Interacting quantum particles in localizing potentials

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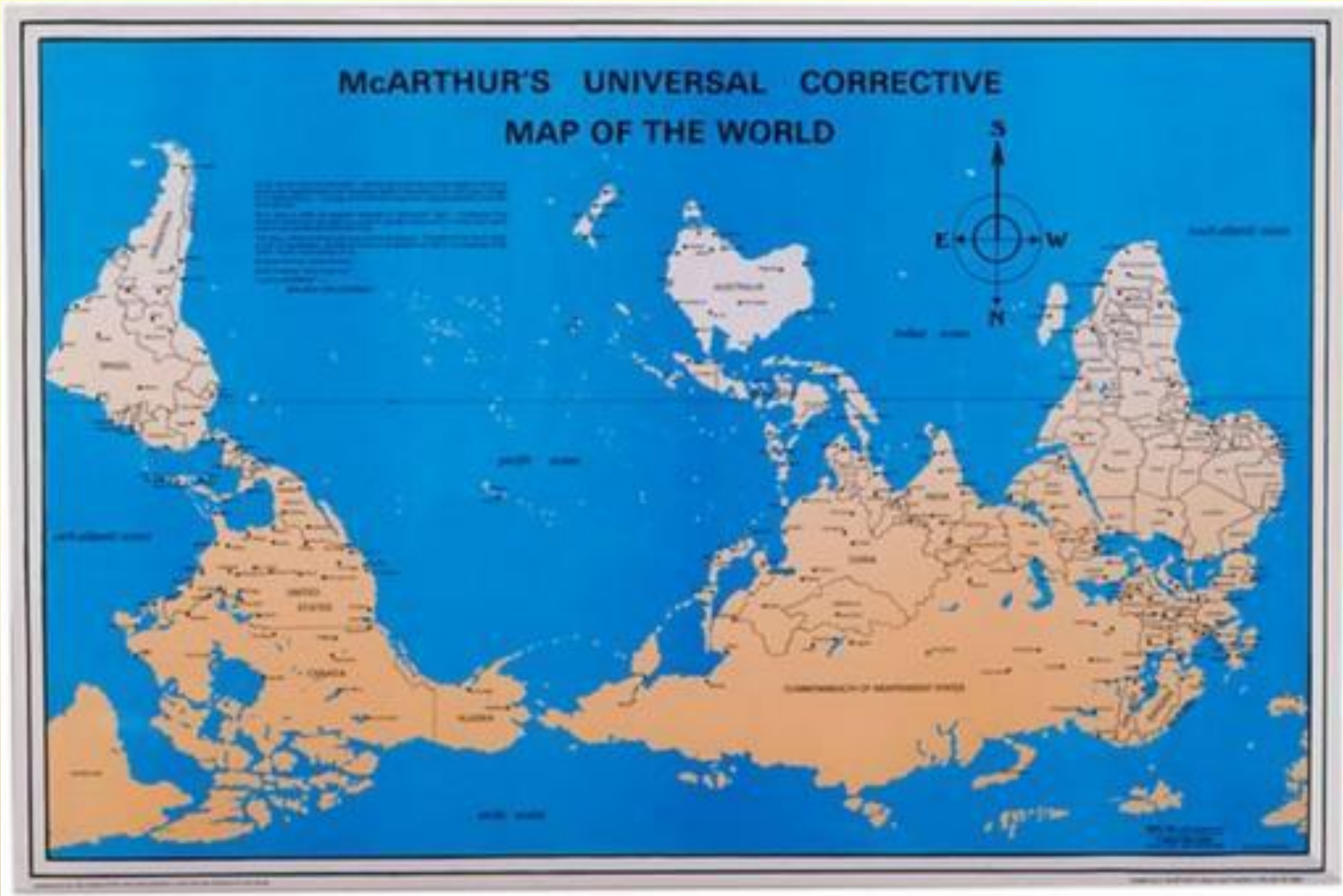
- introduction
- linear and nonlinear waves in localizing potentials
- IQP in disordered chains
- IQP in Wannier Stark ladders
- IQP in quasiperiodic chains


AUCKLAND NEW ZEALAND


McARTHUR'S UNIVERSAL CORRECTIVE
MAP OF THE WORLD
S
E
W
N

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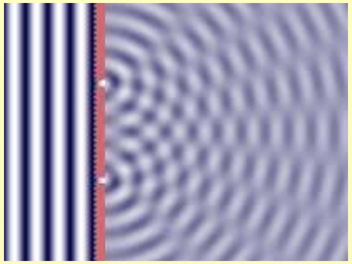
Introduction

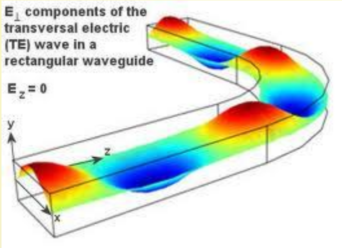
Waves

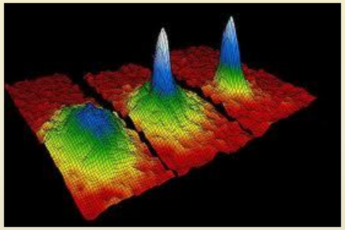
amplitude and phase in space and time

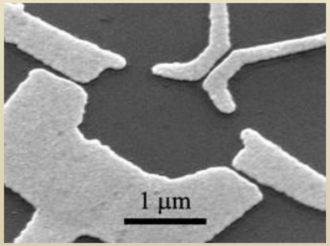


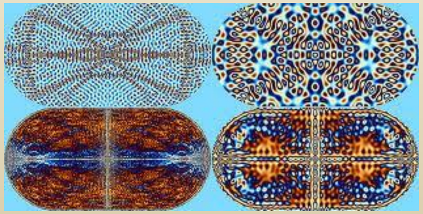
Linear waves: **superposition, interference, phase coherence**

e.g.
optical fibres
microwave cavities
atomic Bose-Einstein condensates
quantum billiards
quantum dots
superconducting networks
molecules, solids





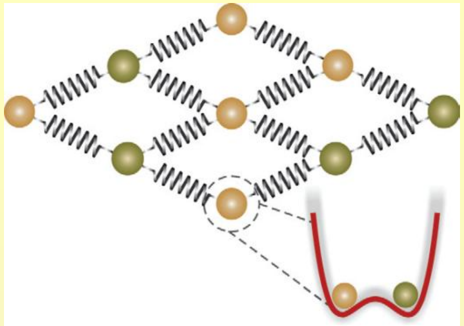


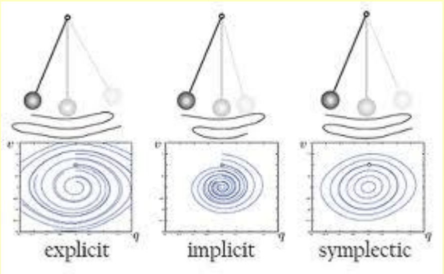


Nonlinear waves?

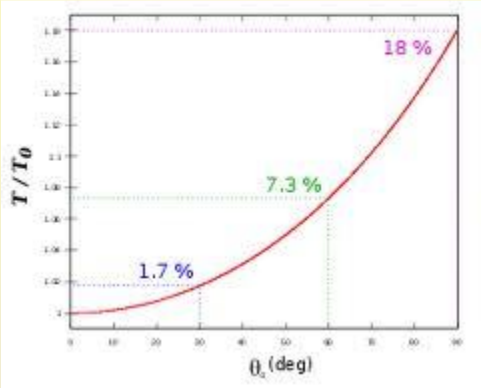
high intensities - qualitatively new properties:
nonlinear response
waves interact with each other
resonances
dynamical chaos
instability
rogue waves ... tsunami ...

Lattice waves





discretize space – introduce lattice
one oscillator per lattice point
oscillator state is defined by amplitude and phase
add interaction between oscillators



anharmonic potential = nonlinear wave equation
intensity increase changes frequency
in quantum world energy levels NOT equidistant

Typical excitations in condensed matter, optics, etc

Linear waves
in localizing potentials

Waves in localizing media

$$i\dot{\psi}_l = \epsilon_l \psi_l \quad - \psi_{l+1} - \psi_{l-1}$$

$$\psi_l = A_l \exp(-i\lambda t) \qquad \lambda A_l = \epsilon_l A_l - A_{l+1} - A_{l-1}$$

- **uncorrelated random potential: Anderson localization**
- **quasiperiodic potential: Aubry-Andre (Harper) localization**
- **dc bias ε(l)=E·l : Wannier-Stark localization (Bloch oscillations)**
- **quantum kicked rotor: localization in momentum space, loosely similar to quasiperiodic potential case**

In all cases all (or almost all) eigenstates are spatially localized, with finite upper bounds on the localization length / volume.

Wannier-Stark ladder

Wannier (1960)

$$i\dot{\psi}_l = lE\psi_l - \psi_{l+1} - \psi_{l-1}$$

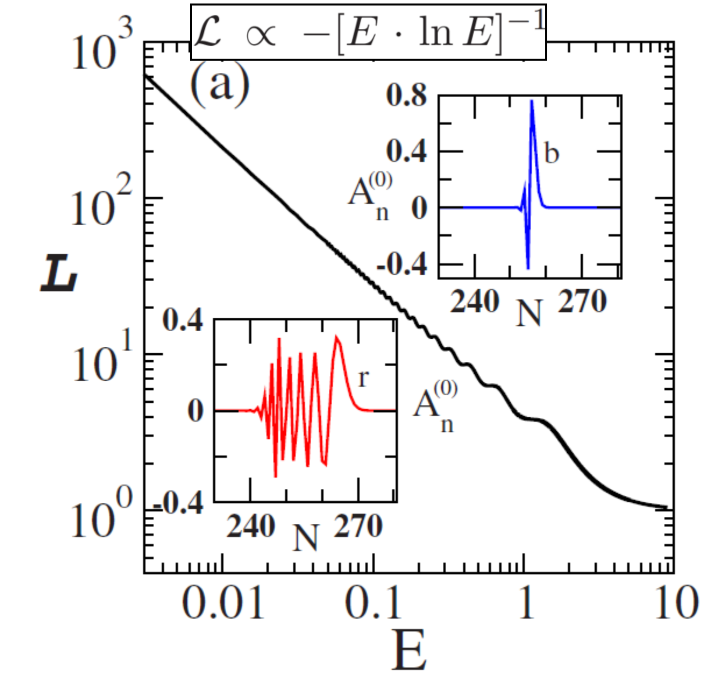
Superexponential localization

$$\lambda_\nu = E\nu \qquad A_{l+\mu}^{\nu+\mu} = A_l^\nu \qquad A_l^{(0)} = J_l(2/E)$$

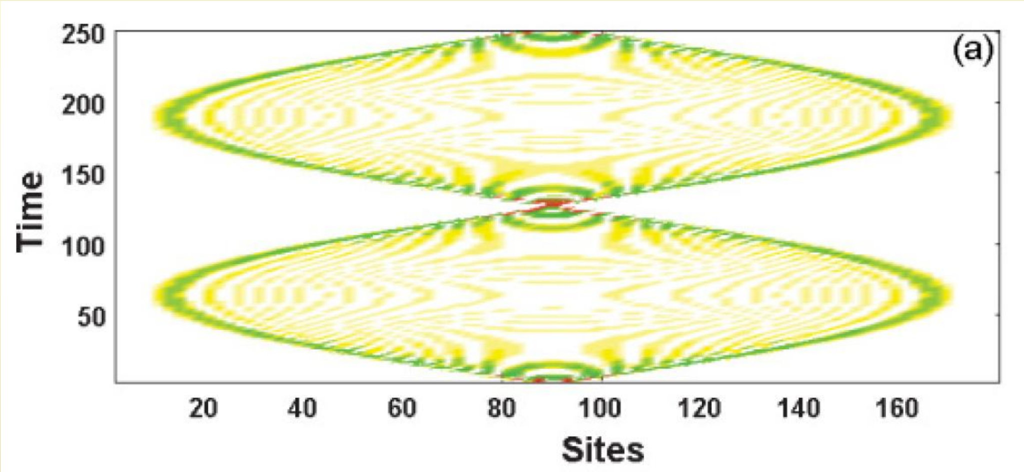
$$|A_{l\to\infty}^{(0)}| \to \frac{\left(\frac{1}{E}\right)^l}{l!}$$

Eigenfunctions

Localization volume



Bloch oscillations for E=0.05



Krimer,Khomeriki,SF (2009)

Aubry-Andre model

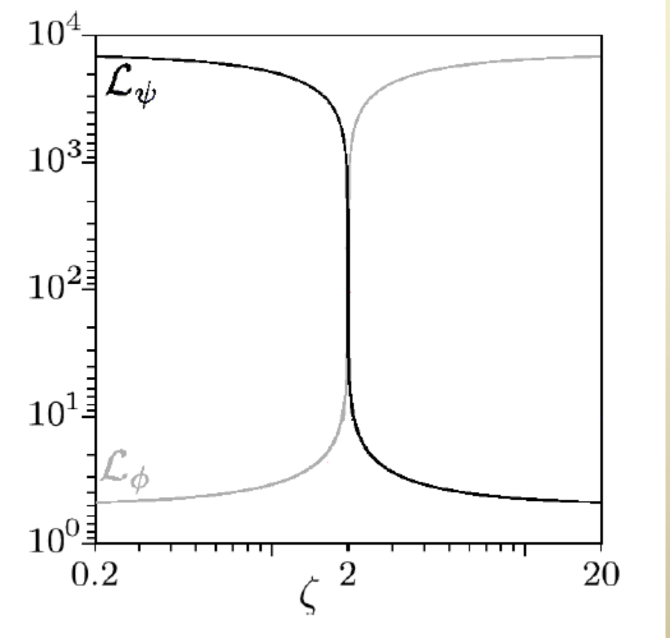
Aubry,Andre (1980)

$$i\frac{\partial \psi_l}{\partial t} = \zeta \cos(2\pi\alpha l)\cdot\psi_l - \psi_{l+1} - \psi_{l-1}$$

$$\alpha = (\sqrt{5}-1)/2$$

Self duality: $\psi_l = \sum_k e^{2\pi i\alpha k l}\phi_k$

$$i\frac{\partial \phi_k}{\partial t} = 2\cos(2\pi\alpha k)\cdot\phi_k - \frac{\zeta}{2}\phi_{k+1} - \frac{\zeta}{2}\phi_{k-1}$$



Metal insulator transition: $\zeta = 2$

Exponential localization.
Localization length = 1 / ln[ζ/2]

adapted from Aulbach et al (2004)

Anderson localization

Anderson (1958)

$$i\frac{\partial \psi_l}{\partial t} = \epsilon_l \psi_l - \psi_{l+1} - \psi_{l-1}$$

$\{\epsilon_l\}$ **in** $[-W/2, W/2]$

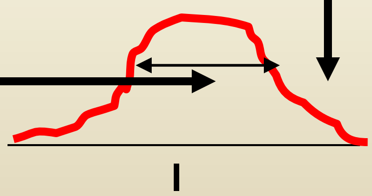
Eigenvalues: $\lambda_\nu \in \left[-2 - \frac{W}{2}, 2 + \frac{W}{2}\right]$

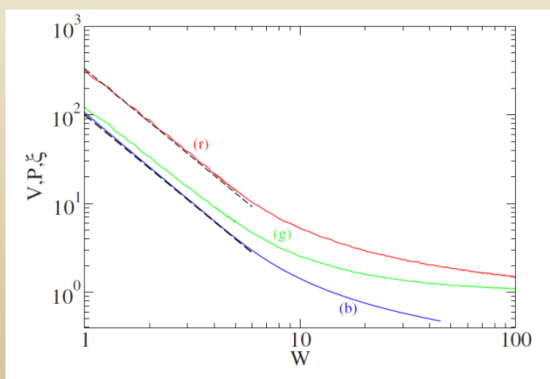
Width of EV spectrum: $\Delta = 4 + W$

Eigenvectors: $A_{\nu,l} \sim \mathrm{e}^{-l/\xi(\lambda_\nu)}$

Localization length: $\xi(\lambda_\nu) \leq \xi(0) \approx 100/W^2$

Localization volume of NM: L





Krimer,SF (2010)

Anderson localization

Anderson (1958)

$$\lambda A_l = \epsilon_l A_l - A_{l-1} - A_{l+1}$$

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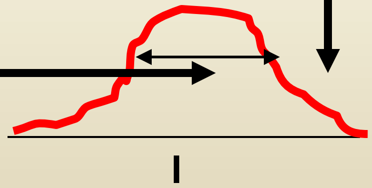
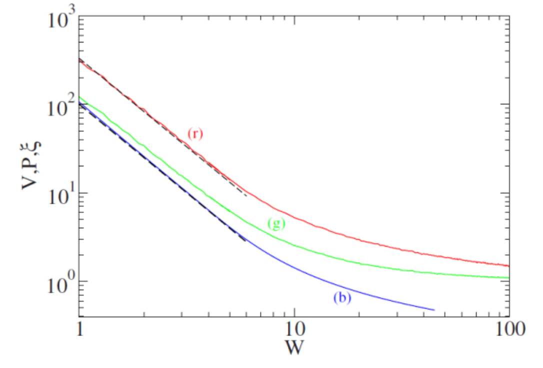
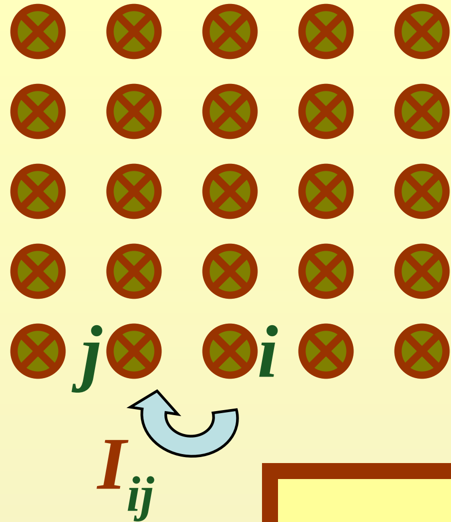




Krimer,SF (2010)

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i - *random*
- *Hopping matrix elements* t_{ij}

$$-W/2 < \varepsilon_i < W/2$$

uniformly distributed

$$t_{ij} = \begin{cases} t & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$t < t_c$

Insulator

All eigenstates are localized

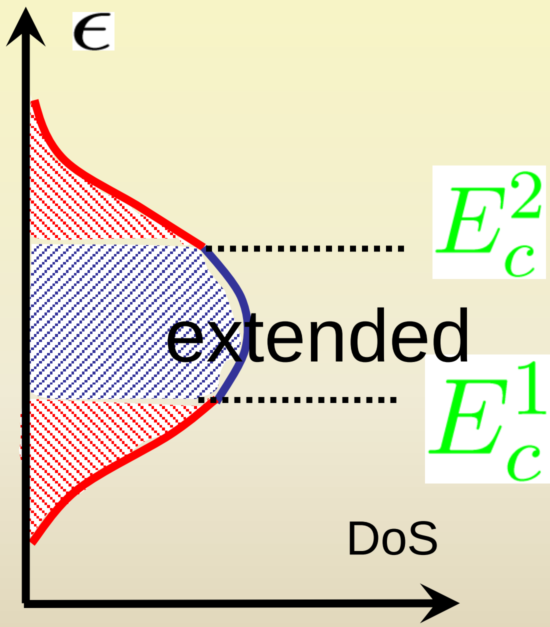
$t > t_c$

Metal

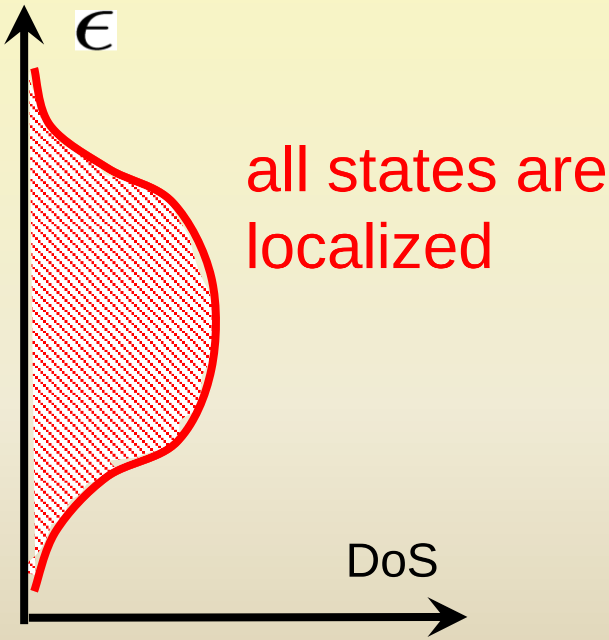
There appear states extended all over the whole system

Anderson Transition

$t > t_c$



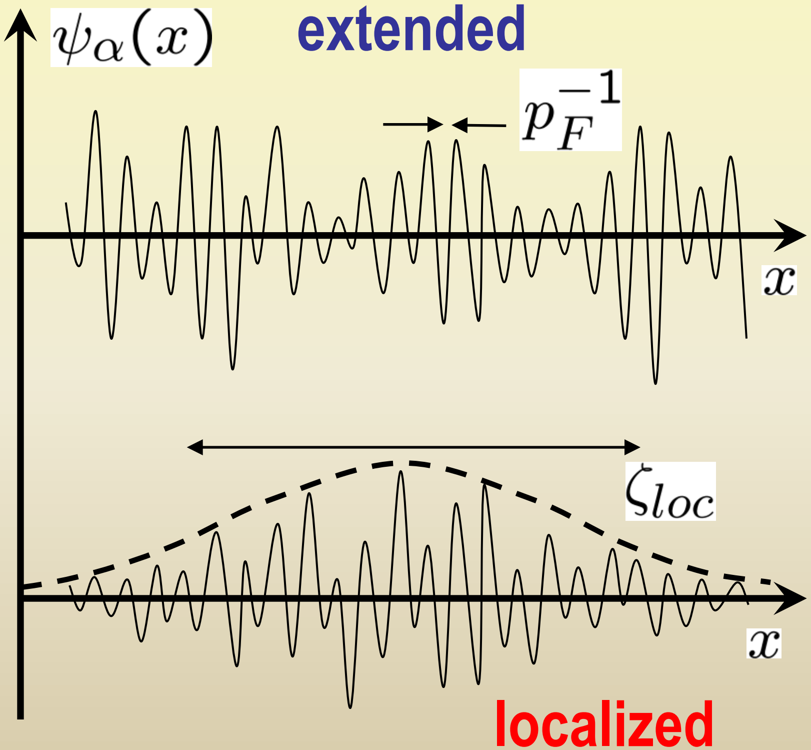
$t < t_c$



E_c - mobility edges (one particle)

Localization of single-particle wave-functions. Continuous limit:

$$\left[-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right] \psi_\alpha(\boldsymbol{r}) = \xi_\alpha \psi_\alpha(\boldsymbol{r})$$

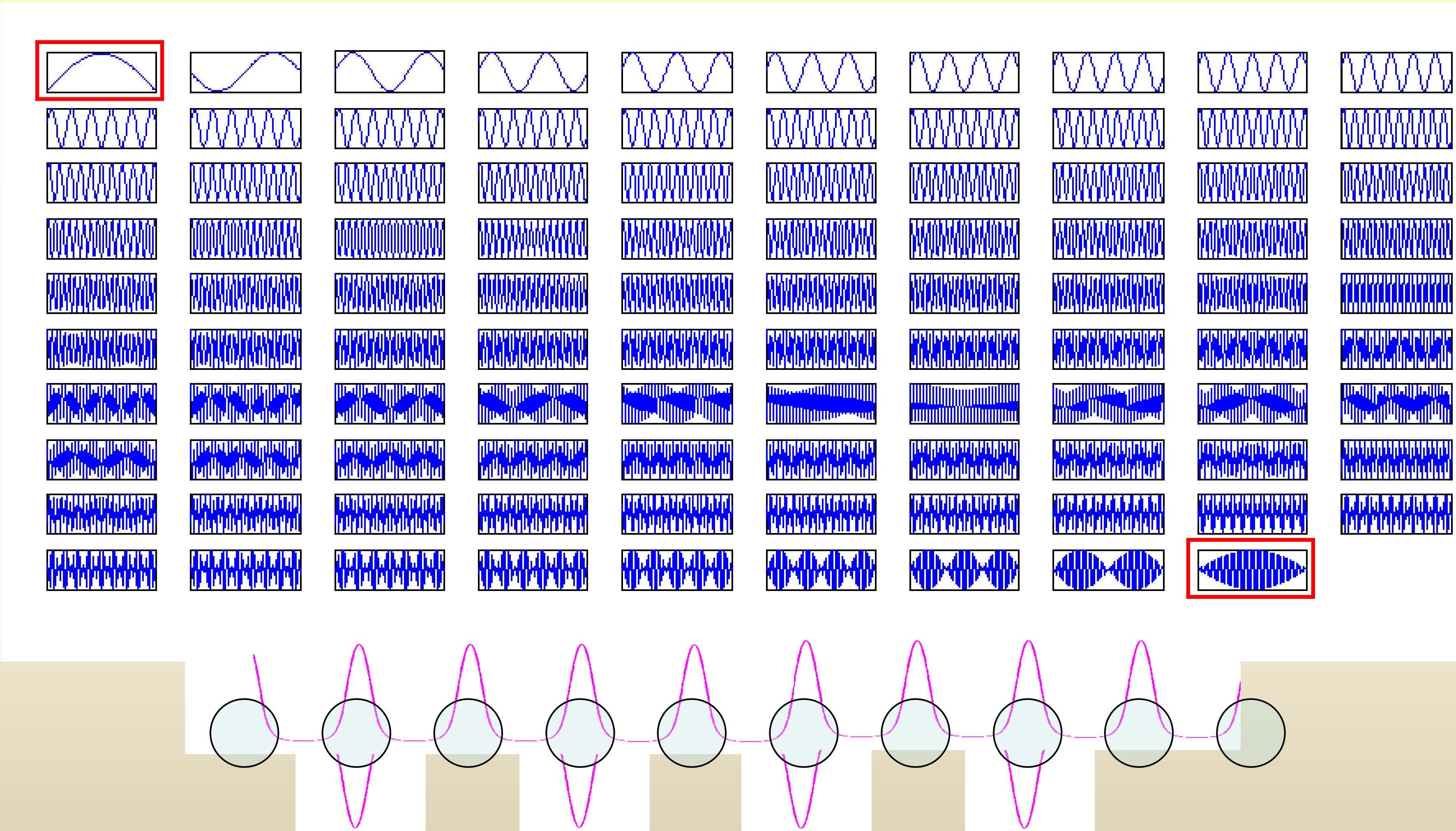


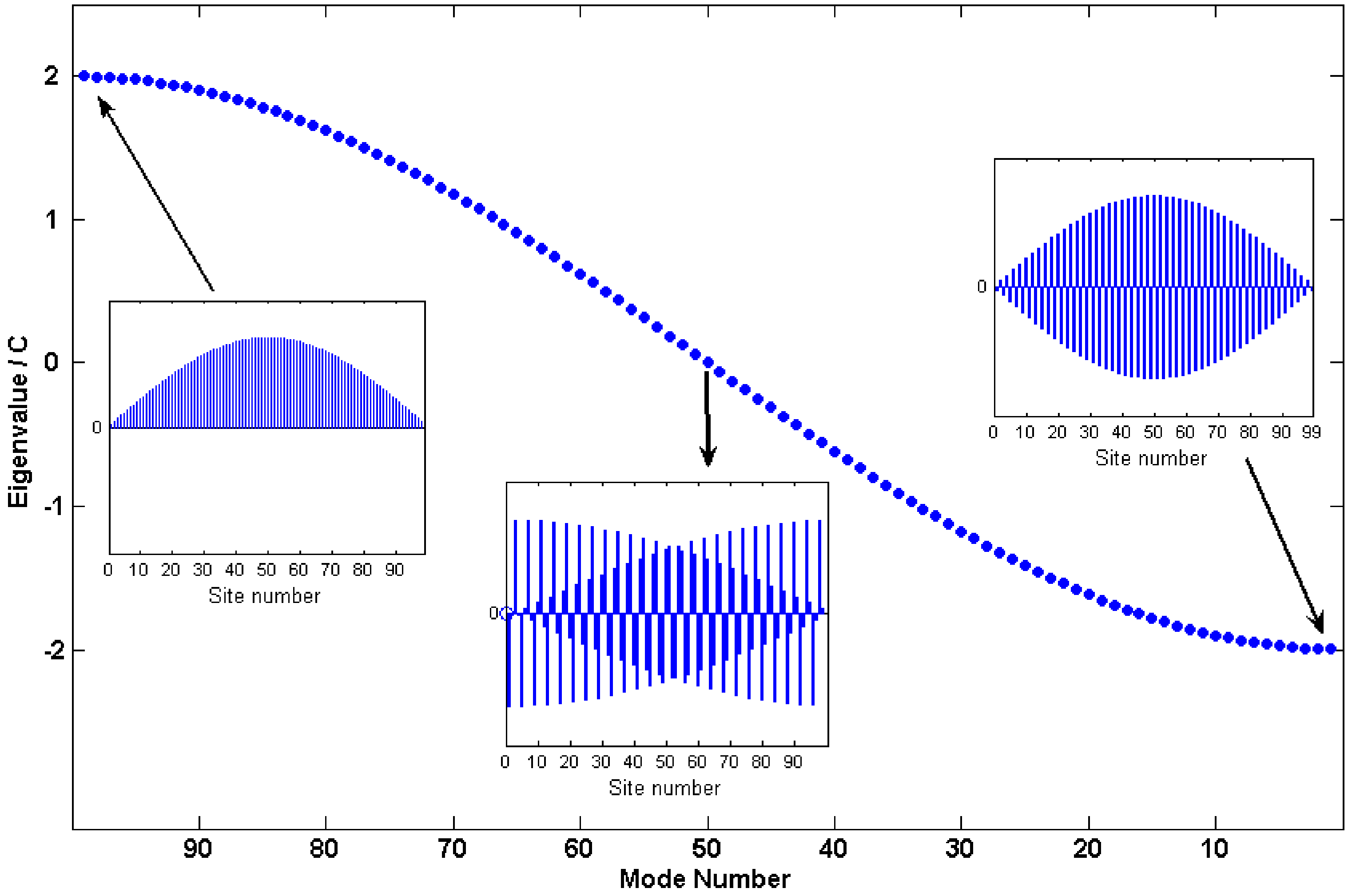
d=1: **All states are localized**

d=2: **All states are localized**

d >2: **Anderson transition**

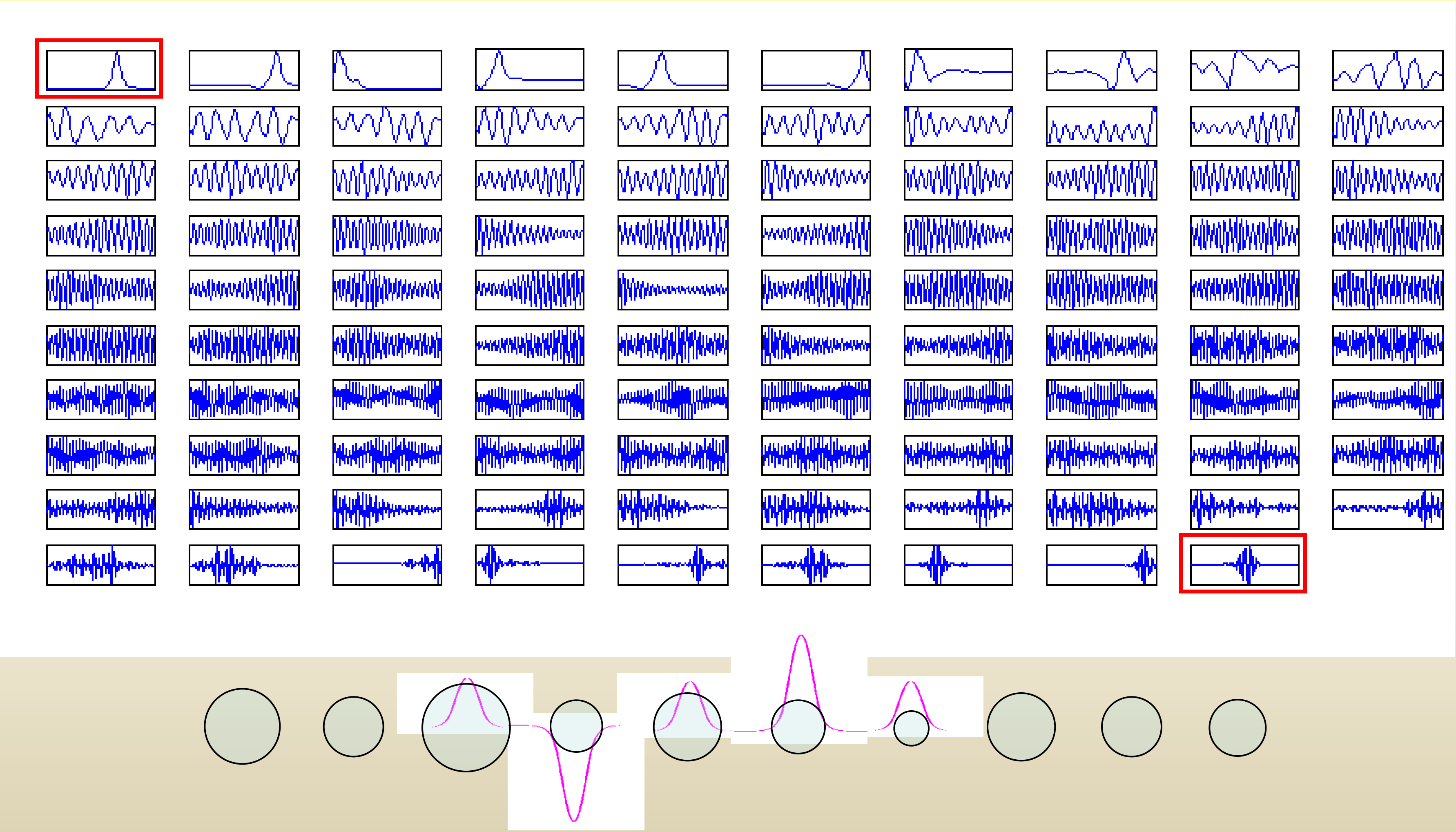
Eigenmodes of a periodic lattice, N=99





Eigenvalue / C
2
1
0
-1
-2
90
80
70
60
50
40
30
20
10
Mode Number
0
0 10 20 30 40 50 60 70 80 90
Site number
0
0 10 20 30 40 50 60 70 80 90
Site number
0
0 10 20 30 40 50 60 70 80 90 99
Site number

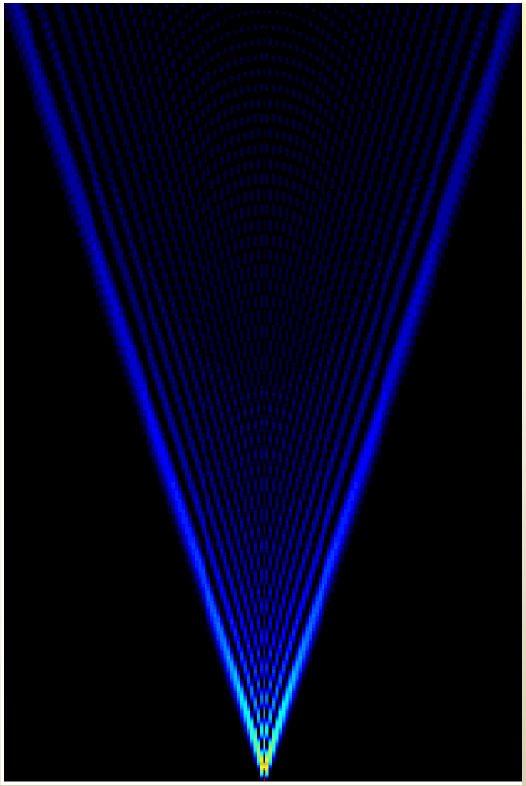
Eigenmodes of a disordered lattice, N=99

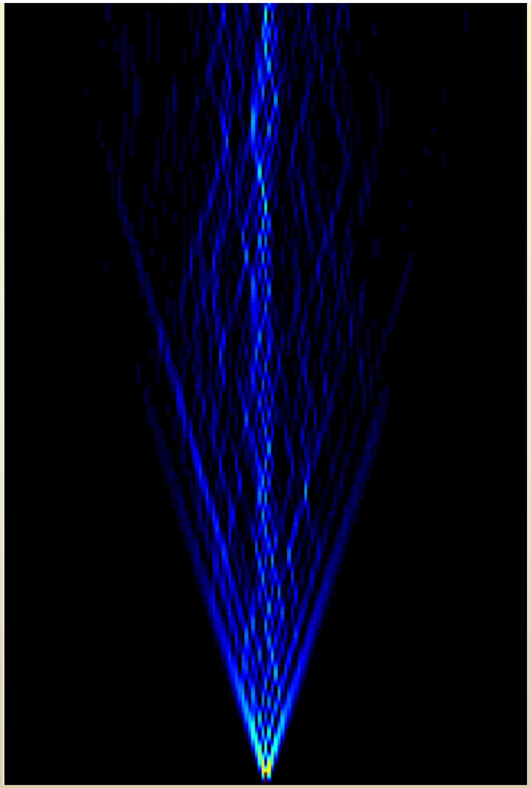


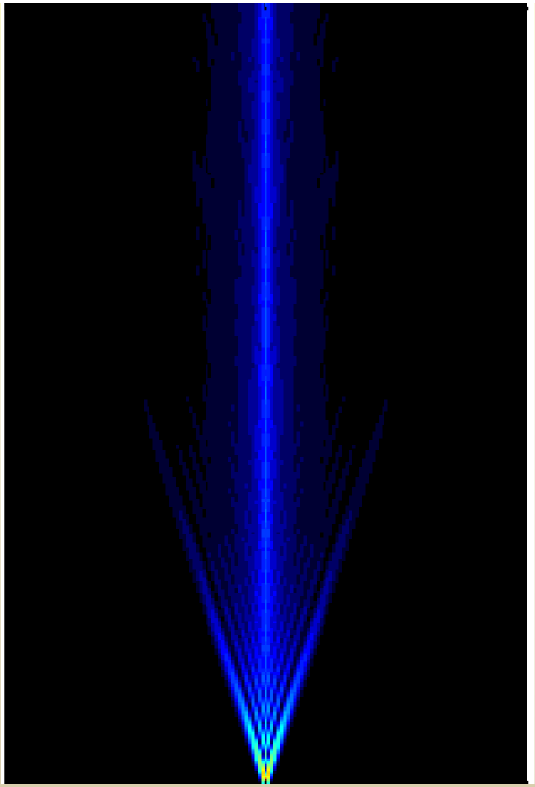
Wave packet evolution

- Exciting a *single* site as an initial condition

Ordered lattice **Disordered lattice** **Disordered lattice - averaged**







Experimental Evidence for Wave Localization

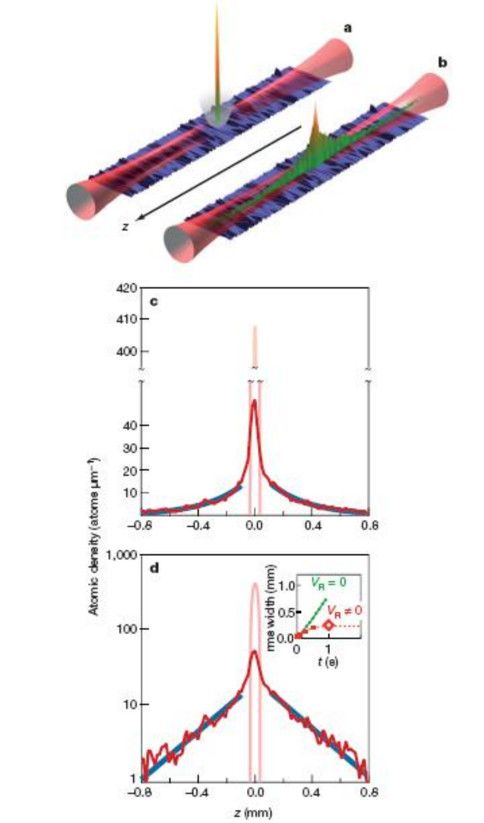


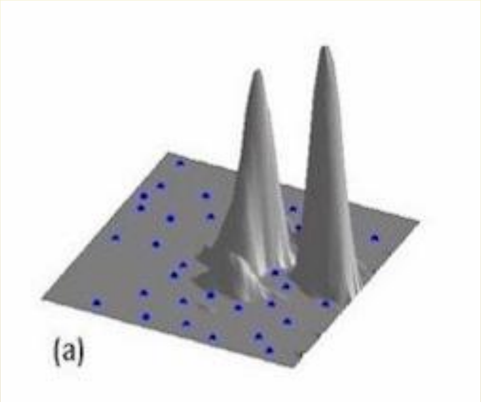
Figure 1 | Observation of exponential localization. a, A small BEC

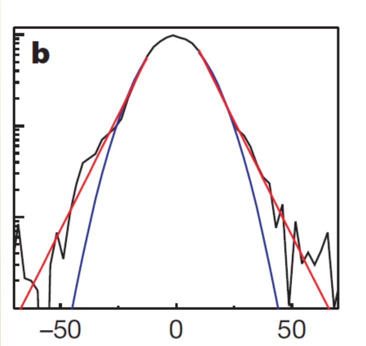
Ultrasound:* *Weaver 1990

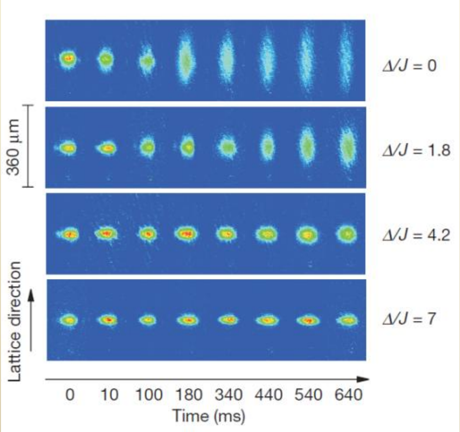
Microwaves:* *Dalichaoush et al 1991, Chabanov et al 2000

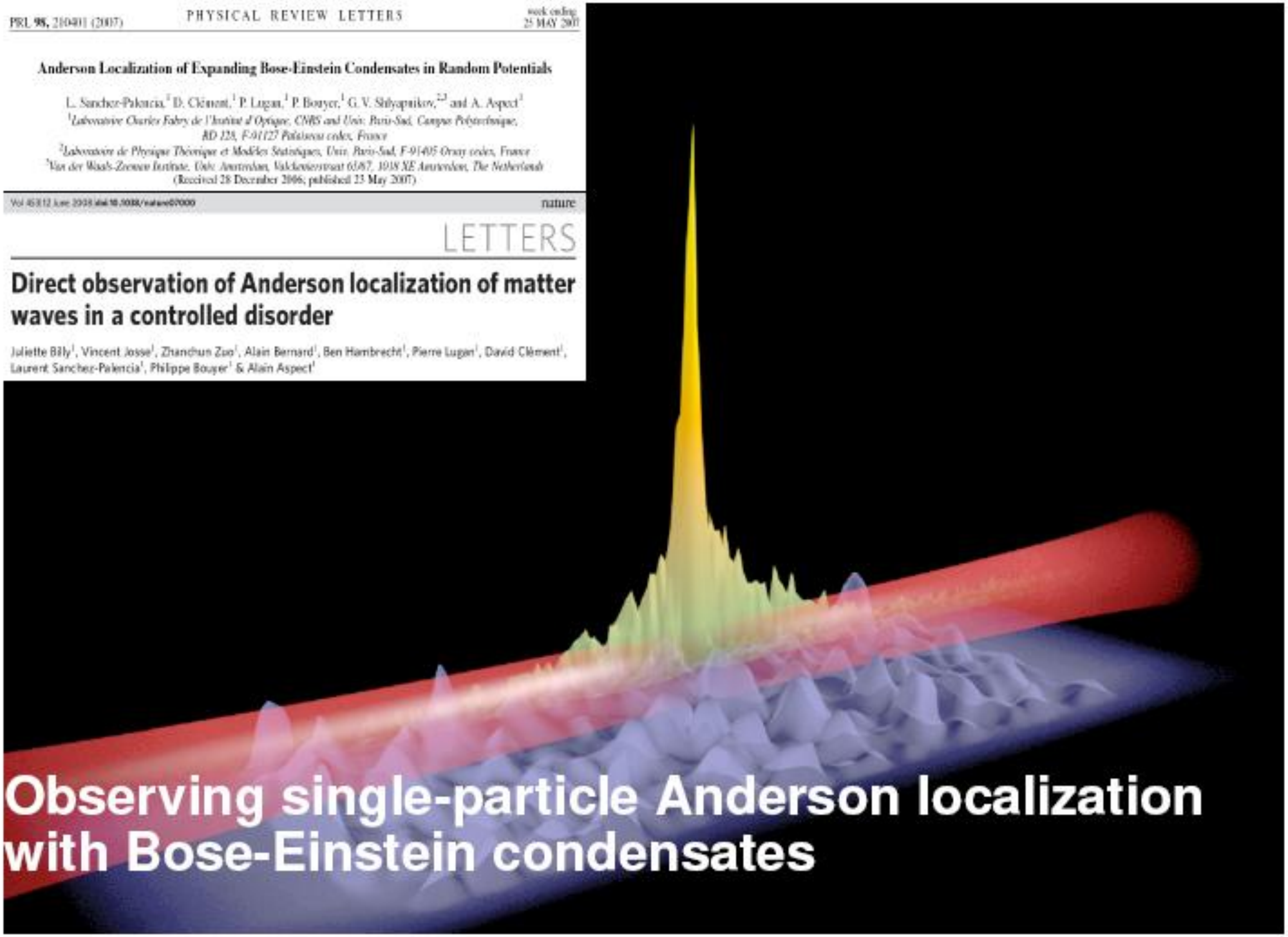
Light:* *Wiersma et al 1997, Scheffold et al 1999, Pertsch et al (1999), Morandotti et al (1999), Stoerzer et al 2006, Schwartz et al 2007, Lahini et al 2008

BEC:* *Moore et al (1994) Anderson et al (1988), Morsch et al (2001), Billy et al 2008, Roati et al 2008









PRL 98, 210401 (2007) PHYSICAL REVIEW LETTERS week ending 25 MAY 2007

Anderson Localization of Expanding Bose-Einstein Condensates in Random Potentials

L. Sanchez-Palencia,[1] D. Clément,[1] P. Lugan,[1] P. Bouyer,[1] G. V. Shlyapnikov,[2,3] and A. Aspect[1]
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LETTERS

Direct observation of Anderson localization of matter waves in a controlled disorder

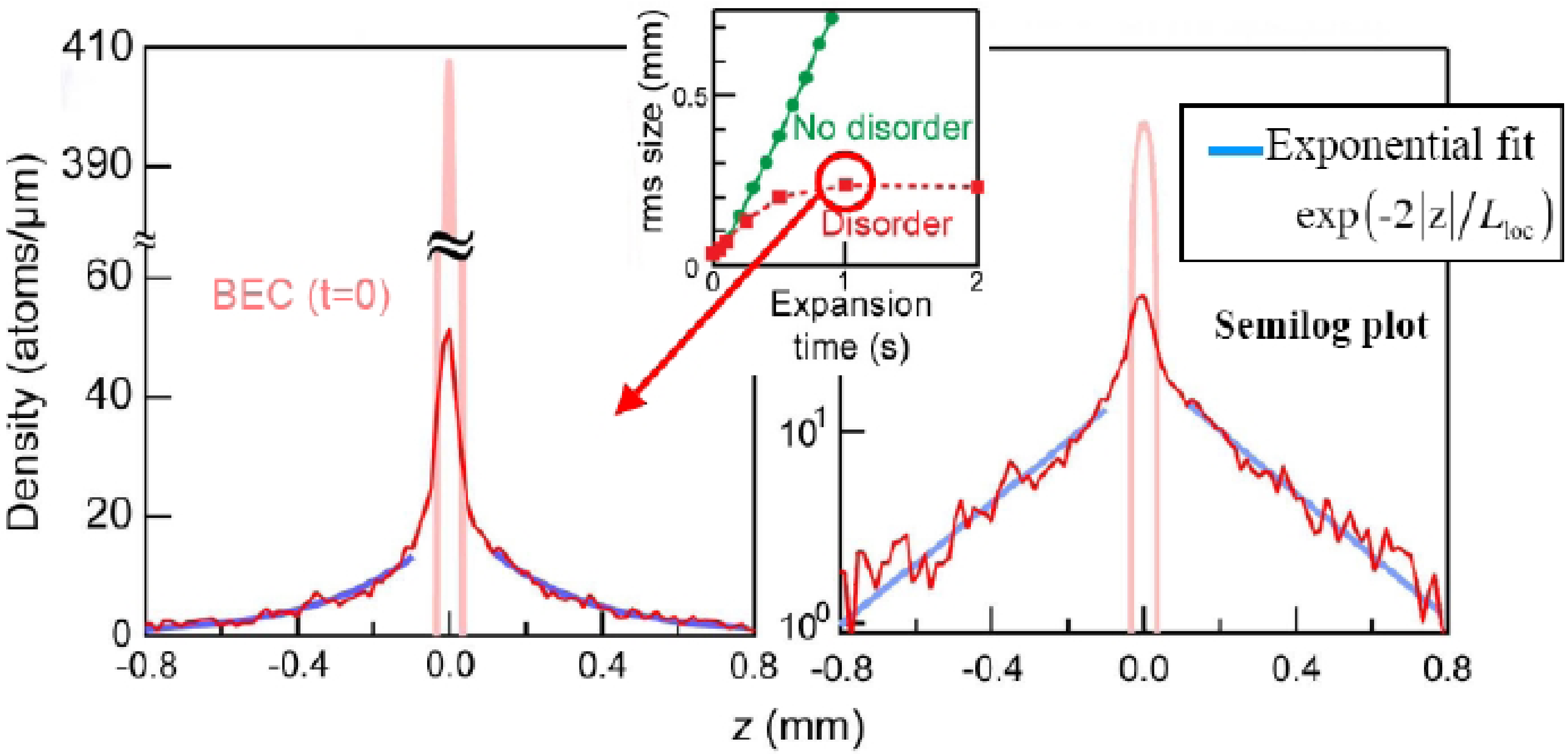
Juliette Billy[1], Vincent Josse[1], Zhanchun Zuo[1], Alain Bernard[1], Ben Hambrecht[1], Pierre Lugan[1], David Clément[1], Laurent Sanchez-Palencia[1], Philippe Bouyer[1] & Alain Aspect[1]

Observation of the signature of AL

BEC parameters : N=1.7 10^4 atoms, (μ_{in}=220Hz)

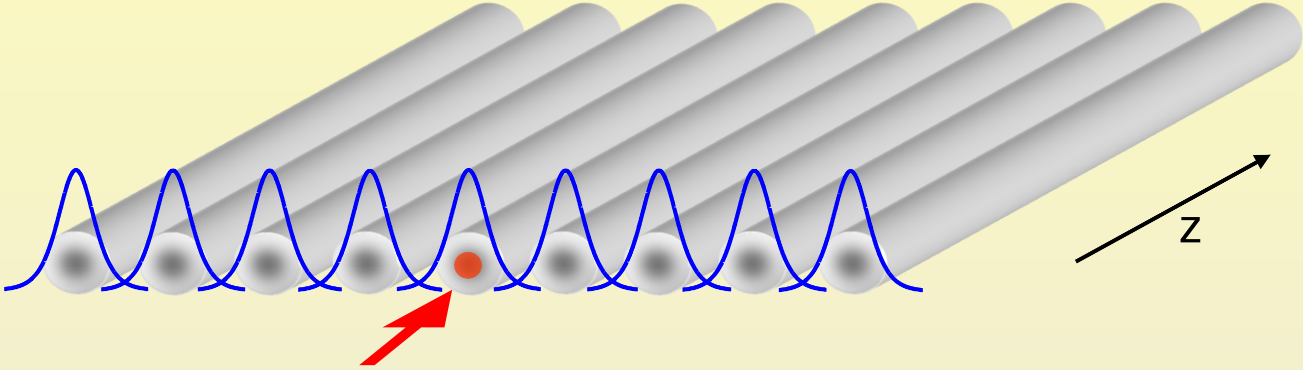
Weak disorder : V_R/μ_{in} =0.12 << 1

$k_{max}/\ k_c$ = 0.63 +/-0.09



⇒ *Exponential decay of the density in the wings : L_{loc}=530 +/- 80 μm*

An optical one-dimensional waveguide lattice (Silberberg et al ’08)

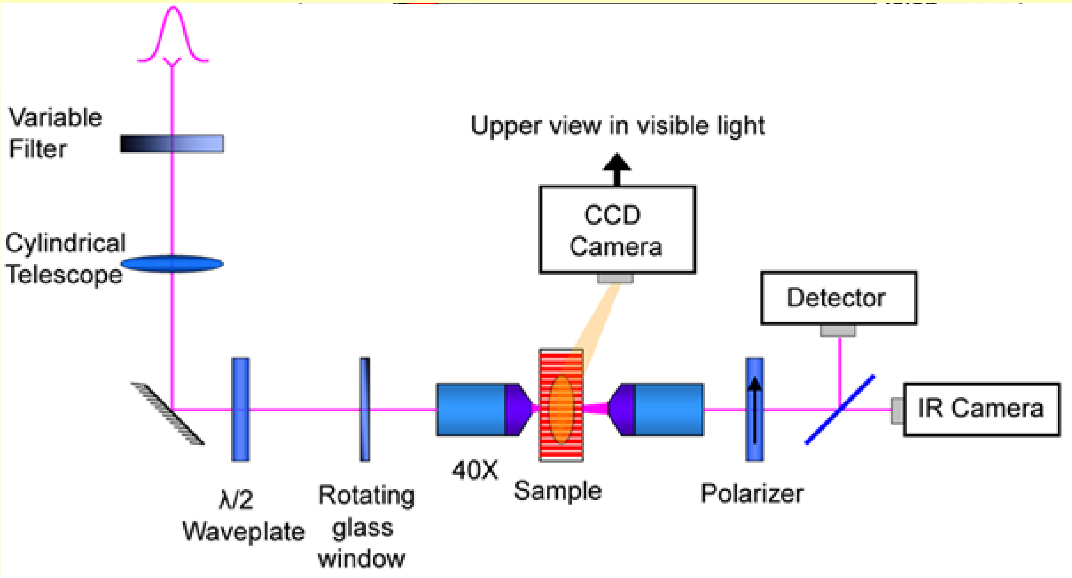


- Evanescent coupling between waveguides
- Light coherently tunnels between neighboring waveguides
- Dynamics is described by the Tight-Binding model

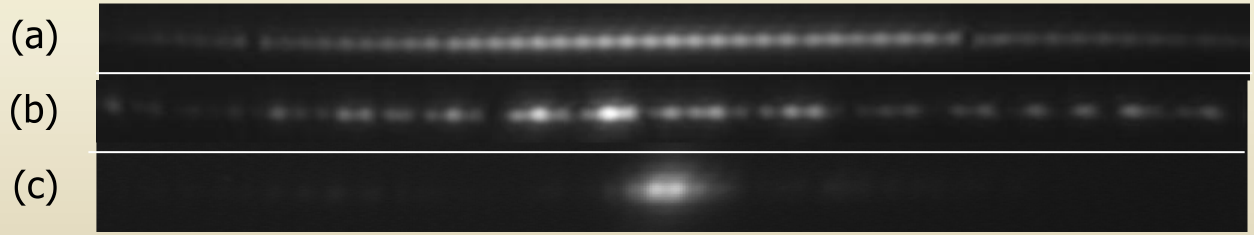
$$i\frac{\partial U_n}{\partial z} = \beta_n U_n + C_{n,n\pm 1}\left[U_{n+1} + U_{n-1}\right]$$

β_n – waveguide’s refraction index /width

$C_{n,n\pm 1}$ – separation between waveguides



- Injecting a narrow beam (~3 sites) at different locations across the lattice



(a) Periodic array – *expansion*
(b) Disordered array - *expansion*
(c) Disordered array - *localization*

Nonlinear waves
in localizing potentials

Defining the problem

- a disordered medium
- linear equations of motion: all eigenstates are Anderson localized
- add short range nonlinearity (interactions)
- follow the spreading of an initially localized wave packet

$$i\dot{\psi}_l = \epsilon_l \psi_l \qquad\qquad - \psi_{l+1} - \psi_{l-1}$$

Defining the problem

- a disordered medium
- linear equations of motion: all eigenstates are Anderson localized
- add short range nonlinearity (interactions)
- follow the spreading of an initially localized wave packet

$$i\dot{\psi}_l = \epsilon_l \psi_l + \beta |\psi_l|^2 \psi_l - \psi_{l+1} - \psi_{l-1}$$

Defining the problem

- a disordered medium
- linear equations of motion: all eigenstates are Anderson localized
- add short range nonlinearity (interactions)
- follow the spreading of an initially localized wave packet

Will it delocalize?

Yes because of nonintegrability and ergodicity

No because of energy conservation – spreading leads to small energy density, nonlinearity can be neglected, dynamics becomes integrable, and Anderson localization is restored

Equations in normal mode space:

$$i\dot{\phi}_\nu = \lambda_\nu \phi_\nu + \beta \sum_{\nu_1,\nu_2,\nu_3} I_{\nu,\nu_1,\nu_2,\nu_3} \phi^*_{\nu_1} \phi_{\nu_2} \phi_{\nu_3}$$

$$I_{\nu,\nu_1,\nu_2,\nu_3} = \sum_l A_{\nu,l} A_{\nu_1,l} A_{\nu_2,l} A_{\nu_3,l}$$

NM ordering in real space: $X_\nu = \sum_l l A^2_{\nu,l}$

Characterization of wavepackets in normal mode space:

$z_\nu \equiv |\phi_\nu|^2 / \sum_\mu |\phi_\mu|^2$ $\bar{\nu} = \sum_\nu \nu z_\nu$

Second moment: $m_2 = \sum_\nu (\nu - \bar{\nu})^2 z_\nu$ → location of tails

Participation number: $P = 1/\sum_\nu z_\nu^2$ → number of strongly excited modes

Compactness index: $\zeta = \dfrac{P^2}{m_2}$

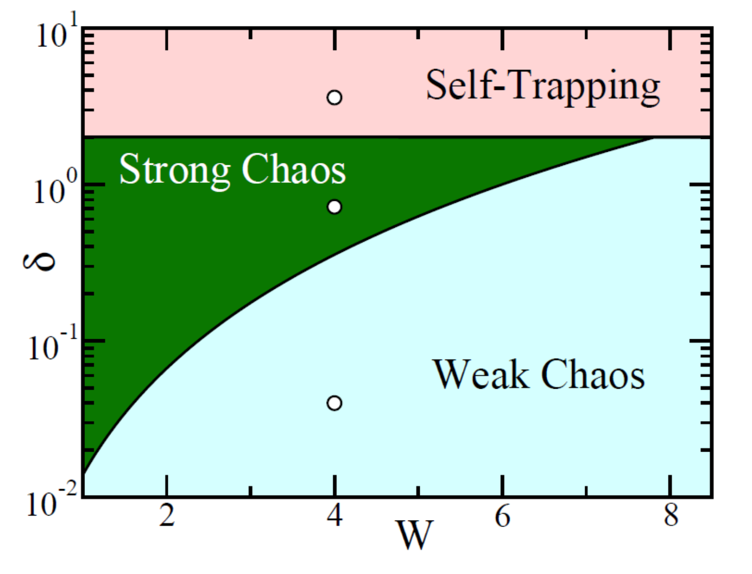
→ K adjacent sites equally excited: $\zeta = 12$

→ K adjacent sites, every second empty or equipartition: $\zeta = 3$

Frequency scales

W=4 :

- Eigenvalue (frequency) spectrum width: $\Delta = W + 4$ — **8**
- Localization volume of eigenstate: $V \approx 360/W^2$ — **~18 (sites)**
- Average frequency spacing inside localization volume: $d = \Delta/V$ — **0.43**
- Nonlinearity induced frequency shift: $\delta_l = \beta|\psi_l|^2$



Three expected evolution regimes:

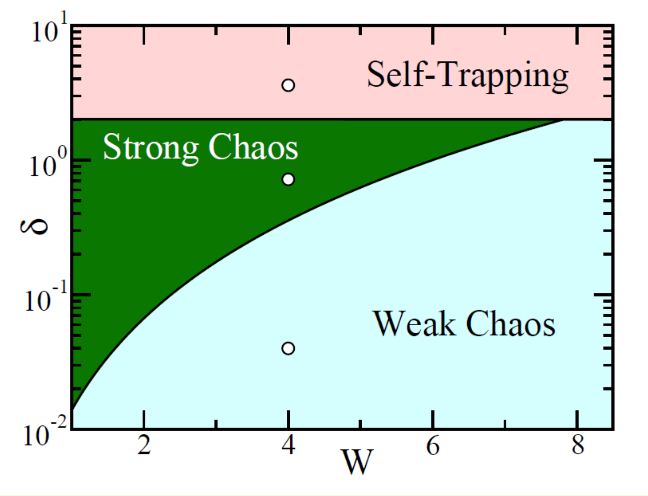
Weak chaos : $\delta < d$

Strong chaos : $d < \delta < 2$

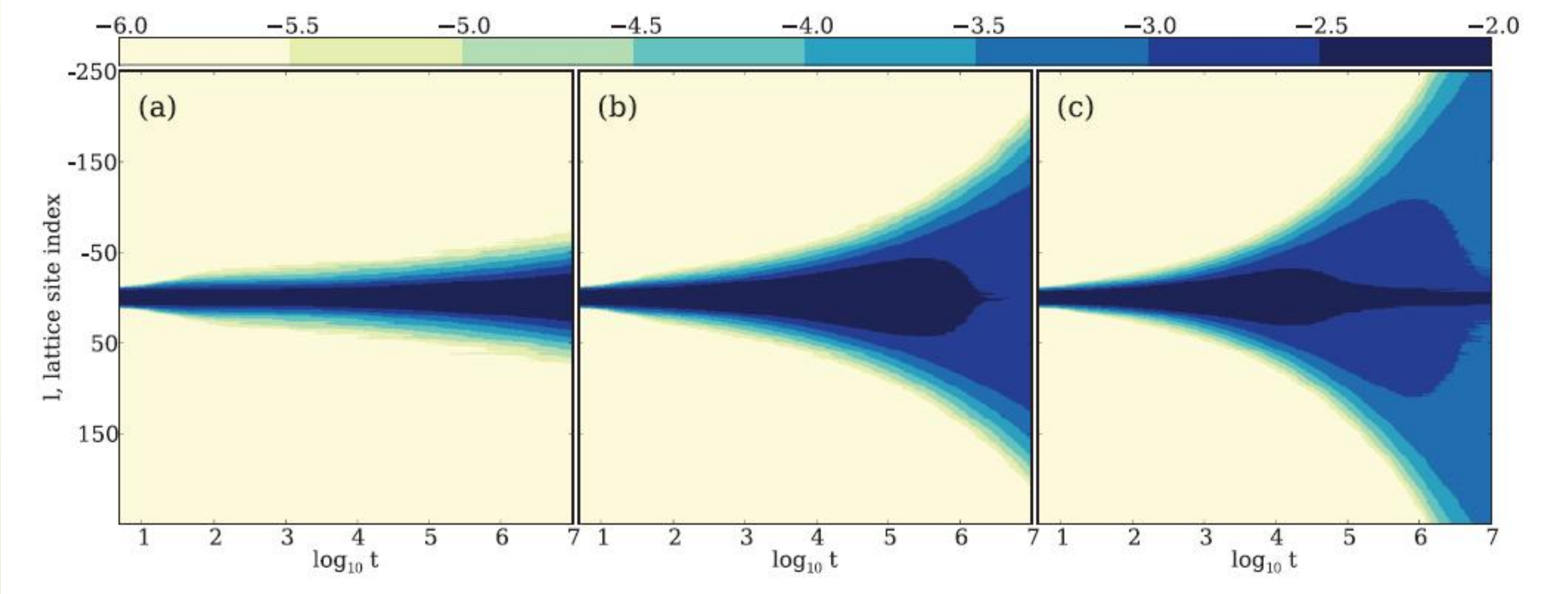
(partial) self trapping : $2 < \delta$

SF Chem Phys 2010, TV Laptyeva et al EPL 2010

W=4
Wave packet with 20 sites
Norm density = 1
Random initial phases
Averaging over 1000 realizations



J Bodyfelt et al PRE 2011



Asymptotic regime of weak chaos

SF et al PRL 2009, Ch. Skokos et al PRE 2009

We averaged the measured exponent over 20 realizations:

α = 0.33 ± 0.02 (DNLS)
α = 0.33 ± 0.05 (KG)

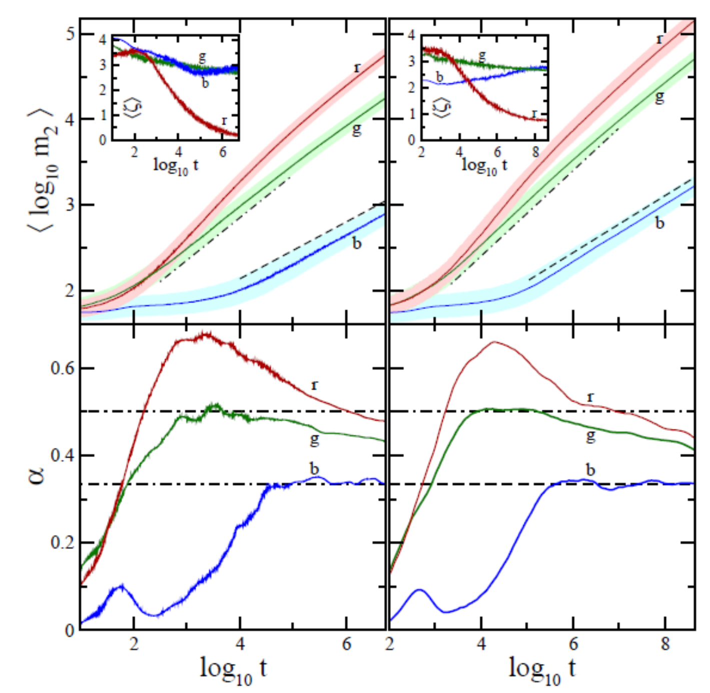
Strong chaos and crossover to weak chaos

TV Laptyeva et al EPL 2010

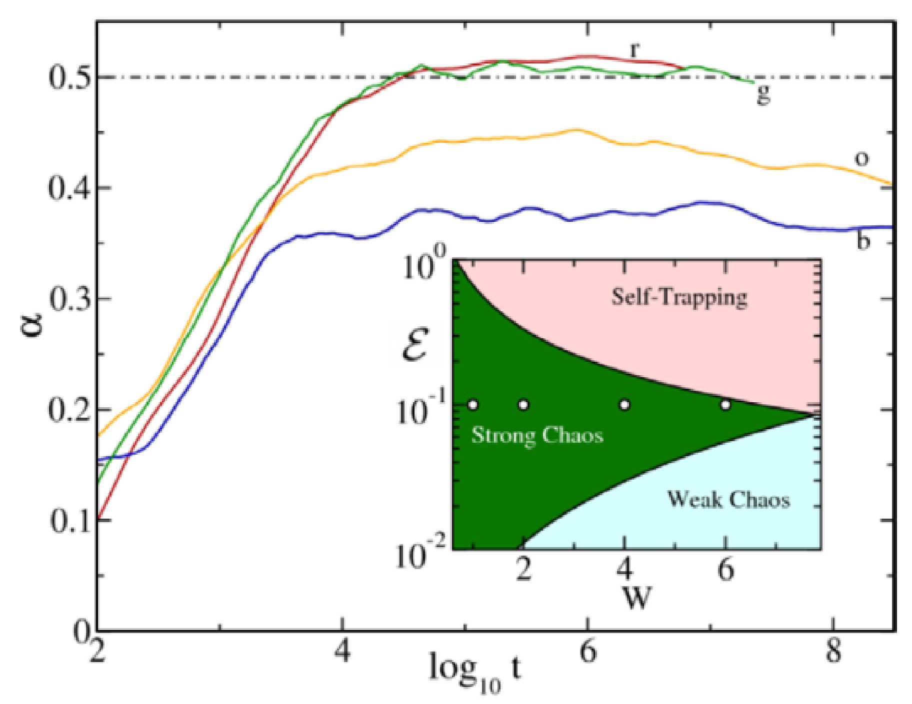
Averaging over 1000 realizations, measuring

$$\alpha(\log t) = \frac{\mathrm{d}\langle \log m_2 \rangle}{\mathrm{d} \log t}$$

DNLS, W=4 KG, W=4



KG



Chaos in wave packet generates nonlinear diffusion:

$$m_2 \sim \begin{cases} \beta t^{1/2}, & \beta n/d > 1 \text{ (strong chaos)} \\ d^{-2/3}\beta^{4/3}t^{1/3}, & \beta n/d < 1 \text{ (weak chaos)} \end{cases}$$

SF ChemPhys 2010

Generalizations: higher dimensions, nonlinearity exponent σ:

$$i\dot{\psi}_{\boldsymbol{l}} = \epsilon_{\boldsymbol{l}}\psi_{\boldsymbol{l}} - \beta|\psi_{\boldsymbol{l}}|^{\sigma}\psi_{\boldsymbol{l}} - \sum_{\boldsymbol{m}\in D(\boldsymbol{l})}\psi_{\boldsymbol{m}}$$

$$D \sim \beta^2 n^{\sigma}\left(\mathscr{P}(\beta n^{\sigma/2})\right)^2$$

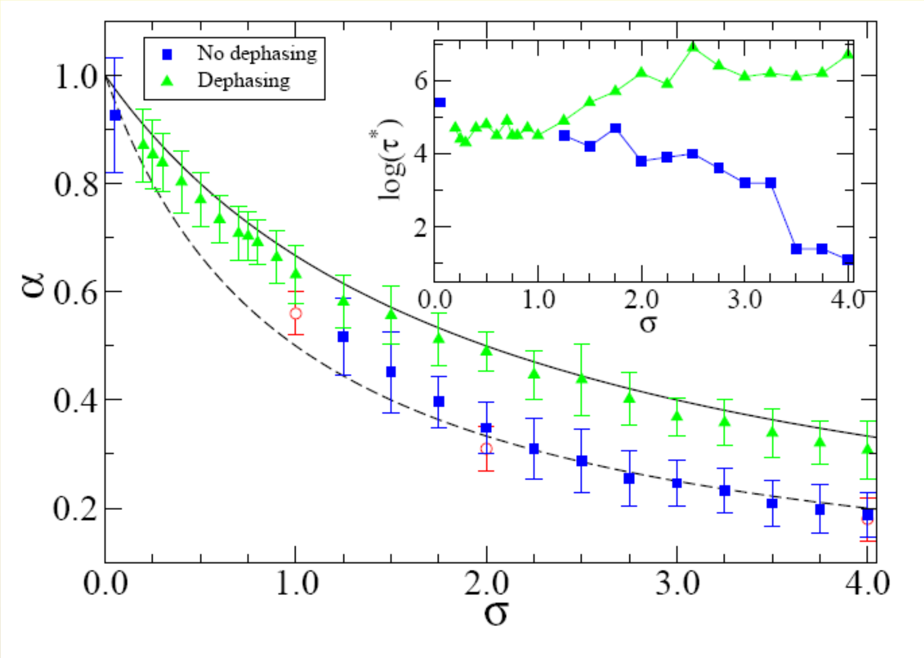
$$m_2 \sim (\beta^2 t)^{\frac{2}{2+\sigma D}}, \text{ strong chaos,}$$

$$m_2 \sim (\beta^4 t)^{\frac{1}{1+\sigma D}}, \text{ weak chaos.}$$

Generalizations: higher dimensions, nonlinearity exponent σ:

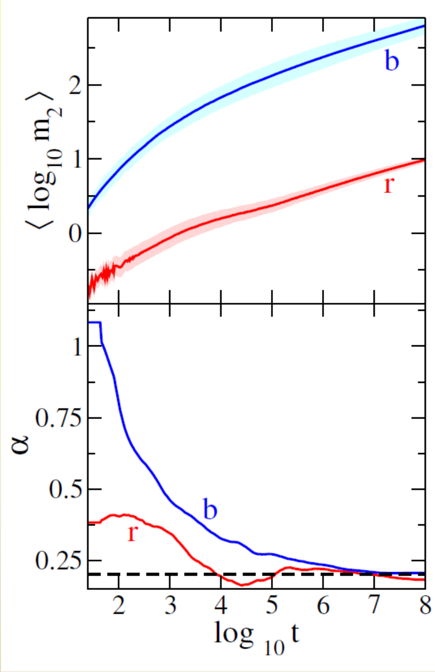
$$i\dot{\psi}_{\boldsymbol{l}} = \epsilon_{\boldsymbol{l}}\psi_{\boldsymbol{l}} - \beta|\psi_{\boldsymbol{l}}|^{\sigma}\psi_{\boldsymbol{l}} - \sum_{\boldsymbol{m}\in D(\boldsymbol{l})}\psi_{\boldsymbol{m}}$$

D=1, 0 < σ < 4 :



Ch Skokos et al PRE 2010

D=2, σ = 2 :

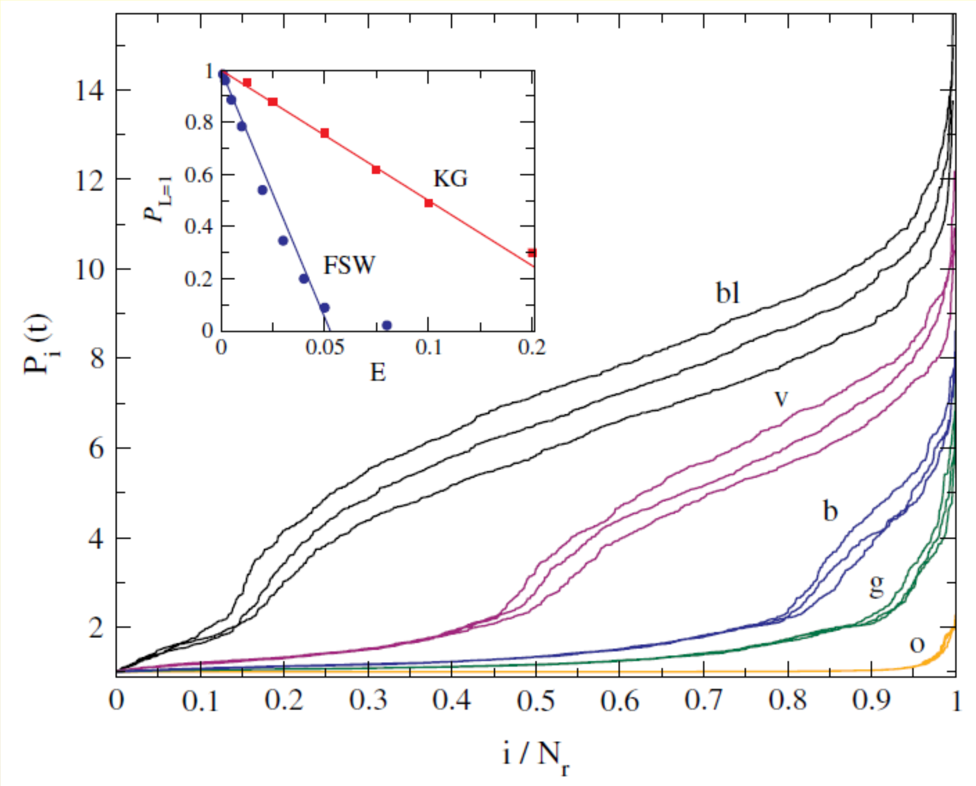


TV Laptyeva et al, EPL 2012

Related results by M Mulansky

Restoring Anderson localization? A matter of probability and KAM!

MV Ivanchenko et al PRL 2011

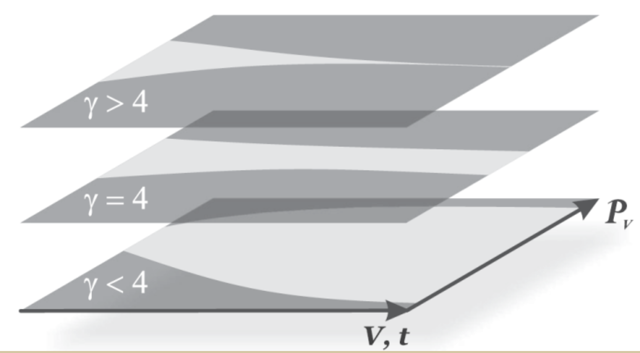


E : total energy

L : size of initial wave packet

$$\mathcal{P}_L = \left(1 - \frac{3\kappa E}{L}\right)^{2L}$$

$$\mathcal{P}_\infty = e^{-6\kappa E}$$



Generalizing:

d: dimension

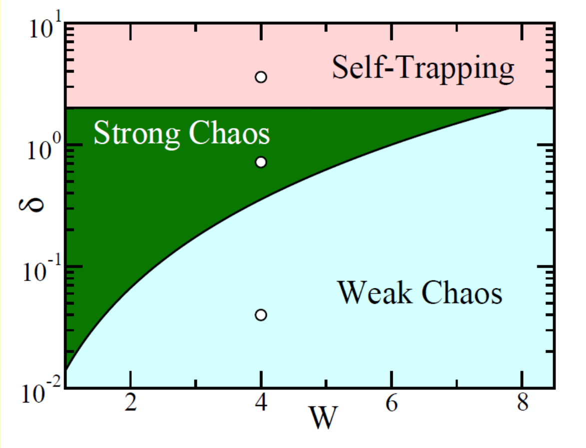
V: volume of wave packet

γ : = 2σ

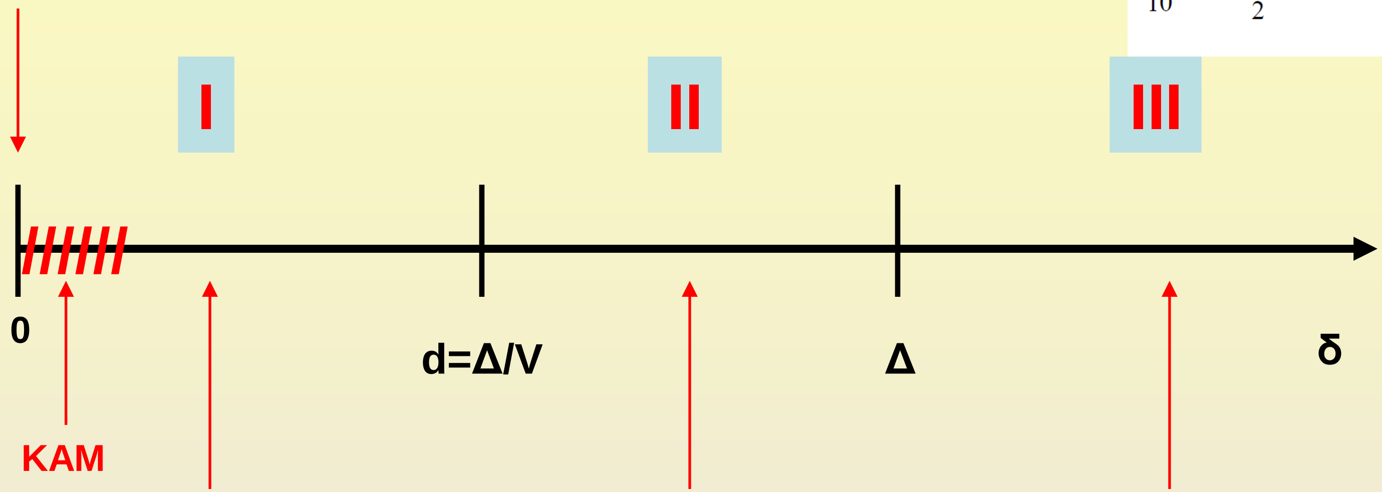
$$\mathcal{P}_V = \left(1 - \frac{\kappa_\gamma E^{\gamma/2-1}}{V^{\gamma/2-1}}\right)^{2Vd}$$

Related results by Aubry,Johansson

The emerging picture



Anderson Localization



Weak Chaos

Strong Chaos

Selftrapping, part of packet IS spreading

SF,Krimer,Skokos (2009)
Shepelyansky and Pikovsky (2008)
Molina (1998)

SF (2010)
Bodyfelt,Lapteva,Krimer, Skokos,SF (2010)

Kopidakis, Komineas, SF, Aubry (2008)

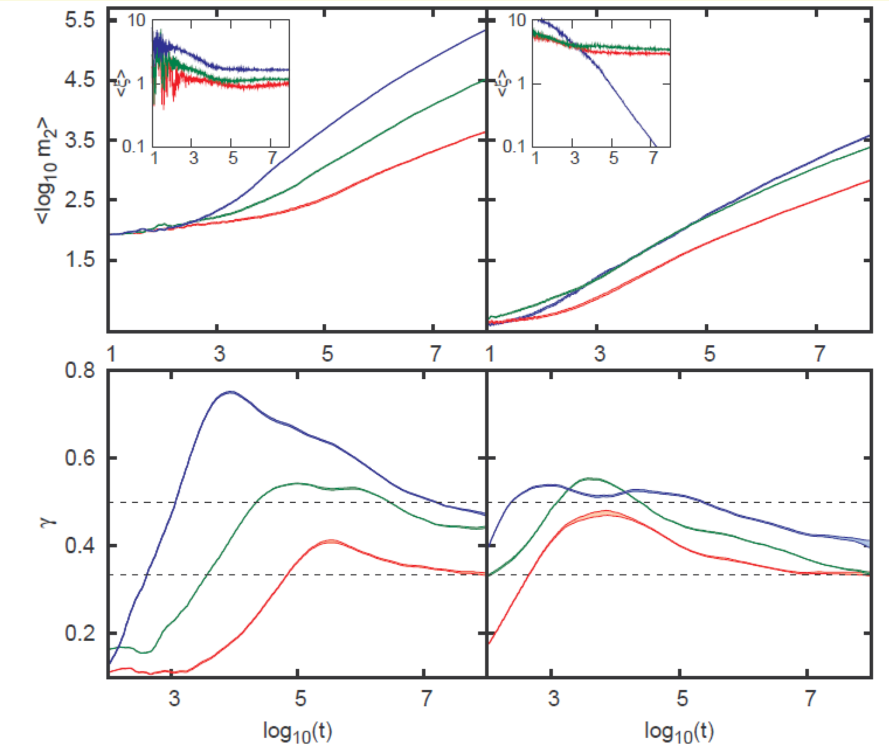
In all cases subdiffusive spreading

Quasiperiodic potentials (Aubry-Andre):

M Larcher et al, arXiv1206.0833

$$i\frac{\partial \psi_j}{\partial t} = -(\psi_{j+1} + \psi_{j-1}) + V_j \psi_j + \beta |\psi_j|^2 \psi_j$$

$$V_j = \lambda \cos(2\pi\alpha j + \varphi)$$



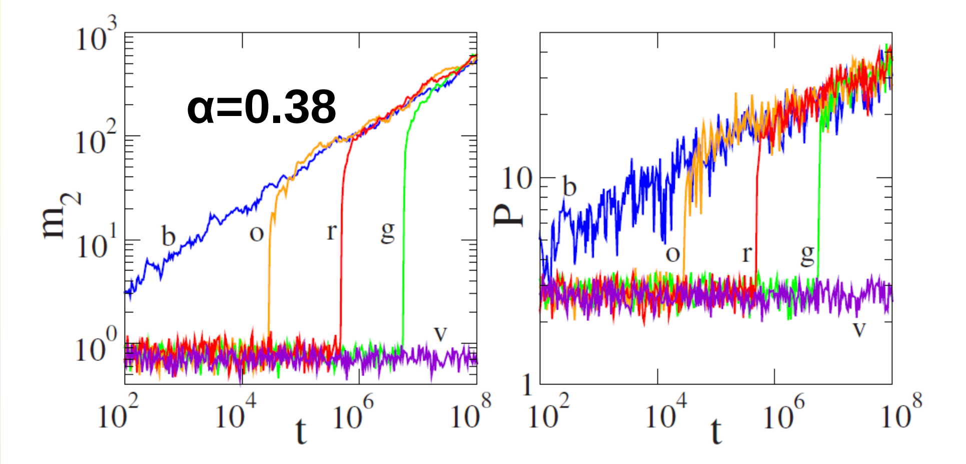
Pecularities:

- **spectrum with gaps**
- **subgaps etc**
- **fractal properties**
- **gap selftrapping**
- **hierarchy of level spacings**
- **strong and weak chaos**
- **α = 1/3**

Nonlinear Wannier-Stark ladder

D Krimer et al PRE 2009

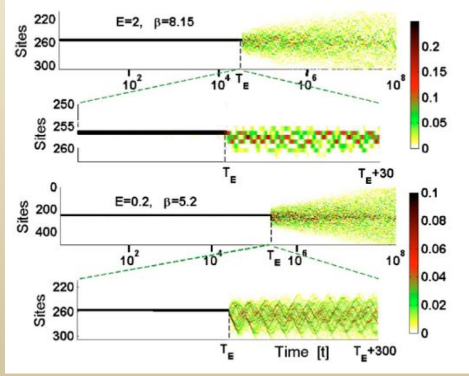
$$i\dot{\Psi}_n = -(\Psi_{n+1} + \Psi_{n-1}) + nE\Psi_n + \beta|\Psi_n|^2\Psi_n$$



E=2, β=8, ...,9

Pecularities:

- spectrum is equidistant
- exact resonances
- absence of universality
- exponents depend on E



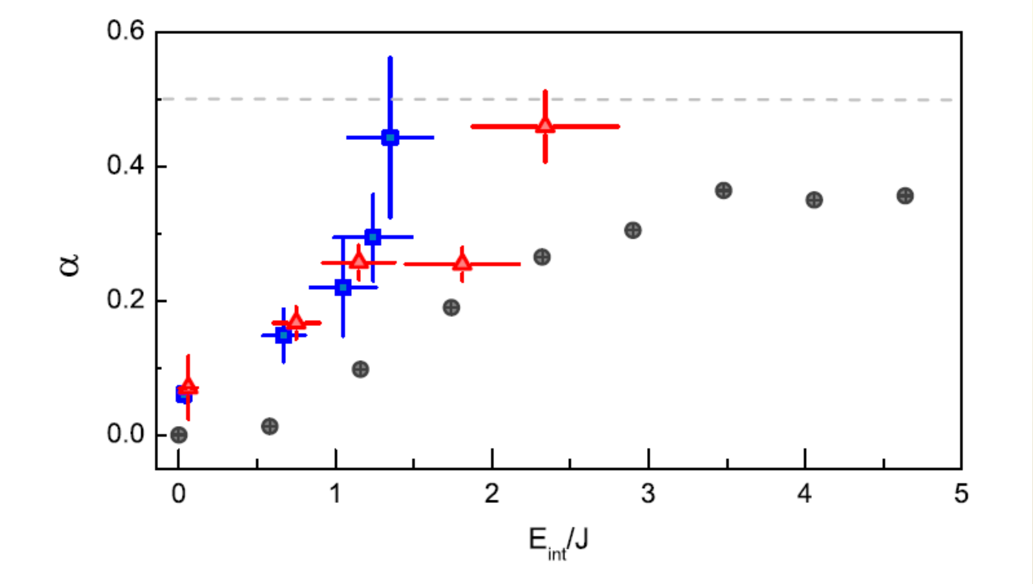
1st experimental confirmation from Firenze

PRL **106,** 230403 (2011) PHYSICAL REVIEW LETTERS wee
10 J

Observation of Subdiffusion in a Disordered Interacting System

E. Lucioni,[1,*] B. Deissler,[1] L. Tanzi,[1] G. Roati,[1] M. Zaccanti,[1,†] M. Modugno,[2,3] M. Larcher,[4]
F. Dalfovo,[4] M. Inguscio,[1] and G. Modugno[1,‡]

Bose-Einstein condensate of ^{39}K atoms

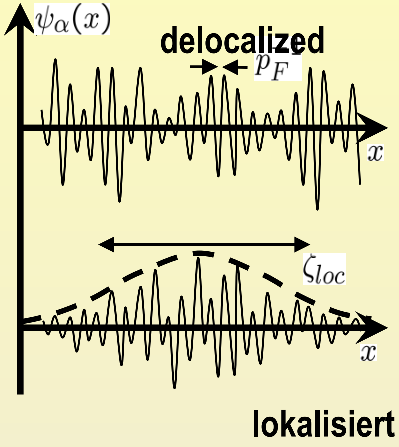


Many body localization?

(what I will not further talk about)

Localization of one-particle wave functions in disordered potentials

$$\left[-\frac{\boldsymbol{\nabla}^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right] \psi_\alpha(\boldsymbol{r}) = \xi_\alpha \psi_\alpha(\boldsymbol{r})$$



d=1: All states are localized

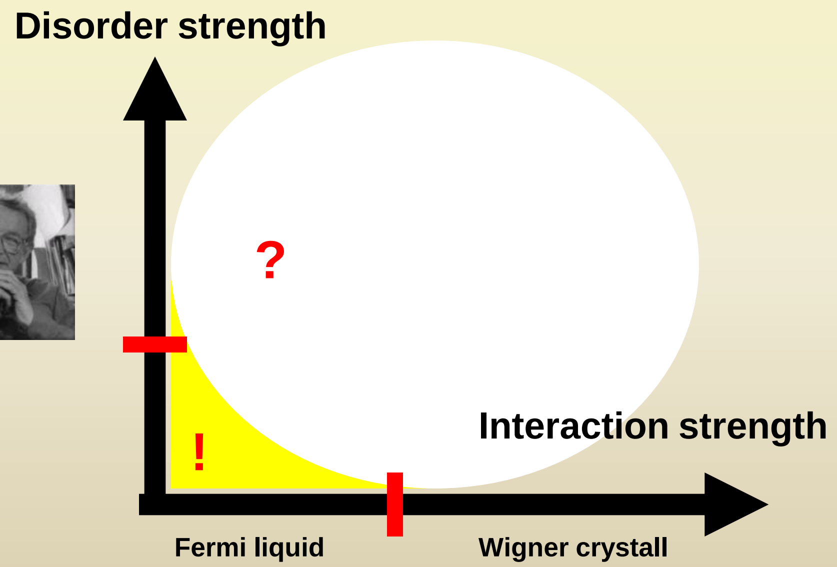
d=2: All states are localized

d >2: Anderson transition

Interacting Fermions:

! – more or less understood

? – not really



Basko, Aleiner, Altshuler (2006):

$$V(\vec{r}_1 - \vec{r}_2) = \frac{\lambda}{\nu}\,\delta(\vec{r}_1 - \vec{r}_2)$$

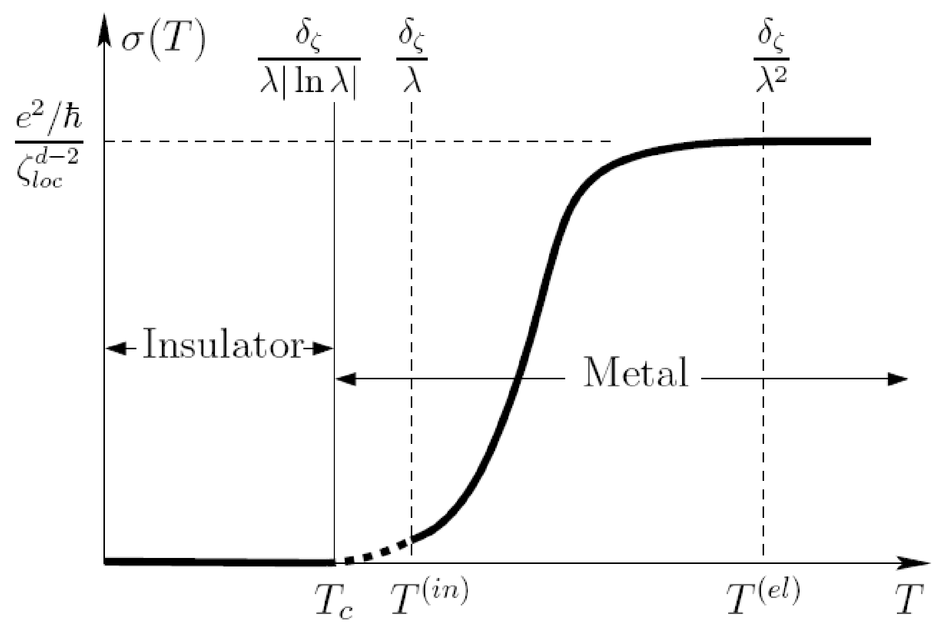
- all single particle states are localized
- no phonons
- short range interaction only

$$\hat{H} = \sum_{\alpha} \xi_\alpha \hat{c}^\dagger_\alpha \hat{c}_\alpha + \frac{1}{2}\sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta}\,\hat{c}^\dagger_\alpha \hat{c}^\dagger_\beta \hat{c}_\gamma \hat{c}_\delta$$

Average level spacing of single particle states within one localization volume:

$$\delta_\zeta = \frac{1}{\nu \zeta_{loc}^d}$$

- critical temperature for MIT
- in the metallic phase fermions need a minimum number of excited partner particles
- in the limit of large localization length and weak disorder: Tc $\rightarrow$ 0 , classical MF description?



Interacting quantum particles in disordered chains

$$\hat{\mathcal{H}} \equiv \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{\text{int}}, \quad \hat{\mathcal{H}}_{\text{int}} = \sum_l \left[\frac{U}{2} \hat{a}_l^+ \hat{a}_l^+ \hat{a}_l \hat{a}_l \right],$$

$$\hat{\mathcal{H}}_0 = \sum_l [\epsilon_l \hat{a}_l^+ \hat{a}_l + V(\hat{a}_{l+1}^+ \hat{a}_l + \hat{a}_l^+ \hat{a}_{l+1})],$$

$\{\epsilon_l\}$ **random uncorrelated from** $[-W/2, W/2]$

$$a_l a_m^\dagger - a_m^\dagger a_l = \delta_{lm}$$

Model describes interacting bosons in one dimension

$$\hat{N} = \sum_n \hat{n}_n \quad , \quad \hat{n}_n = a_n^+ a_n \quad , \quad [H, \hat{N}] = 0$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$|n\rangle = \frac{1}{\sqrt{n!}}\,(a^+)^n|0\rangle$$

One particle

Real space basis: $|l\rangle \equiv a_l^+|0\rangle$ with $l = 1, \ldots, N$

Eigenstates = normal modes: $|\nu\rangle = \sum_l^N A_l^{(\nu)}|l\rangle$

Eigenvalue problem: $\lambda_\nu A_l^{(\nu)} = \epsilon_l A_l^{(\nu)} + V(A_{l+1}^{(\nu)} + A_{l-1}^{(\nu)})$

Single quantum particle identical with classical linear wave equation.

Here: Anderson localization

Two particles

Real space basis:

$$|l, m\rangle \equiv a_l^+ a_m^+ |0\rangle / (\sqrt{1 + \delta_{lm}})$$

Eigenstates = normal modes:

$$|q\rangle = \sum_{m,\, l \le m}^{N} \mathcal{L}_{l,\,m}^{(q)} |l, m\rangle$$

Eigenvectors:

$$\mathcal{L}_{l,\,m}^{(q)} = \langle l, m|q\rangle$$

Indistinguishable particles:

$$l \le m$$

PDF of particle number:

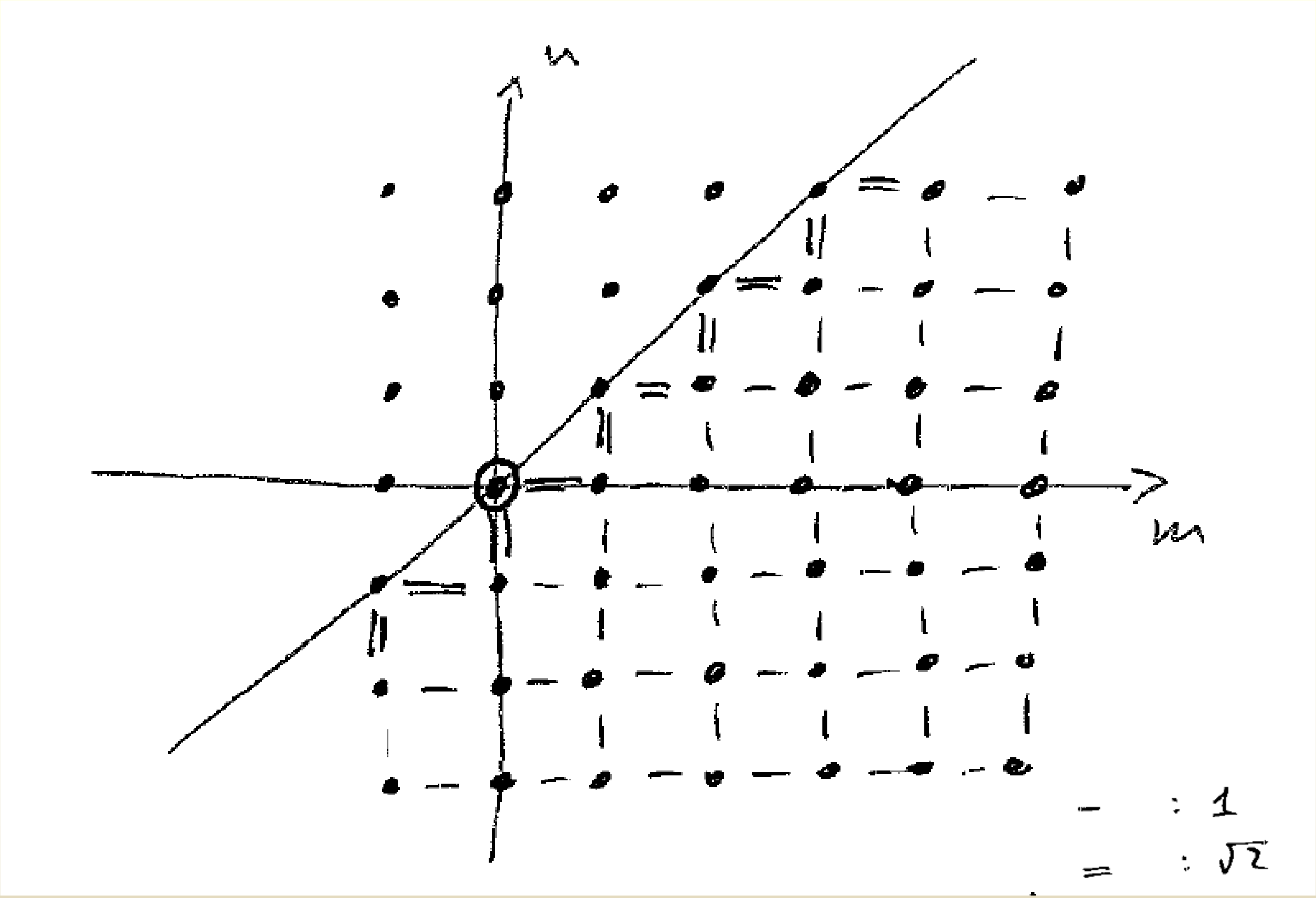
$$p_l = \langle q|\hat{a}_l^+ \hat{a}_l|q\rangle / 2$$

$$p_l^{(q)} = \frac{1}{2}\left(\sum_{k,\, l \le k}^{N} \mathcal{L}_{l,\,k}^{(q)2} + \sum_{m,\, l \ge m}^{N} \mathcal{L}_{m,\,l}^{(q)2} \right)$$

$$\langle \ell m | H | \ell m \rangle = \varepsilon_\ell + \varepsilon_m + U \delta_{\ell m}$$

$$(n'm' | H | nm) = \begin{cases} 1 & \text{if } \begin{pmatrix} n' \pm 1 = n \\ m' = m \end{pmatrix}, \quad m > n+1 \\ 1 & \text{if } \begin{pmatrix} n' = n \\ m' \pm 1 = m \end{pmatrix}, \quad m > n+1 \\ \sqrt{2} & \text{if } \begin{pmatrix} n' \pm 1 = n \\ m' = m \end{pmatrix}, \quad m = n+1 \\ \sqrt{2} & \text{if } \begin{pmatrix} n' = n \\ m' + 1 = m \end{pmatrix}, \quad m = n+1 \\ 1 & \text{if } \begin{pmatrix} n' + 1 = n \\ m' = m \end{pmatrix}, \quad m = n+1 \\ 1 & \text{if } \begin{pmatrix} n' = n \\ m' - 1 = m \end{pmatrix}, \quad m = n+1 \\ \sqrt{2} & \text{if } \begin{pmatrix} n' + 1 = n \\ m' = m \end{pmatrix}, \quad m = n \\ \sqrt{2} & \text{if } \begin{pmatrix} n' = n \\ m' - 1 = m \end{pmatrix}, \quad m = n \end{cases}$$

V=1



N particles in one dimension are equivalent to one fictuous particle in N dimensions

Unfolding irreducible space into full two-dimensional plane:

$$\varphi_{nm} = \varphi_{mn}$$

$$\begin{cases} \psi_{n,n} = \sqrt{2}\,\varphi_{n,n} & \\ \psi_{n,m} = \varphi_{n,m} & n \neq m \end{cases}$$

$$E\,\psi_{n,m} = \psi_{n,m\pm1} + \psi_{n\pm1,m} + (\varepsilon_n + \varepsilon_m + U\delta_{nm})\,\psi_{nm}$$

Noninteracting eigenstate basis:

$$|\mu, \nu \geq \mu\rangle = \frac{|\mu\rangle|\nu\rangle}{\sqrt{1+\delta_{\mu,\nu}}},$$

$$\hat{\mathcal{H}}_0|\mu, \nu\rangle = (\lambda_\mu + \lambda_\nu)|\mu, \nu\rangle$$

Eigenstates = normal modes (NM): $|q\rangle = \sum_{\nu,\, \mu \leq \nu}^{N} \phi_{\mu\nu}^{(q)}|\mu, \nu\rangle$

$$\lambda_q \phi_{\mu\nu}^{(q)} = \lambda_{\mu\nu}\phi_{\mu\nu}^{(q)} + 2U\sum_{\mu',\nu'} \bar{I}_{\mu\nu}^{\mu'\nu'}\phi_{\mu'\nu'}^{(q)}$$

$$\lambda_{\mu\nu} \equiv \lambda_\mu + \lambda_\nu$$

$$\bar{I}_{\mu\nu}^{\mu'\nu'} = I_{\mu\nu}^{\mu'\nu'}/(\sqrt{1+\delta_{\mu\nu}}\sqrt{1+\delta_{\mu'\nu'}})$$

$$I_{\mu\nu}^{\mu'\nu'} = \sum_l A_l^\mu A_l^\nu A_l^{\mu'} A_l^{\nu'}$$

- The overlap integrals are same as in classical nonlinear wave theory
- connectivity is L*L (instead of L in classical theory)
- phase space is 4*N*N (instead of 2*N in classical theory)
- differential equations are linear (instead of nonlinear in classical theory)
- width of spectrum: $\Delta_2 = 2\Delta_1$
- average spacing of connected eigenstates: $d = \Delta_2 / L^2$
- energy mismatch = effective disorder in NM space: $\overline{W} \equiv d$
- effective hopping: $\overline{V} = 2U\langle I \rangle$
- weak disorder: $\xi_2/\xi_1 \approx 100\,\overline{V}^2/\overline{W}^2 = 400\,U^2\langle I\rangle^2 L^4/\Delta_2^2$
- U > W + V : bound states separate into narrow band, loc length small, separation into strongly localized bound states and spinless fermions
- small U: perturbation regime, strong disorder in NM space: $UI_0 \lesssim d$
- relevant regime: $\Delta_2/L \lesssim U \lesssim V.$

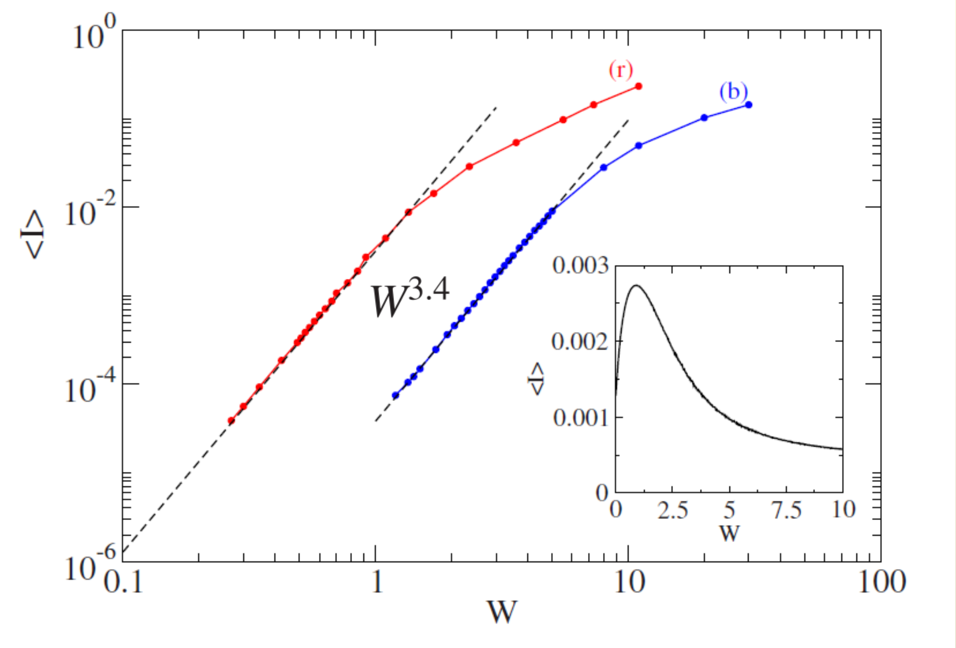
- **Analytics boils down to getting control over the overlap integrals I**

- **Shepelyansky / Imry: neglecting phase correlations in eigenvectors**

$$\langle I \rangle_{SI} \sim L^{-3/2}$$

- **Schreiber / Roemer: depends how to average,** $\langle I \rangle_{R} \sim L^{-2}$

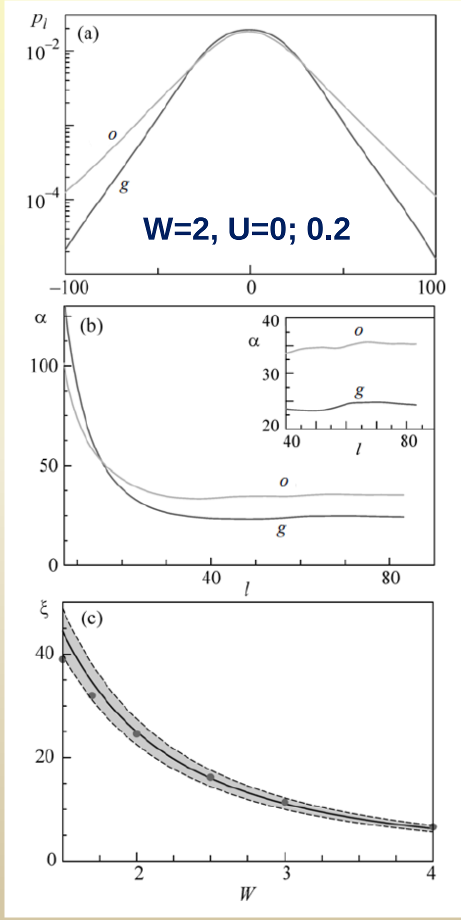
- **Krimer / Flach: subset of Is conserving momentum: < I > ~ 1/L**
  rest of Is : $\langle I \rangle_{SI} \sim \ln(L) L^{-2}$

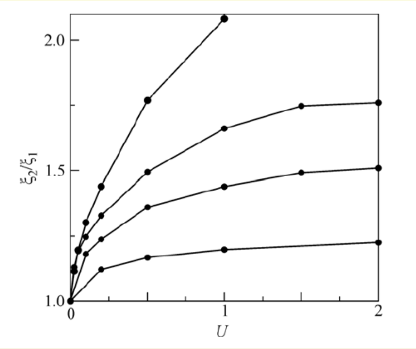


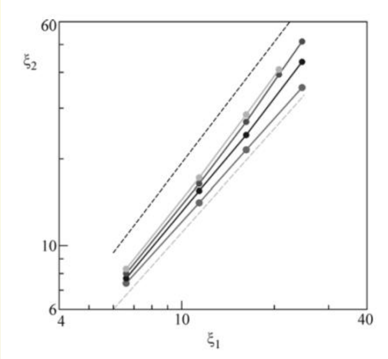
Inconclusive, needs further and more intelligent studies

Direct computation of the new localization length

Choose eigenstates with centers close to the diagonal, and maximum NM contribution from state with both single particle energies close to zero

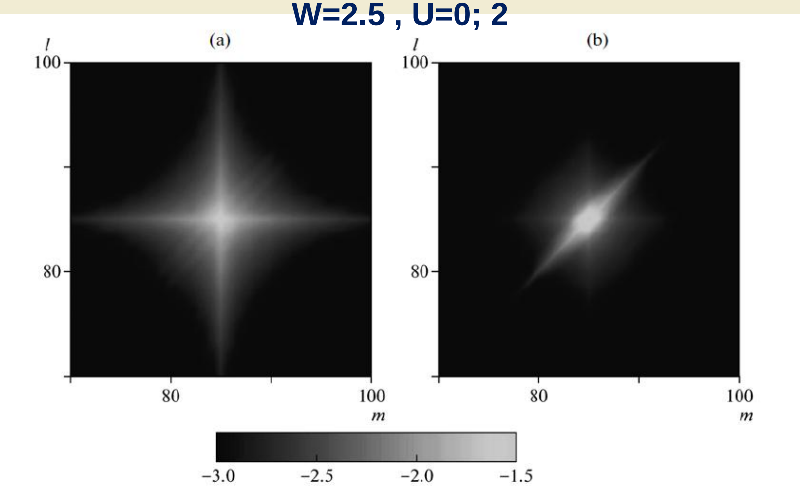






Exponents 1.3 ... 1.4

W=2.5 , U=0; 2

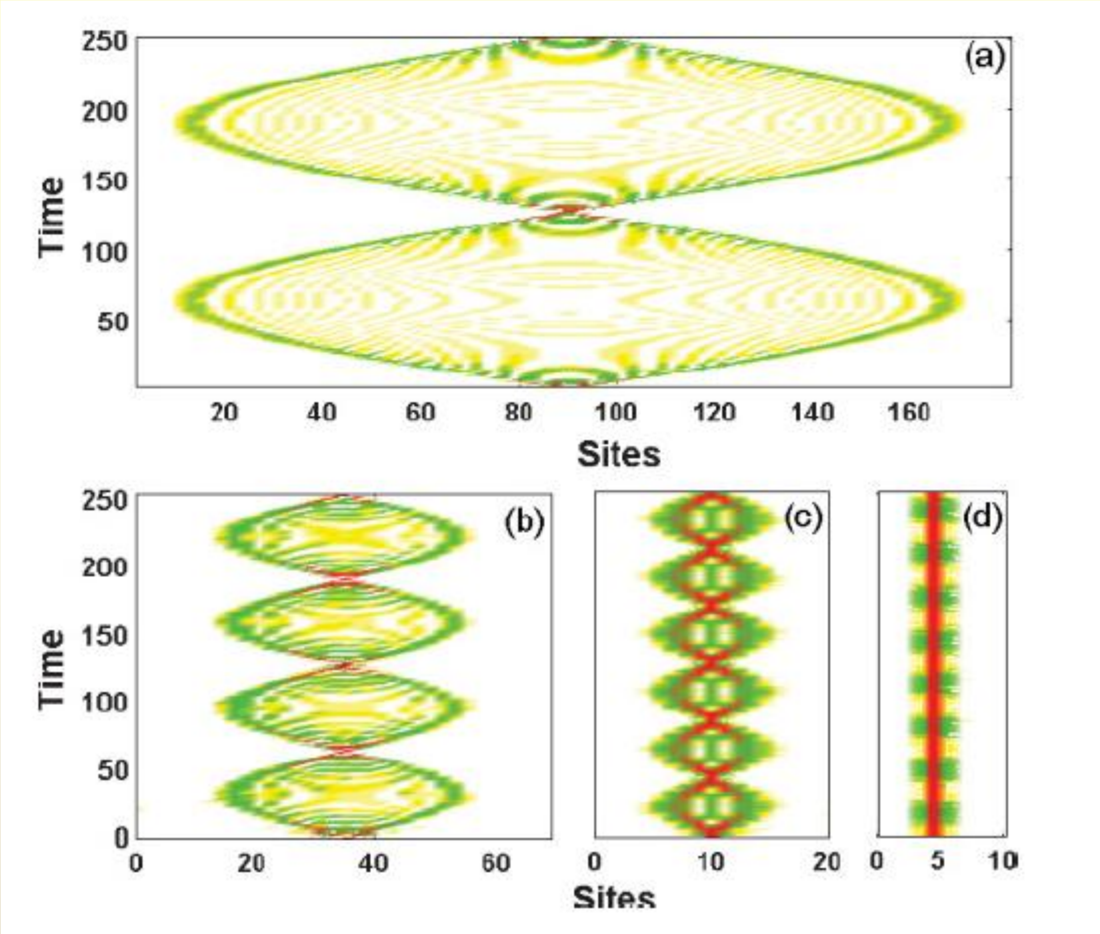


Conclusions?

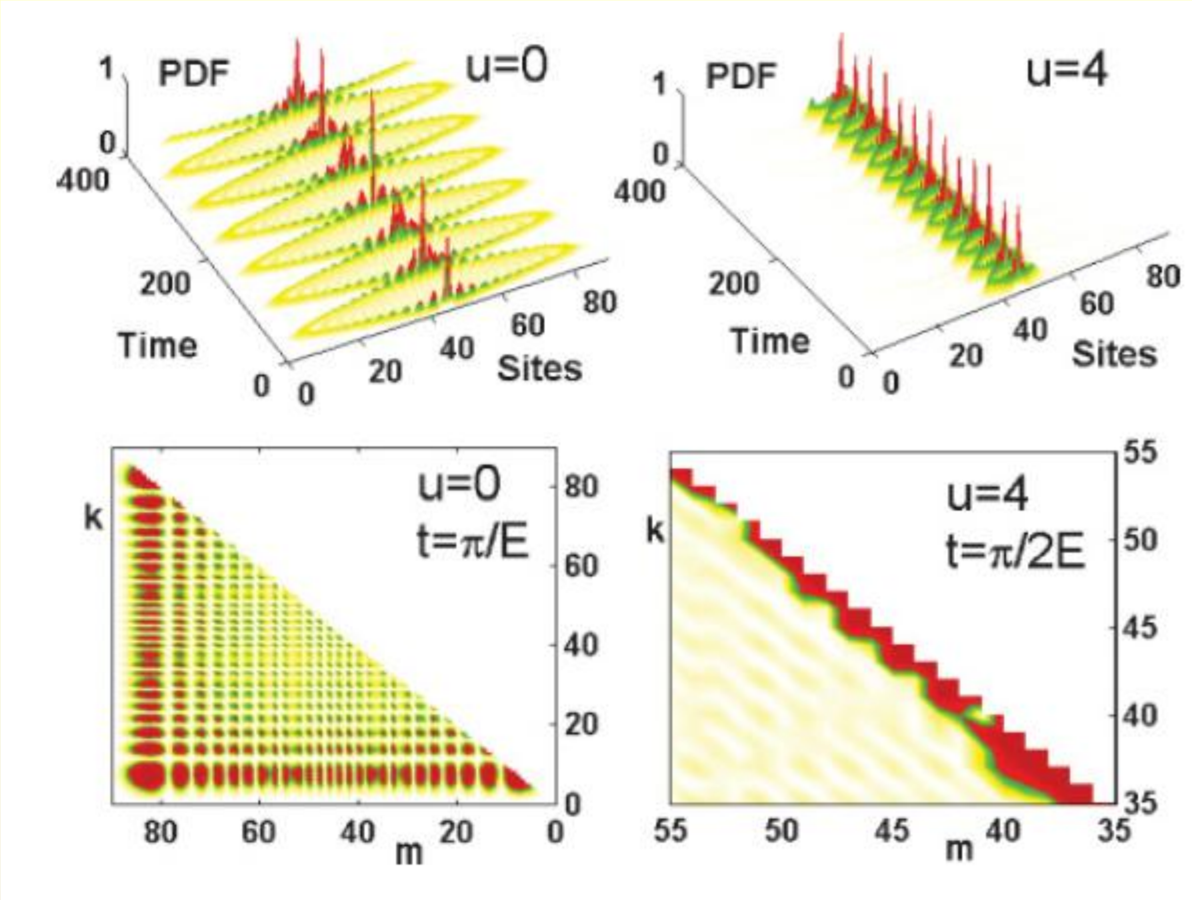
- **previous estimates probably wrong**
- **numerics of direct localziation length calculation is not getting into the relevant scaling regime so far**
- **question remains completely open**

Two (and more) interacting particles in a Wannier-Stark ladder

Interaction induced fractional Bloch and tunneling oscillations



U=3, E=0.05, N=1,2,3,4



Two particles form a bound state and Bloch oscillate with double frequency

Effective model for bound state dynamics with n particles:

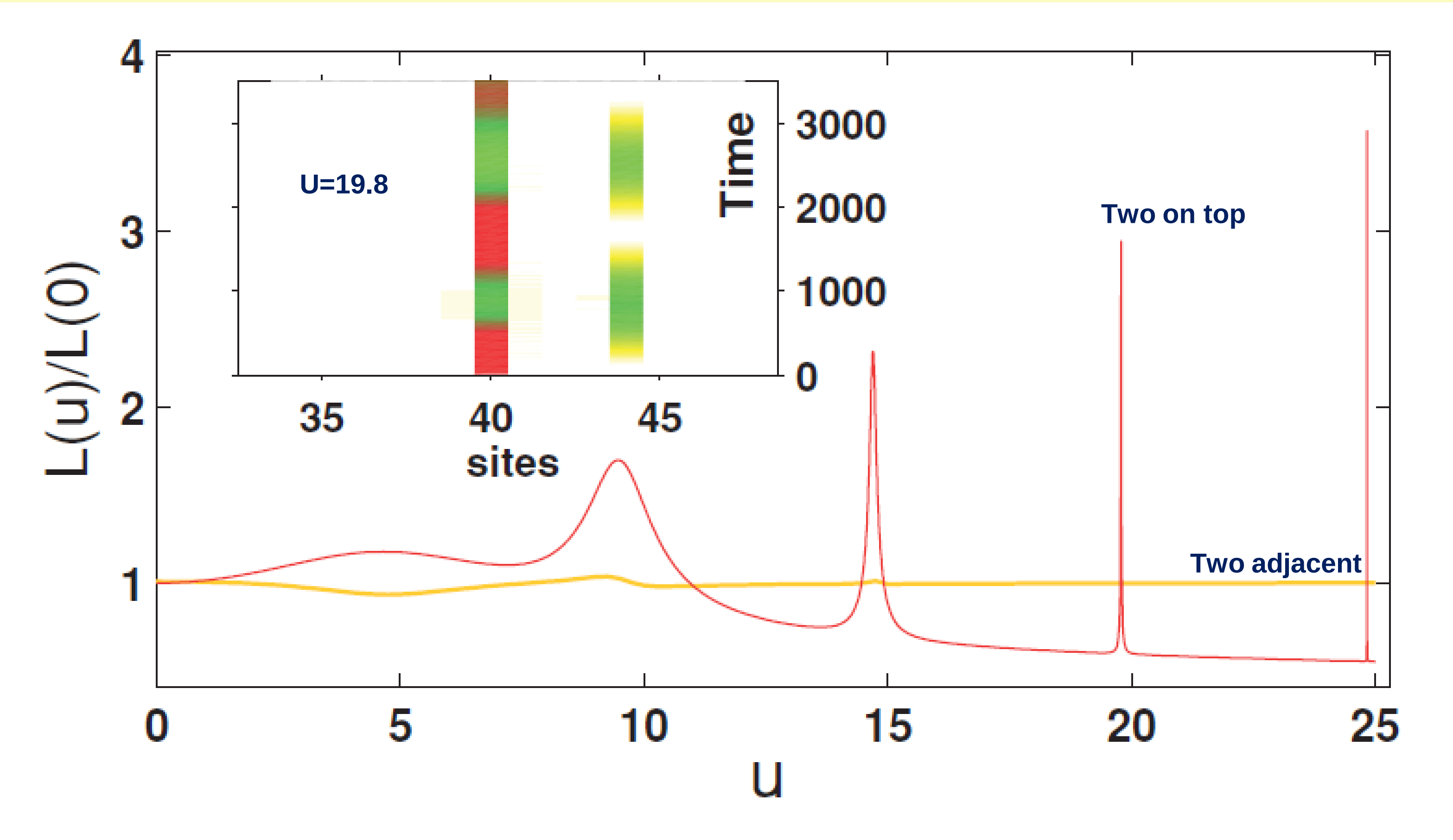
$$t_n \simeq \frac{n}{U^{n-1}(n-1)!}.$$

$$\hat{\mathcal{H}} \approx \sum_j [t_n(\hat{R}_{j+1}^{+}\hat{R}_j + \hat{R}_j^{+}\hat{R}_{j+1}) + nEj\hat{R}_j^{+}\hat{R}_j].$$

$$P_j(t) = \sum_{\nu,\mu} A_p^{\nu} A_p^{\mu} A_j^{\nu} A_j^{\mu} e^{inE(\mu-\nu)t}$$

$$A_p^{\nu} = J_{\nu-p}[2t_n/(nE)]$$

Resonant tunneling



$$(n-1)U = dE, \quad d = q - p$$

$$\tau_{\rm tun} \simeq \frac{\pi}{\sqrt{n}} E^{d-1}(d-1)!$$

Two interacting particles in a quasiperiodic potential

$$\hat{\mathcal{H}} = \sum_j \left[\hat{b}^+_{j+1}\hat{b}_j + \hat{b}^+_j \hat{b}_{j+1} + \epsilon_j \hat{b}^+_j \hat{b}_j + \frac{U}{2}\hat{b}^+_j \hat{b}^+_j \hat{b}_j \hat{b}_j \right]$$

basis:

$$|q\rangle = \sum_{m,l\leqslant m}^{N} \mathcal{L}^{(q)}_{l,m}|l,m\rangle, \qquad |l,m\rangle \equiv \frac{b^+_l b^+_m |0\rangle}{\sqrt{1+\delta_{lm}}}$$

M.V. Ivanchenko
R. Khomeriki
S. Flach
EPL 98 66002 (2012)

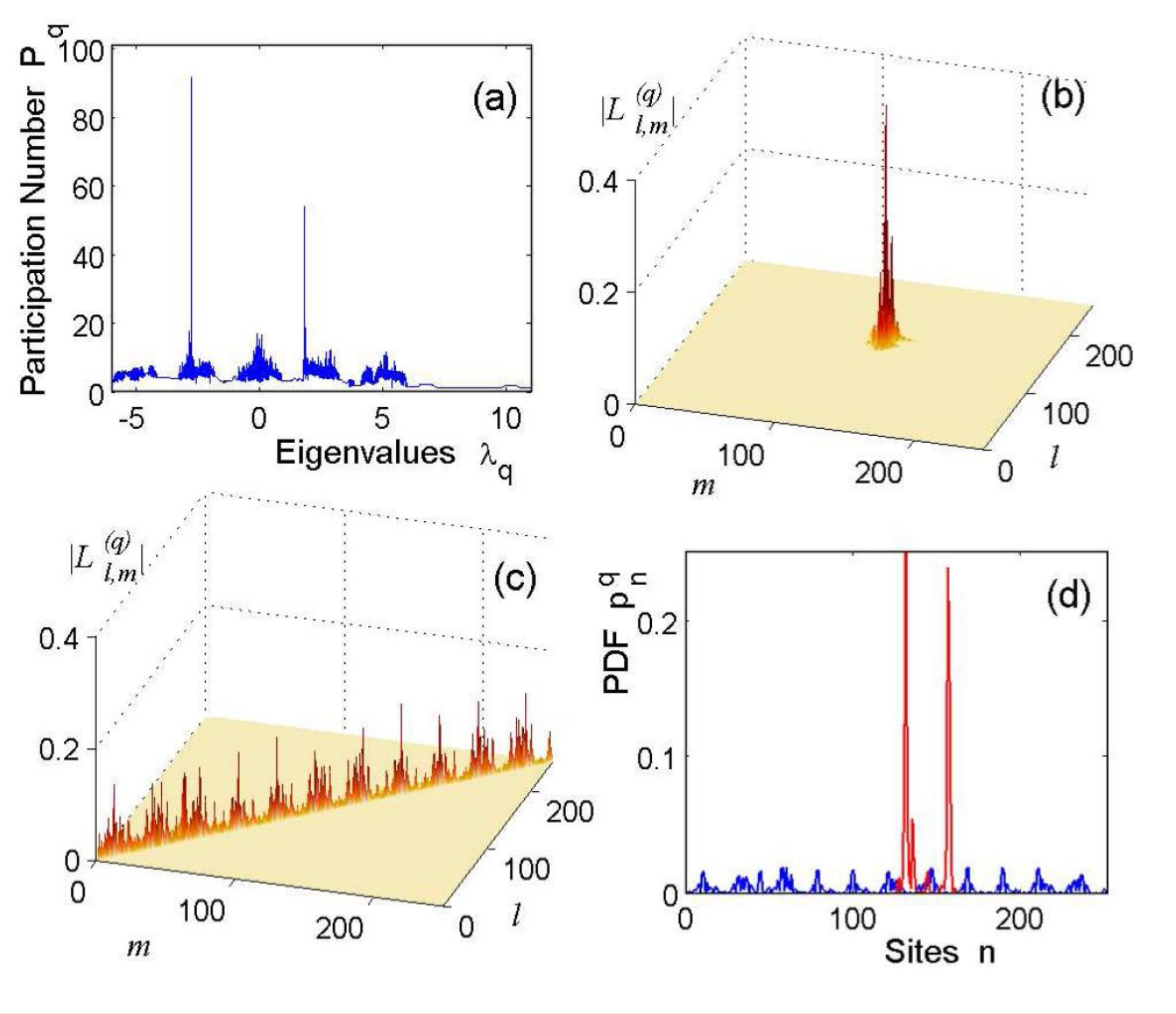
pdf of particle density:

$$p^{(q)}_l = \frac{\langle q|\hat{b}^+_l \hat{b}_l|q\rangle}{2} = \frac{1}{2}\left(\sum_{k,l\leqslant k}^{N} \mathcal{L}^{(q)2}_{l,k} + \sum_{m,l\geqslant m}^{N} \mathcal{L}^{(q)2}_{m,l} \right)$$

Participation number of density pdf:

$$P_q = 1/\sum_l^N (p^{(q)}_l)^2$$

Results: eigenfunctions



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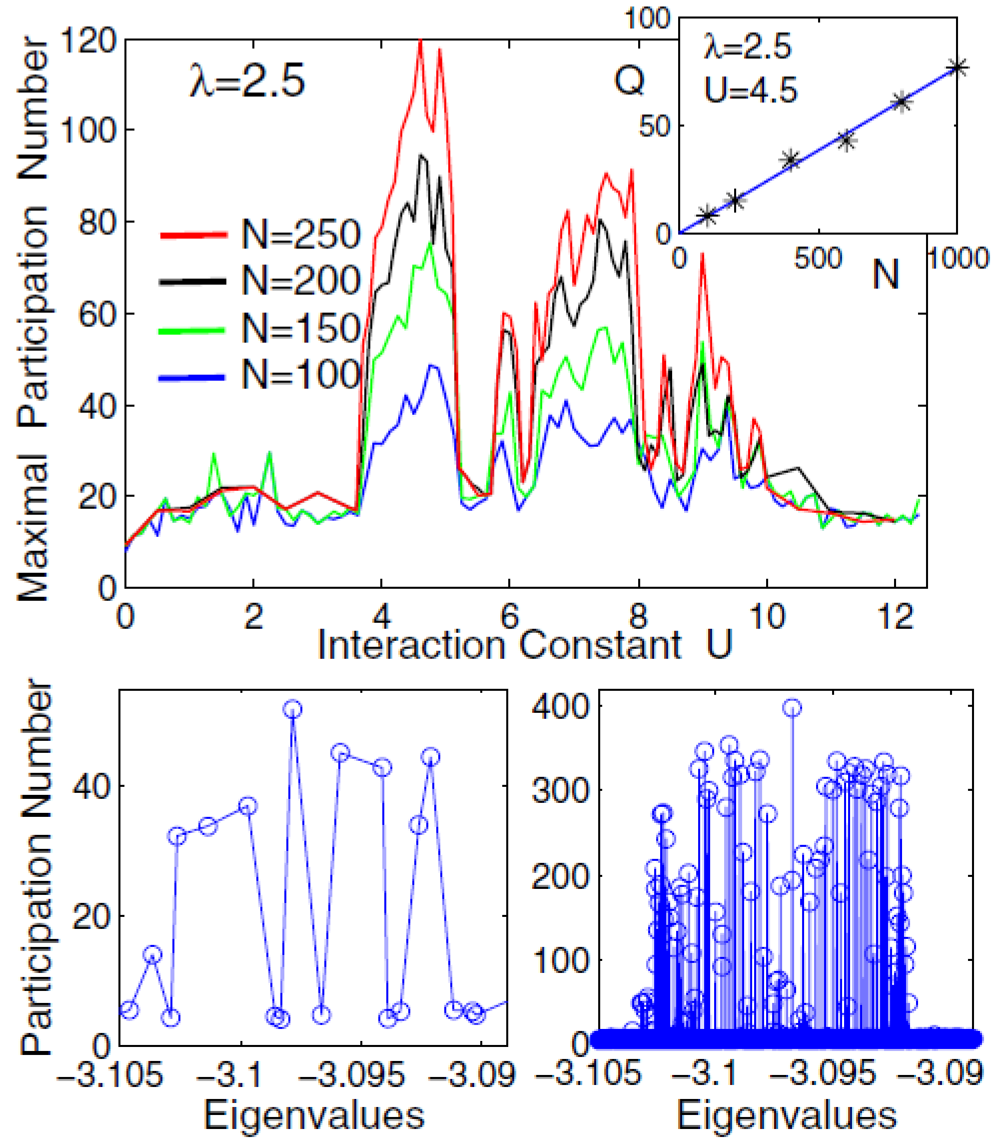
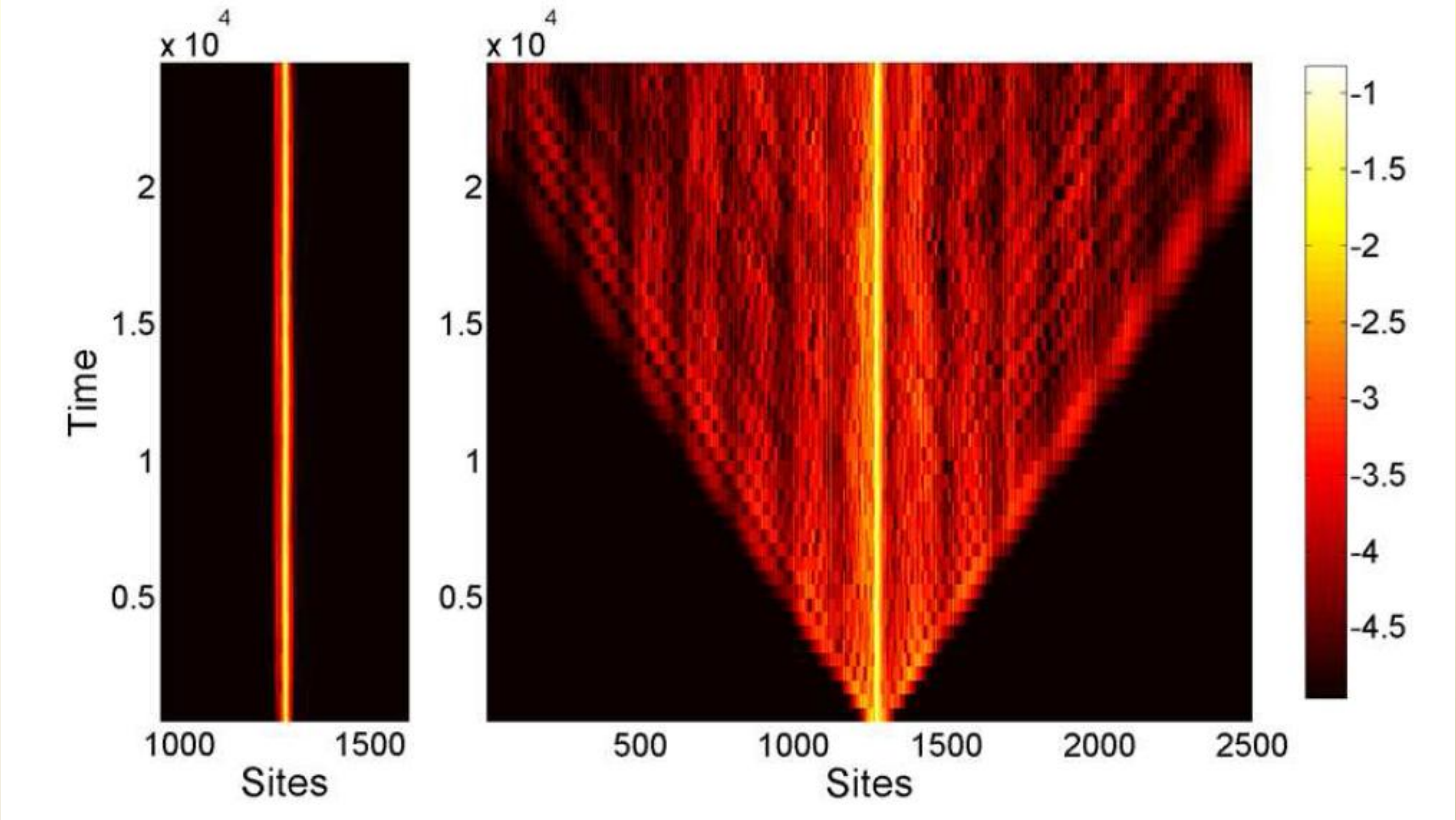
EPL 98 66002 (2012)

$U = 7.9$ and $\lambda = 2.5$


Maximal Participation Number
λ=2.5
Q
N=250
N=200
N=150
N=100
Interaction Constant U
0
2
4
6
8
10
12
20
40
60
80
100
120
λ=2.5
U=4.5
500
1000
50
N
Participation Number
Eigenvalues
−3.105
−3.1
−3.095
−3.09
300
200
400

Results: PDF of spreading of wave packet with λ=2.5 and N=2500 and two particles initially at adjacent sites

U=2 **U=4.5**



Results: the complete picture from spreading wave packets: square rooted 2^{nd} moment for 60 different realizations