QUANTUM CHROMODYNAMICS
PERTURBATIVE ASPECTS

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LECTURE OUTLINE

- **Perturbative QCD running coupling**

  For details see Chapter 3 of [1] and references therein

- **Electron–positron annihilation into hadrons**

  For details see Chaps. 1, 2, Sects. 4.1, 4.2 of [1] and references therein

[1] **А.В.Нестеренко, *Теоретическое описание функции Адлера и электрон–позитронной аннигиляции в адроны.* Дубна: ОИЯИ, УНЦ–2011–49, 144 с., 2011.**

PERTURBATIVE QCD RUNNING COUPLING

The QCD running coupling $\alpha_s(\mu^2) = g^2(\mu^2)/(4\pi)$ satisfies the renormalization group equation

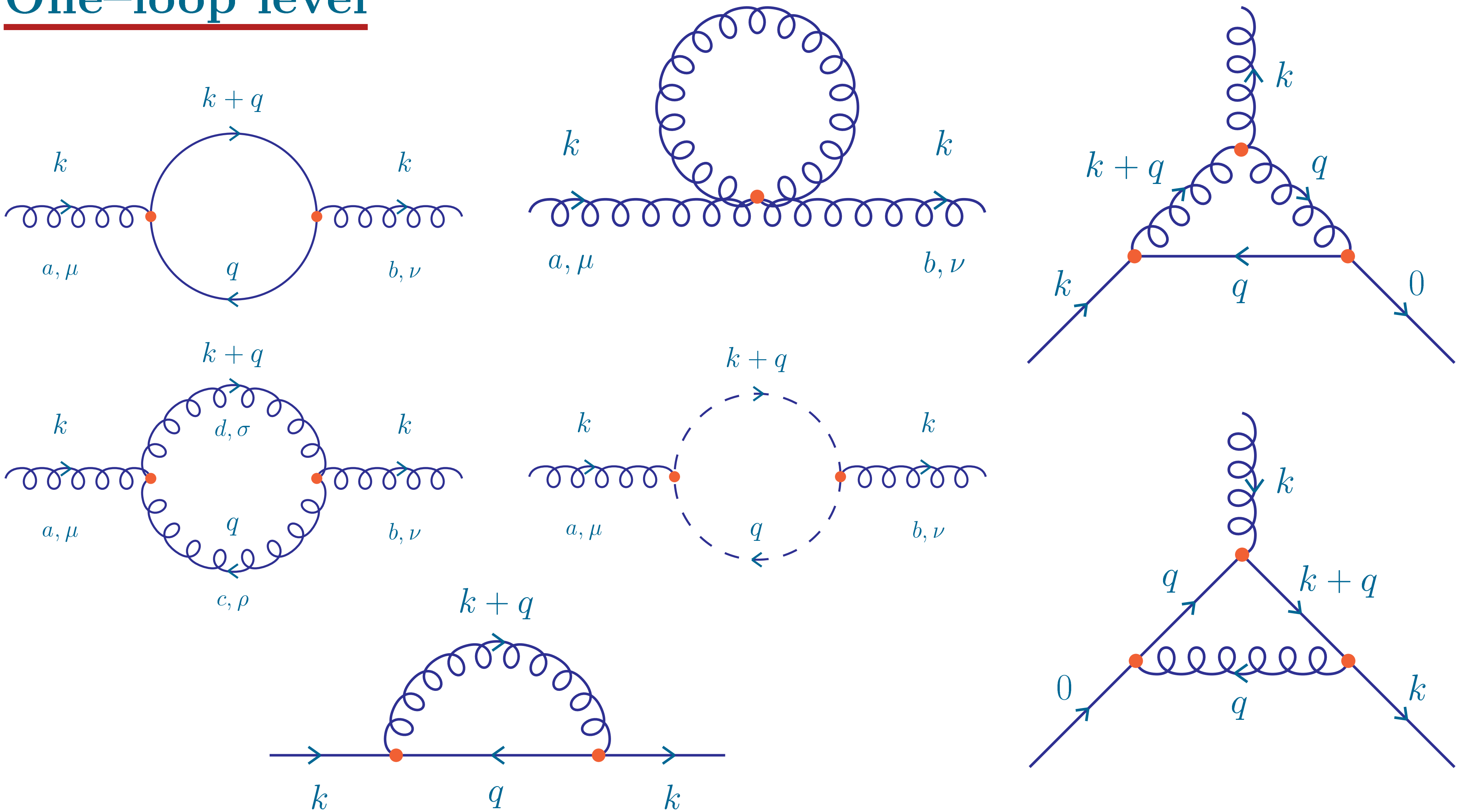
$$\frac{d\,\ln\left[g^2(\mu^2)\right]}{d\,\ln\mu^2} = \beta\bigl(g(\mu^2)\bigr).$$

Within perturbative approach, assuming $\alpha_s(\mu^2)$ being sufficiently small, one can approximate the β–function by

$$\underset{[\,g^2(\mu^2)\to 0\,]}{\beta\bigl(g(\mu^2)\bigr) \simeq \beta_{\text{pert}}\bigl(g(\mu^2)\bigr)} = -\left\{\beta_0\left[\frac{g^2(\mu^2)}{16\pi^2}\right] + \beta_1\left[\frac{g^2(\mu^2)}{16\pi^2}\right]^2 + \cdots\right\}.$$

The perturbative β–function is known up to four–loop level.

One–loop level



$$\beta_0 = 11 - \frac{2}{3} n_{\mathrm{f}}$$

't Hooft (1972); Gross, Wilczek (1973); Politzer (1973).

Two–loop level

$$\beta_1 = 102 - \frac{38}{3}\, n_{\mathrm{f}}$$

Caswell (1974); Jones (1974); Egorian, Tarasov (1979).

Three–loop level

$$\beta_2 = \frac{2857}{2} - \frac{5033}{18}\, n_{\mathrm{f}} + \frac{325}{54}\, n_{\mathrm{f}}^2$$

Tarasov, Vladimirov, Zharkov (1980); Larin, Vermaseren (1993).

Four–loop level

$$\beta_3 = \frac{149753}{6} + 3564\,\zeta(3) - \left[\frac{1078361}{162} + \frac{6508}{27}\,\zeta(3)\right] n_{\mathrm{f}} + \\ + \left[\frac{50065}{162} + \frac{6472}{81}\,\zeta(3)\right] n_{\mathrm{f}}^2 + \frac{1093}{729}\, n_{\mathrm{f}}^3$$

van Ritbergen, Vermaseren, Larin 1997; Chetyrkin, Kniehl, Steinhauser (1997).

In these equations n_{f} is the number of active flavours and $\zeta(x)$ denotes the Riemann ζ–function, $\zeta(3) \simeq 1.202$.

The one– and two–loop coefficients (β_0 and β_1) are scheme–independent, whereas the expressions given for β_2 and β_3 correspond to $\overline{\mathbf{MS}}$ scheme.

For practical purposes it is convenient to deal with the so–called "couplant" $a(\mu^2) \equiv \alpha(\mu^2)\beta_0/(4\pi)$. In this case the ℓ–loop renormalization group equation for QCD running coupling takes the following form:

$$\frac{d\,\ln\bigl[a_{\mathrm{s}}^{(\ell)}(\mu^2)\bigr]}{d\,\ln\mu^2} = -\sum_{j=0}^{\ell-1} B_j \bigl[a_{\mathrm{s}}^{(\ell)}(\mu^2)\bigr]^{j+1}, \qquad B_j = \frac{\beta_j}{\beta_0^{j+1}}.$$

One–loop level

The renormalization group equation for running coupling:

$$\frac{d\,\ln\left[a_{\mathrm{s}}^{(1)}(\mu^2)\right]}{d\,\ln\mu^2} = -B_0\, a_{\mathrm{s}}^{(1)}(\mu^2), \qquad B_0 = 1.$$

After the separation of variables

$$-\frac{d\,a_{\mathrm{s}}^{(1)}(\mu^2)}{\left[a_{\mathrm{s}}^{(1)}(\mu^2)\right]^2} = d\,\ln\mu^2$$

and integration of this result in finite limits, one arrives at

$$\frac{1}{a_{\mathrm{s}}^{(1)}(Q^2)} - \frac{1}{a_{\mathrm{s}}^{(1)}(Q_0^2)} = \ln\left(\frac{Q^2}{Q_0^2}\right).$$

This equation can be solved explicitly:

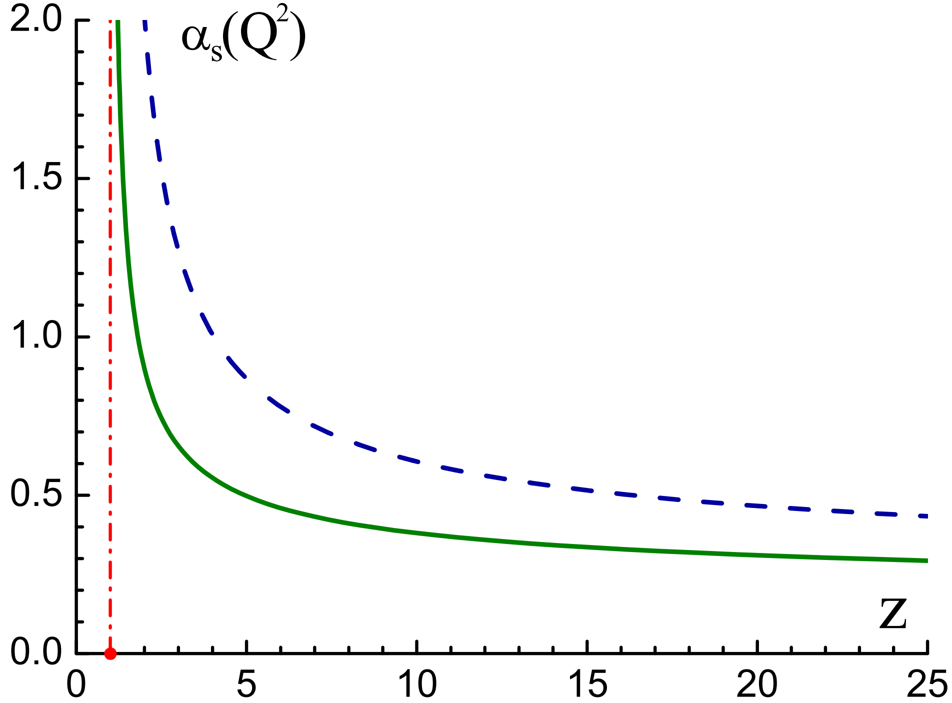
$$\alpha_{\mathrm{s}}^{(1)}(Q^2) = \frac{\alpha_{\mathrm{s}}^{(1)}(Q_0^2)}{1 + \alpha_{\mathrm{s}}^{(1)}(Q_0^2)\,\ln(Q^2/Q_0^2)\,\beta_0/(4\pi)}.$$

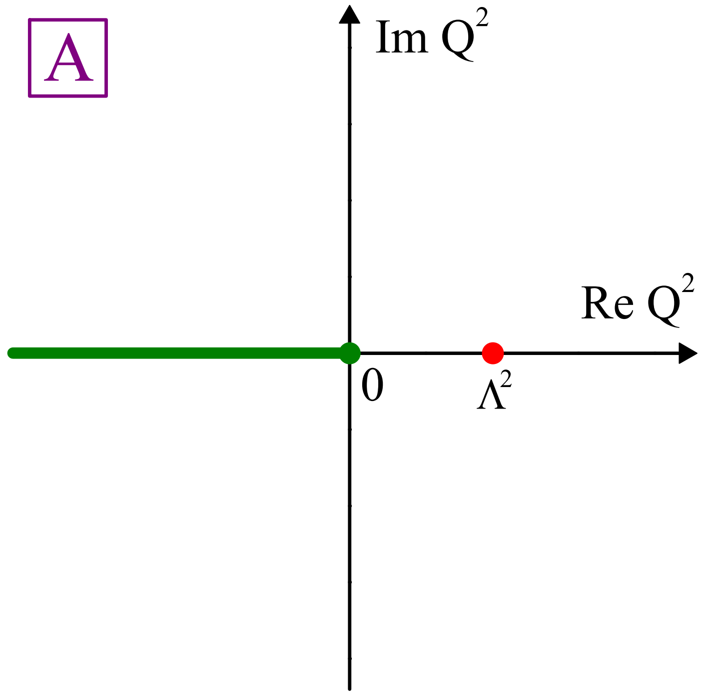
Equivalently, one can absorb all the dependence on $\alpha_s^{(1)}(Q_0^2)$ and Q_0^2 into the so–called QCD scale parameter:

$$\frac{1}{a_s^{(1)}(Q^2)} \equiv \ln\left(\frac{Q^2}{\Lambda^2}\right), \qquad \Lambda^2 = Q_0^2 \exp\left[-\frac{4\pi}{\beta_0}\frac{1}{\alpha_s^{(1)}(Q_0^2)}\right].$$

In this case the one–loop QCD running coupling reads

$$\alpha_s^{(1)}(Q^2) = \frac{4\pi}{\beta_0}\frac{1}{\ln z}, \qquad z = \frac{Q^2}{\Lambda^2}, \qquad Q^2 = -q^2 > 0.$$





Two–loop level

The renormalization group equation for running coupling:

$$\frac{d\,\ln\left[a_{\rm s}^{(2)}(\mu^2)\right]}{d\,\ln\mu^2} = -a_{\rm s}^{(2)}(\mu^2) - B_1\left[a_{\rm s}^{(2)}(\mu^2)\right]^2, \qquad B_0 = 1, \quad B_1 = \frac{\beta_1}{\beta_0^2}.$$

Similarly to the one–loop case, after separation of variables

$$-\frac{d\,a_{\rm s}^{(2)}(\mu^2)}{\left[a_{\rm s}^{(2)}(\mu^2)\right]^2\left[1 + B_1\,a_{\rm s}^{(2)}(\mu^2)\right]} = d\,\ln\mu^2$$

and integration of this result in finite limits, one arrives at

$$\frac{1}{a_{\rm s}^{(2)}(Q^2)} - B_1\ln\left[1 + \frac{1}{B_1\,a_{\rm s}^{(2)}(Q^2)}\right] = \ln\left(\frac{Q^2}{\Lambda^2}\right),$$

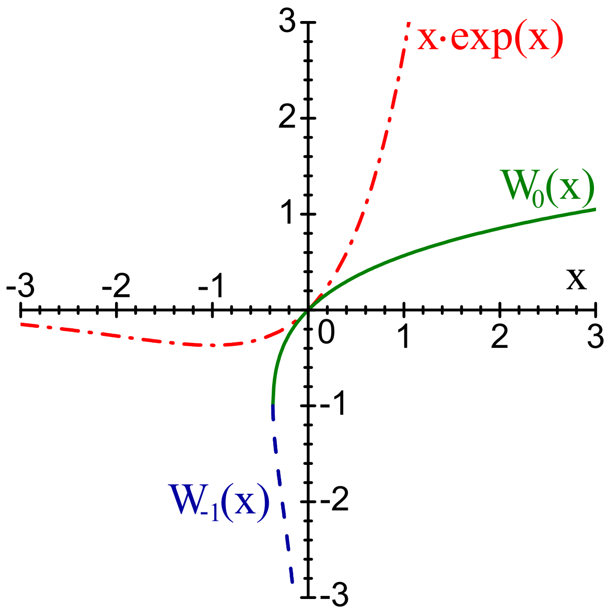
where

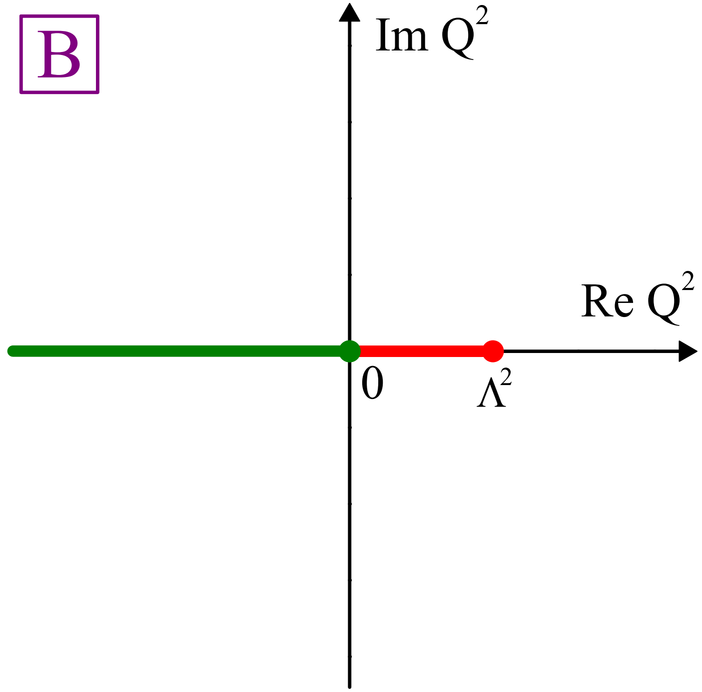
$$\Lambda^2 = \mu^2\exp\left\{-\frac{4\pi}{\beta_0}\frac{1}{\alpha_{\rm s}^{(2)}(\mu^2)} + B_1\ln\left[1 + \frac{1}{B_1}\frac{4\pi}{\beta_0}\frac{1}{\alpha_{\rm s}^{(2)}(\mu^2)}\right]\right\}.$$

The obtained equation for two–loop running coupling can be solved explicitly in terms of the Lambert W–function:

$$\alpha_{\rm s}^{(2)}(Q^2) = -\frac{4\pi}{\beta_0}\frac{1}{B_1}\frac{1}{1+W_{-1}\Bigl\{-\exp\bigl[-(1+B_1^{-1}\ln z)\bigr]\Bigr\}}, \qquad z=\frac{Q^2}{\Lambda^2}.$$

$$W_k(x)\,\exp\bigl[W_k(x)\bigr] = x$$





$$\alpha_{\rm s}^{(2)}(Q^2) \simeq \frac{4\pi}{\beta_0}\left[\frac{1}{\ln z} - B_1\frac{\ln(\ln z)}{\ln^2 z}\right], \qquad Q^2\to\infty.$$

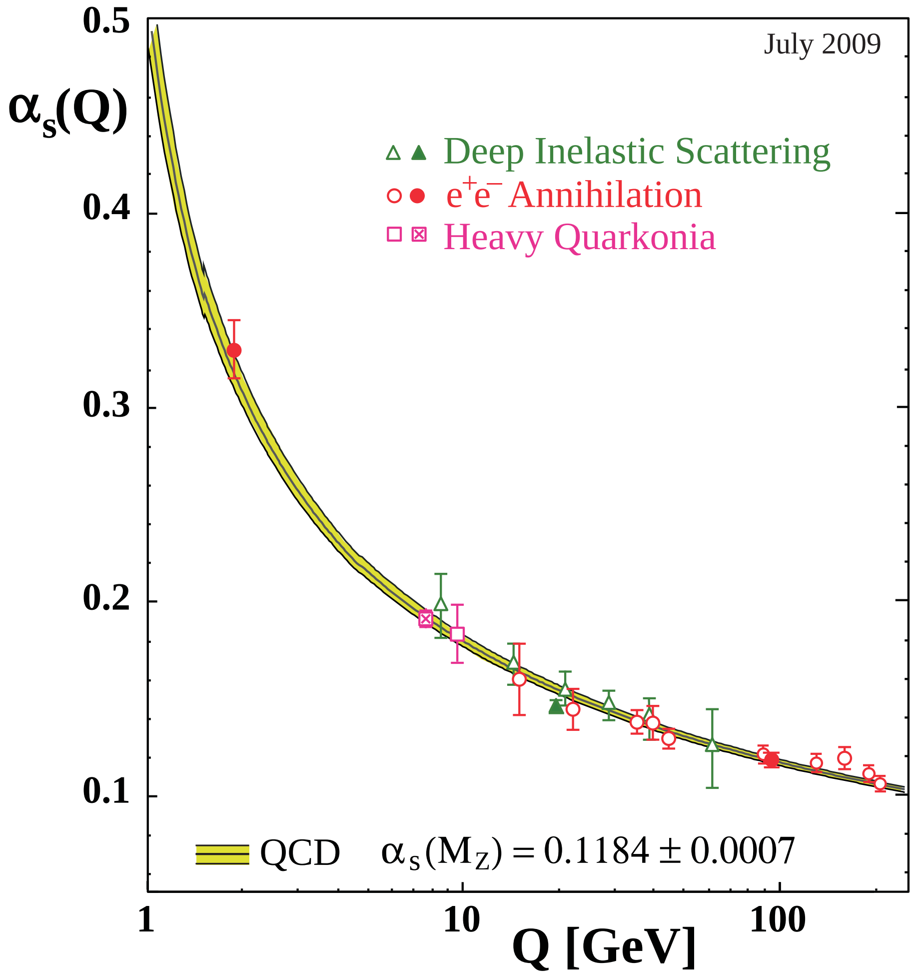
Three–loop level

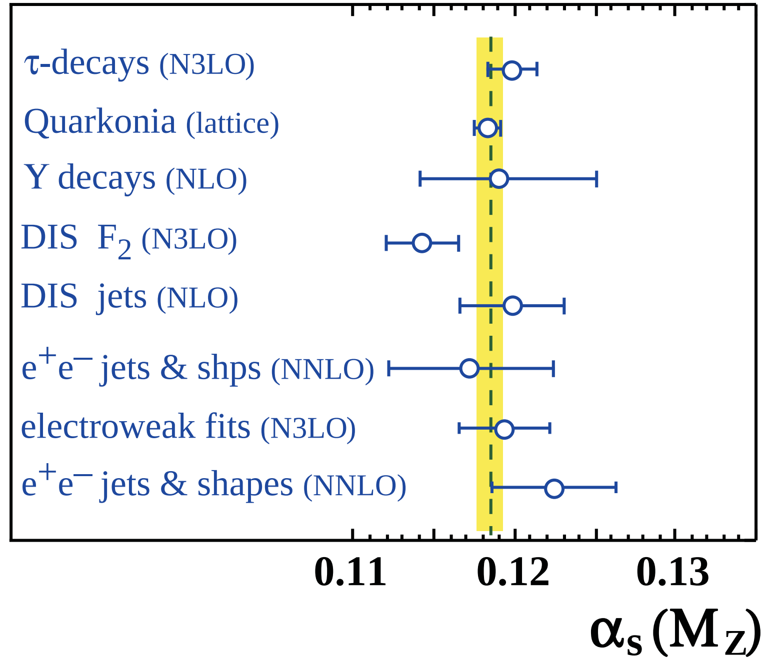
$$\alpha_{\mathrm{s}}^{(3)}(Q^2) \simeq \frac{4\pi}{\beta_0}\left\{\frac{1}{\ln z} - B_1\frac{\ln(\ln z)}{\ln^2 z} + \right.$$
$$\left. + \frac{1}{\ln^3 z}\left[B_1^2\Big(\ln^2(\ln z) - \ln(\ln z) - 1\Big) + B_2\right]\right\}.$$

Four–loop level

$$\alpha_{\mathrm{s}}^{(4)}(Q^2) \simeq \frac{4\pi}{\beta_0}\left\{\frac{1}{\ln z} - B_1\frac{\ln(\ln z)}{\ln^2 z} + \right.$$
$$+ \frac{1}{\ln^3 z}\left[B_1^2\Big(\ln^2(\ln z) - \ln(\ln z) - 1\Big) + B_2\right] +$$
$$+ \frac{1}{\ln^4 z}\left[B_1^3\left(-\ln^3(\ln z) + \frac{5}{2}\ln^2(\ln z) + 2\ln(\ln z) - \frac{1}{2}\right) - \right.$$
$$\left.\left. - 3B_1B_2\ln(\ln z) + \frac{B_3}{2}\right]\right\}.$$

QCD running coupling world average





$$M_Z = 91.188\,\mathrm{GeV}$$

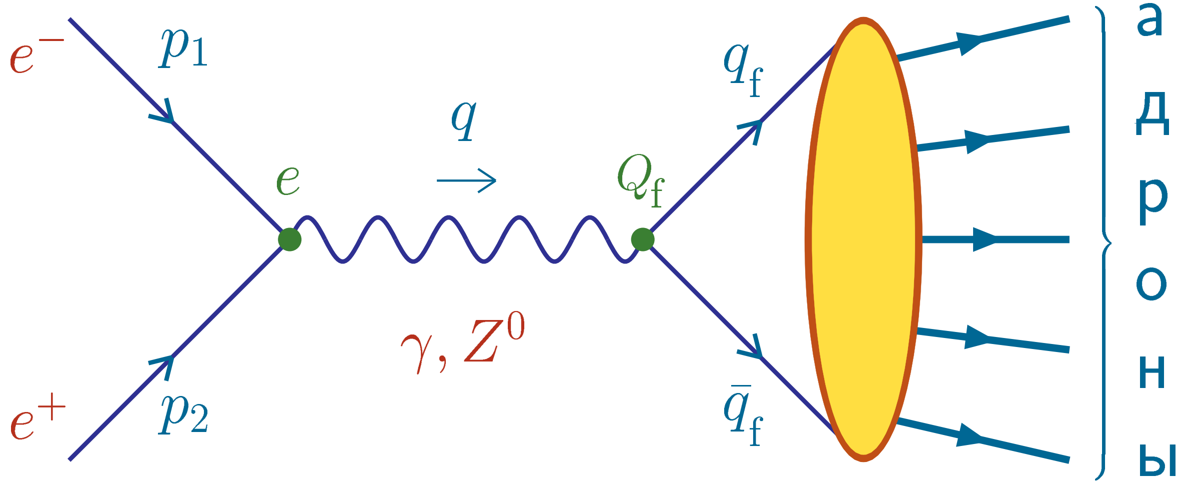
$$\alpha_s(M_Z) = 0.1184 \pm 0.0007$$

$$\Lambda = (213 \pm 9)\ \mathrm{MeV}\ \left[\, n_f = 5,\ \overline{\mathrm{MS}}\,\right]$$

Figures taken from: *S.Bethke, EPJC64, 689 (2009); arXiv:1110.0016 [hep-ph];*

See also Chap. 9 of *Review of particle physics, JPG37, 075021 (2010); http://pdg.lbl.gov*

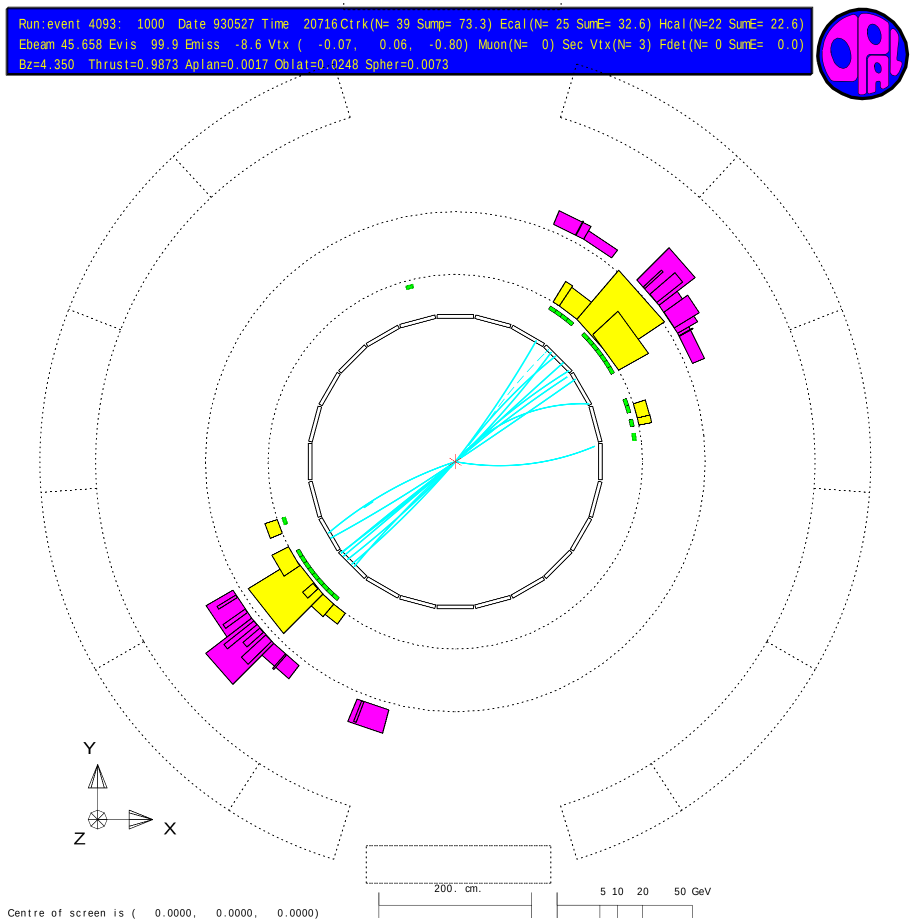
ELECTRON–POSITRON ANNIHILATION



- **No need in phenomenological hadronization models**
- **Tests of QCD and entire Standard Model**
- **Constraints on "New physics" beyond Standard Model**
- **Rather accurate experimental data**

DELPHI Collaboration

OPAL Collaboration



Cross–section of $e^+e^- \to$ hadrons

The corresponding Feynman amplitude reads

$$M_{\rm if} = i\,\bar{v}(p_2,\sigma_2)\,e\gamma^{\mu}\,u(p_1,\sigma_1)\,\frac{g_{\mu\nu}}{q^2}\,\langle\Gamma|J^{\nu}(q)|0\rangle,$$

where $q^2=(p_1+p_2)^2 \equiv s>0$ **and** $J_{\nu}=\sum_{f=1}^{n_f} :\bar{q}_f\,Q_f\gamma_{\nu}\,q_f:$.

The total cross–section takes the form

$$\sigma\bigl(e^+e^- \to \text{hadrons}; s\bigr) = 8\pi^2\,\frac{\alpha_{\rm em}^2}{s^3}\,L_{\mu\nu}H_{\mu\nu},$$

where

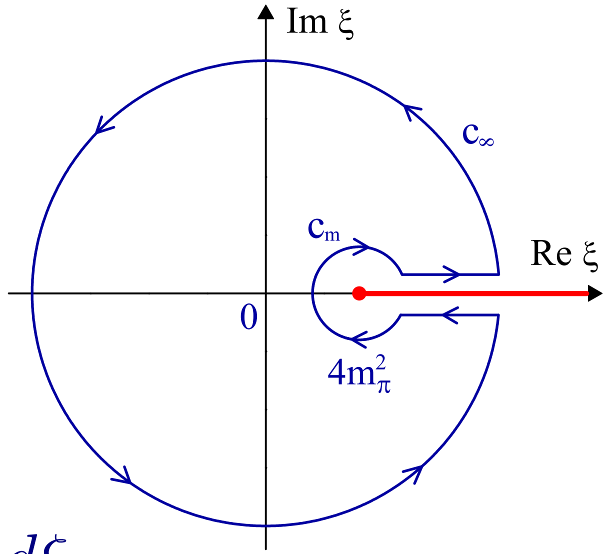
$$L_{\mu\nu} = \frac{1}{2}\Bigl[q_{\mu}q_{\nu} - g_{\mu\nu}q^2 - (p_1-p_2)_{\mu}(p_1-p_2)_{\nu}\Bigr],$$

$$H_{\mu\nu}(q^2) = (2\pi)^4\sum_{\Gamma}\delta(p_1+p_2-p_{\Gamma})\langle 0|J_{\mu}(-q)|\Gamma\rangle\langle\Gamma|J_{\nu}(q)|0\rangle.$$

The latter can be represented as $H_{\mu\nu}(q^2)=2\,{\rm Im}\,\Pi_{\mu\nu}(q^2)$**, with**

$$\Pi_{\mu\nu}(q^2) = i\int d^4x\,e^{iqx}\langle 0|\,T\bigl\{J_{\mu}(x)J_{\nu}(0)\bigr\}\,|0\rangle \equiv \frac{i}{12\pi^2}(q_{\mu}q_{\nu}-g_{\mu\nu}q^2)\Pi(q^2).$$

Hadronic vacuum polarization function



Cauchy integral formula:

$$f(x) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - x} d\xi.$$

Function $\Pi(q^2)$ has the only cut $q^2 \geq 4m_\pi^2$:

$$\Pi(q^2) = \Pi(q_0^2) + \frac{1}{2\pi i}(q^2 - q_0^2) \int_C \frac{\Pi(\xi)}{(\xi - q^2)(\xi - q_0^2)} d\xi.$$

Eventually, dispersion relation for $\Pi(q^2)$ acquires the form

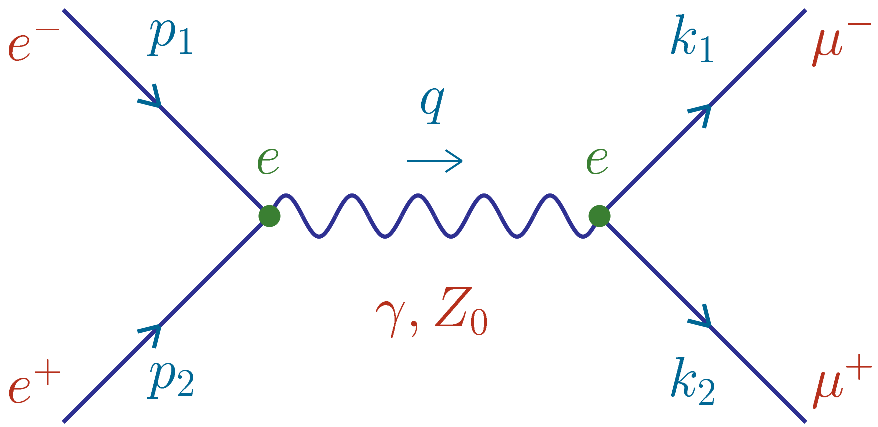
$$\Pi(q^2) = \Pi(q_0^2) + (q^2 - q_0^2) \int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s - q^2)(s - q_0^2)} ds,$$

where

$$R(s) \equiv \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[\Pi(s + i\varepsilon) - \Pi(s - i\varepsilon)\Big] = \frac{1}{\pi} \operatorname{Im} \lim_{\varepsilon \to 0_+} \Pi(s + i\varepsilon).$$

R–ratio of $e^+e^- \to$ hadrons

$$R(s) = \frac{\sigma\left(e^+e^- \to \text{hadrons}; s\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-; s\right)}$$



$$\sigma\left(e^+e^- \to \mu^+\mu^-; s\right) = \frac{4\pi}{3}\frac{\alpha_{\text{em}}^2}{s}$$

$$\alpha_{\text{em}} = \frac{e^2}{4\pi}$$

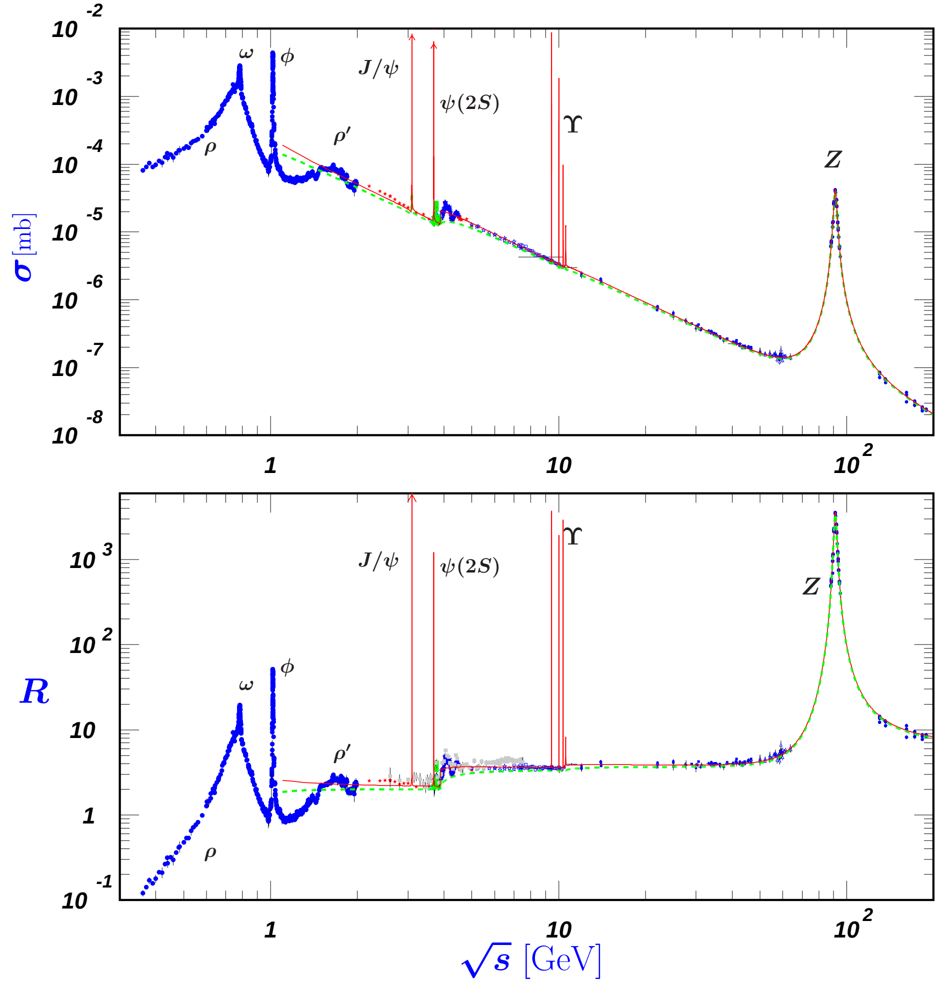


Figure taken from: ***Review of particle physics JPG37, 075021 (2010); http://pdg.lbl.gov***

$$R^{(0)}(s) = N_{\text{c}} \sum_{f=1}^{n_f} Q_f^2, \qquad s \to \infty.$$

Adler function

For practical purposes it proves to be convenient to deal with the Adler function

$$D(Q^2) = -\frac{d\,\Pi(-Q^2)}{d\ln Q^2}.$$

Dispersion relation for $D(Q^2)$ **has the following form**

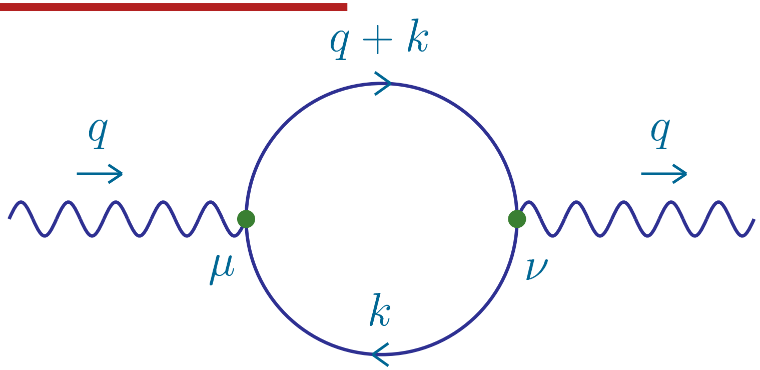
$$D(Q^2) = Q^2 \int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s+Q^2)^2}\, ds,$$

whereas inverse relation between $D(Q^2)$ **and** $R(s)$ **reads**

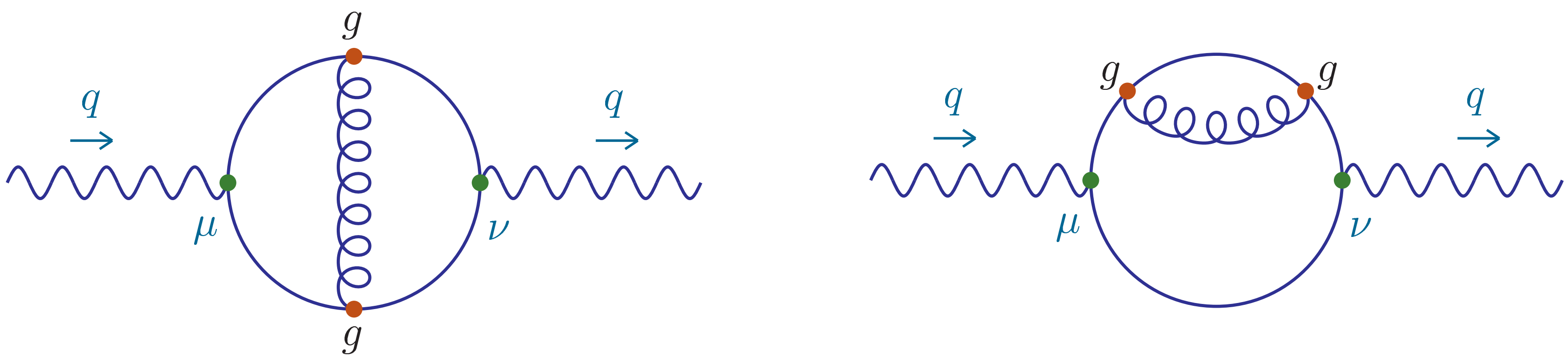
$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\, \frac{d\zeta}{\zeta}.$$

Adler (1974); Radyushkin (1982); Krasnikov, Pivovarov (1982)

Perturbative Adler function



$$\Pi^{(0)}(q^2) = -\,N_{\rm c}\sum_{f=1}^{n_f} Q_f^2\,\ln\left(\frac{-q^2}{\mu^2}\right) \quad \longrightarrow \quad D^{(0)}_{\rm pert}(Q^2) = N_{\rm c}\sum_{f=1}^{n_f} Q_f^2.$$

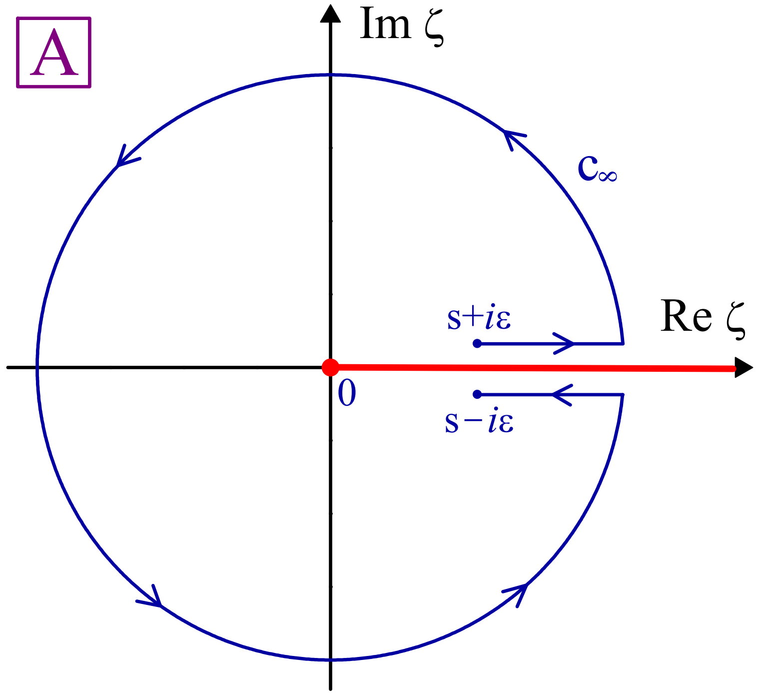


$$D^{(1)}_{\rm pert}(Q^2) = N_{\rm c}\sum_{f=1}^{n_f} Q_f^2\left[1+\frac{1}{\pi}\,\alpha_{\rm s}^{(1)}(Q^2)\right], \qquad \alpha_{\rm s}^{(1)}(Q^2) = \frac{4\pi}{\beta_0}\,\frac{1}{\ln(Q^2/\Lambda^2)}\,.$$

Theoretical prediction for R–ratio

In what follows massless limit will be assumed and the factor $N_c \sum_{f=1}^{n_f} Q_f^2$ will be omitted.

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta}.$$



$$D(Q^2) = 1 + d(Q^2)$$

$$\Downarrow$$

$$R(s) = 1 + r(s),$$

where

$$r(s) = \int_s^\infty \rho(\sigma) \frac{d\sigma}{\sigma}, \qquad \rho(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[d(-\sigma - i\varepsilon) - d(-\sigma + i\varepsilon) \Big].$$

Calculation of $D(Q^2)$ and $R(s)$ at the one–loop level

$$D^{(1)}_{\text{pert}}(Q^2) = 1 + \frac{1}{\pi}\alpha^{(1)}_{\text{s}}(Q^2), \qquad \alpha^{(1)}_{\text{s}}(Q^2) = \frac{4\pi}{\beta_0}\frac{1}{\ln(Q^2/\Lambda^2)}$$

$$\Downarrow$$

$$R^{(1)}_{\text{pert}}(s) = 1 + \frac{1}{\pi}\alpha^{(1)}_{\text{TL}}(s), \qquad \alpha^{(1)}_{\text{TL}}(s) = \frac{4\pi}{\beta_0}\left\{\frac{1}{2} - \frac{1}{\pi}\arctan\left[\frac{\ln(s/\Lambda^2)}{\pi}\right]\right\}.$$

B.Schrempp, F.Schrempp (1980); A.V.Radyushkin (1982).

In the ultraviolet asymptotic $s \to \infty$

$$a^{(1)}_{\text{TL}}(s) \simeq a^{(1)}_{\text{s}}(|s|) - \frac{\pi^2}{3}\frac{1}{\ln^3 w} + \mathcal{O}\left(\frac{1}{\ln^5 w}\right), \qquad w = \frac{s}{\Lambda^2},$$

hence

$$R^{(1)}_{\text{pert}}(s) \simeq 1 + \frac{1}{\pi}\alpha^{(1)}_{\text{s}}(|s|) + \mathcal{O}\left(\frac{1}{\ln^3 w}\right), \qquad s \to \infty.$$

Adler function and R–ratio at the ℓ–loop level

$$D^{(\ell)}_{\text{pert}}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left(\frac{4}{\beta_0}\right)^j \left[a_{\text{s}}^{(\ell)}(Q^2)\right]^j, \qquad Q^2 \to \infty$$

$$R^{(\ell)}_{\text{pert}}(s) = 1 + \sum_{j=1}^{\ell} r_j \left(\frac{4}{\beta_0}\right)^j \left[a_{\text{s}}^{(\ell)}(|s|)\right]^j, \qquad s \to \infty,$$

where $a_{\text{s}}^{(\ell)}(Q^2) = \alpha_{\text{s}}^{(\ell)}(Q^2)\,\beta_0/(4\pi)$ **and** $r_j = d_j - \delta_j$ **:**

$$\delta_1 = 0,$$

$$\delta_2 = 0,$$

$$\delta_3 = \frac{\pi^2}{3}\left(\frac{\beta_0}{4}\right)^2,$$

$$\delta_4 = \frac{\pi^2}{3}\left(\frac{\beta_0}{4}\right)^2\left(3d_2 + \frac{5}{8}\frac{\beta_1}{\beta_0}\right).$$

$$d_1 = 1,$$

$$d_2 = \frac{365}{24} - 11\,\zeta(3) + n_{\mathrm{f}}\left[-\frac{11}{12} + \frac{2}{3}\,\zeta(3)\right],$$

$$d_3 = \frac{87029}{288} - \frac{1103}{4}\,\zeta(3) + \frac{275}{6}\,\zeta(5) +$$

$$+\,n_{\mathrm{f}}\left[-\frac{7847}{216} + \frac{262}{9}\,\zeta(3) - \frac{25}{9}\,\zeta(5)\right] + n_{\mathrm{f}}^2\left[\frac{151}{162} - \frac{19}{27}\,\zeta(3)\right] +$$

$$+\left(\sum_{f=1}^{n_f} Q_f\right)^2\left[\frac{55}{72} - \frac{5}{3}\,\zeta(3)\right]\left(N_{\mathrm{c}}\sum_{f=1}^{n_f} Q_f^2\right)^{-1},$$

$$d_4 = \frac{144939499}{20736} - \frac{5693495}{864}\,\zeta(3) + \frac{5445}{8}\,\zeta^2(3) + \frac{65945}{288}\,\zeta(5) - \frac{7315}{48}\,\zeta(7) +$$

$$+ n_{\mathrm{f}}\left[-\frac{13044007}{10368} + \frac{12205}{12}\zeta(3) - 55\,\zeta^2(3) + \frac{29675}{432}\zeta(5) + \frac{665}{72}\zeta(7)\right] +$$

$$+ n_{\mathrm{f}}^2\left[\frac{1045381}{15552} - \frac{40655}{864}\,\zeta(3) + \frac{5}{6}\,\zeta^2(3) - \frac{260}{27}\,\zeta(5)\right] +$$

$$+ n_{\mathrm{f}}^3\left[-\frac{6131}{5832} + \frac{203}{324}\,\zeta(3) + \frac{5}{18}\,\zeta(5)\right]$$

[incomplete result]