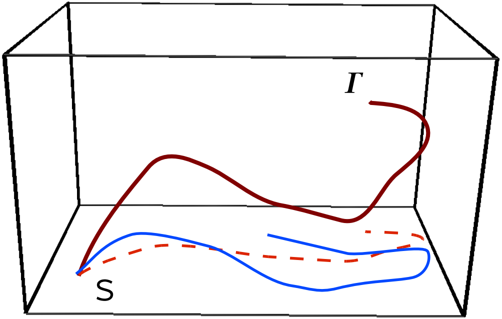

▷ projection of exact flow on subspace of truncation (dashed)

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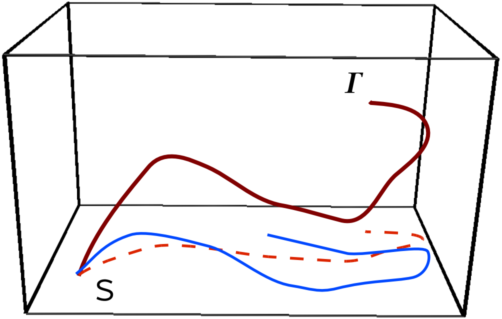

▷ projection of exact flow on subspace of truncation (dashed)

→ does not coincide with approximate flow (blue)

(omission of operator in 3rd direction)

▷ enlarge subspace (of relevant operators)

→ improve approximation

▷ → choose "optimized" IR regulator

[Litim, Pawlowski]

Truncations

example: scalar theory with Z_2-symmetry

$$S_{\text{eff}} = \int d^d x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi^2) \right\}$$

$\rightarrow$ lowest order of derivative expansion (LPA)

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- β-functions for coefficients a_i:

$$\begin{aligned} \partial_t a_2 &= 2a_2 - \frac{\zeta}{2} \frac{a_4}{1+a_2} \\ \partial_t a_4 &= (4-d)a_4 - \zeta \left[\frac{2}{5} \frac{a_6}{1+a_2} - \frac{a_4^2}{(1+a_2)^2} \right] \\ \partial_t a_6 &= \dots \\ &\vdots \end{aligned}$$

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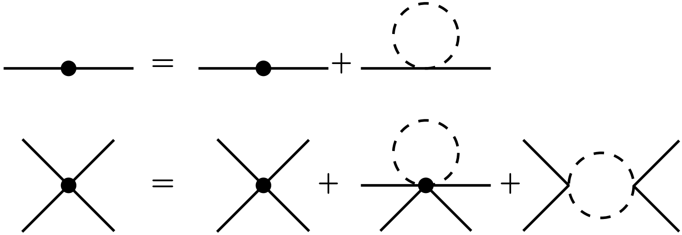
- "Feynman diagram" representation of the β-functions:



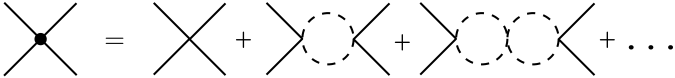
Integrating the β-functions

- consider quartic coupling a_4 in $d = 4$:

ignore a_6 contribution and use $a_2 \ll 1$ at cutoff scale:

$$\partial_t a_4 = \zeta a_4^2$$

$$\rightarrow a_4(t) = \frac{a_4(0)}{1 - \zeta a_4(0) t} = \frac{a_4(0)}{1 - \zeta a_4(0) \ln(k/\Lambda)} = \frac{a_4(0)}{1 - a_4(0)\,\text{[loop]}}$$



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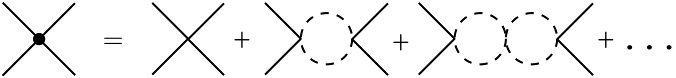

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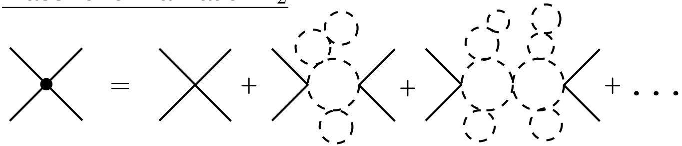

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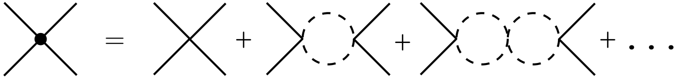

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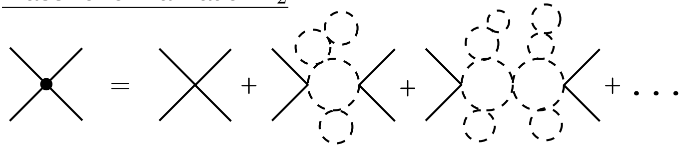

- lowest level approximation in NPRG contains even improved ladder Schwinger-Dyson results

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Summary Part I

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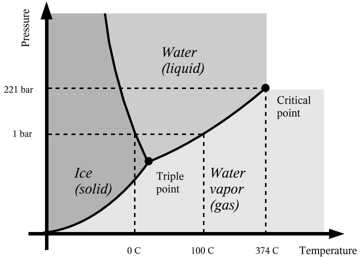
examples: critical opalescence,

critical point in phase diagram



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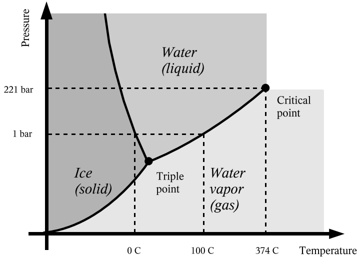

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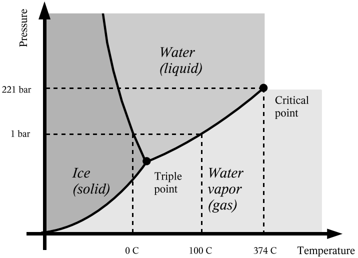

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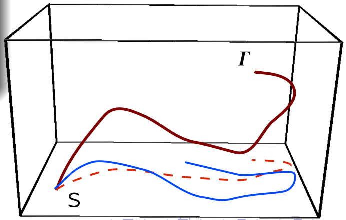

- ▷ exact RG → impossible to solve

  → truncations needed