Discrete spectrum of the three-particle Schrödinger operators.Efimov's effect

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Goal

To give some results (in particular Efimov's effect)for the two and three-particle lattice Hamiltonians in dimension $d = 3$ with emphasis on *new threshold phenomena* that are not present in the continuous case.

The free Hamiltonian $\hat{h}_0$ of a system of two identical quantum mechanical particles on the three dimensional lattice $\mathbb{Z}^3$

$$(\hat{h}^0\hat{\psi})(x_\beta, x_\gamma) = \frac{1}{2}\sum_{|s|=1}[2\hat{\psi}(x_\beta, x_\gamma) - \hat{\psi}(x_\beta + s, x_\gamma) - \hat{\psi}(x_\beta, x_\gamma + s)],$$

The Hamiltonian $\hat{h}_\mu$ of a system of two identical particles (bosons) interacting via a zero-range pair potential $\hat{v}$ is associated with the following self-adjoint operator

$$\hat{h}_\mu = \hat{h}_0 - \hat{v},$$

$$(\hat{v}\hat{\psi})(x_\beta, x_\gamma) = \mu\delta_{x_\beta, x_\gamma}\hat{\psi}(x_\beta, x_\gamma), \quad \hat{\psi} \in \ell_2^{(s)}((\mathbb{Z}^3)^2).$$

The free Hamiltonian $\widehat{H}_0$ of a system of three identical particles(bosons) on the lattice $\mathbb{Z}^3$ is

$$(\widehat{H}_0\hat{\psi})(x_1,x_2,x_3) = \frac{1}{2}\sum_{|s|=1}[3\hat{\psi}(x_1,x_2,x_3) - \hat{\psi}(x_1+s,x_2,x_3) - \hat{\psi}(x_1,x_2+s,x_3) - \hat{\psi}(x_1,x_2,x_3+s)], \hat{\psi}\in \ell_2^{(s)}((\mathbb{Z}^3)^3).$$

The Hamiltonian $\widehat{H}$ of a system of three identical particles with the pair zero-range interaction $\hat{v}=\hat{v}_\alpha=\hat{v}_{\beta\gamma}=\mu, \alpha,\beta,\gamma=1,2,3$ is a bounded perturbation of H_0

$$\widehat{H} = \widehat{H}_0 - \widehat{V}_1 - \widehat{V}_2 - \widehat{V}_3, \qquad (1)$$

where $\widehat{V}_\alpha=\widehat{V}, \alpha=1,2,3$ is multiplication operator:

$$(\widehat{V}\hat{\psi})(x_\alpha,x_\beta,x_\gamma) = \mu\delta_{x_\beta x_\gamma}\psi(x_\alpha,x_\beta,x_\gamma), \hat{\psi}\in \ell_2^{(s)}((\mathbb{Z}^3)^3).$$

Let

$$\mathcal{F}_m : L_2((\mathbb{T}^3)^m) \to \ell_2((\mathbb{Z}^3)^m)$$

be Fourier trans.
The two-resp. three-particle Hamiltonians in the momentum representation are given on $L_2^{(s)}((\mathbb{T}^3)^2)$ resp. $L_2^{(s)}((\mathbb{T}^3)^3)$

$$h = (\mathcal{F}_2^s)^{-1}\hat{h}_\mu \mathcal{F}_2^s, \quad \text{resp.} H_\mu = (\mathcal{F}_3^s)^{-1}\widehat{H}\mathcal{F}_3^s.$$

The two-particle Hamiltonian h_μ is of the form

$$h_\mu = h_0 - v.$$

The operator h_0 is the multiplication operator

$$(h_0 f)(k_\beta, k_\gamma) = (\varepsilon(k_\beta) + \varepsilon(k_\gamma)) f(k_\beta, k_\gamma),\ \ f \in L_2^{(s)}((\mathbb{T}^3)^2),$$

where $k_\alpha, \alpha = 1, 2, 3$ is the *quasi-momentum* of the particle α.

The integral operator v is of convolution type

$$(vf)(k_\beta, k_\gamma) = \frac{\mu}{(2\pi)^{\frac{3}{2}}} \int_{(\mathbb{T}^3)^2} \delta(k_\beta + k_\gamma - k'_\beta - k'_\gamma) f(k'_\beta, k'_\gamma) dk'_\beta dk'_\gamma,$$

where $\delta(\cdot)$ is the three-dimensional Dirac delta-function. The function $\varepsilon(k)$ is given by the Fourier series

$$\varepsilon(k) = \sum_{j=1}^{3} (1 - cos k^{(j)}).$$

The three-particle Hamiltonian H_μ in the momentum representation is given on $L_2^{(s)}((\mathbb{T}^3)^3)$ and is of the form

$$H_\mu = H_0 - V_1 - V_2 - V_3,$$

where H_0 is the multiplication operator:

$$(H_0 f)(k_1, k_2, k_3) = [\sum_{\alpha=1}^{3} \varepsilon(k_\alpha)] f(k_1, k_2, k_3),$$

and V_α is partial integral operator of convolution type

$$(V_\alpha f)(k_\alpha, k_\beta, k_\gamma) = (Vf)(k_\alpha, k_\beta, k_\gamma)$$
$$= \frac{\mu}{(2\pi)^3} \int\limits_{(\mathbb{T}^3)^3} \delta(k_\alpha - k'_\alpha)\delta(k_\beta + k_\gamma - k'_\beta - k'_\gamma) \quad f(k'_\alpha, k'_\beta, k'_\gamma) dk'_\alpha dk'_\beta dk'_\gamma,$$

Note that V is not compact.

Denote by $k = k_1 + k_2 \in \mathbb{T}^3$ resp.$K = k_1 + k_2 + k_3 \in \mathbb{T}^3$ the *two-particle* resp. *three-particle quasi-momentum* and define $\mathbb{F}^2_k$ resp. $\mathbb{F}^3_K$ as follows

$$\mathbb{F}^2_k = \{(k_1, k - k_1) \in (\mathbb{T}^3)^2 : k_1 \in \mathbb{T}^3, k - k_1 \in \mathbb{T}^3\}.$$

$$\mathbb{F}^3_K = \{(k_1, k_2) \in (\mathbb{T}^3)^2 : k_1, k_2 \in \mathbb{T}^3, K - k_1 - k_2 \in \mathbb{T}^3\}.$$

The h and H can be decomposed into the direct integrals

$$h = \int\limits_{k \in \mathbb{T}^3} \oplus \tilde{h}(k) dk \quad H = \int\limits_{K \in \mathbb{T}^3} \oplus \widetilde{H}(K) dK \tag{2}$$

with respect to the decompositions

$$L_2^{(s)}((\mathbb{T}^3)^2) = \int\limits_{k \in \mathbb{T}^3} \oplus L_2^{(s)}(\mathbb{F}_k^2) dk,$$

$$L_2^{(s)}((\mathbb{T}^3)^3) = \int\limits_{K \in \mathbb{T}^3} \oplus L_2^{(s)}(\mathbb{F}_K^3) dK.$$

We introduce the mapping(projector)

$$\pi^{(2)} : (\mathbb{T}^3)^2 \to \mathbb{T}^3, \quad \pi^{(2)}((k_\beta, k_\gamma)) = k_\beta$$

resp.

$$\pi^{(3)} : (\mathbb{T}^3)^3 \to (\mathbb{T}^3)^2, \quad \pi^{(3)}((k_\alpha, k_\beta, k_\gamma)) = (k_\alpha, k_\beta).$$

Denote by $\pi_k^{(2)}$, $k \in \mathbb{T}^3$ resp. $\pi_K^{(3)}$, $K \in \mathbb{T}^3$ the restriction of $\pi^{(2)}$ resp. $\pi^{(3)}$ onto $\mathbb{F}_k^2 \subset (\mathbb{T}^3)^2$ resp. $\mathbb{F}_K^3 \subset (\mathbb{T}^3)^3$, i.e.,

$$\pi_k^{(2)} = \pi^{(2)}|_{\mathbb{F}_k^2} \quad \text{resp.} \quad \pi_K^{(3)} = \pi^{(3)}|_{\mathbb{F}_K^3}.$$

Note that $\mathbb{F}_k^2$, $k \in \mathbb{T}^3$ resp. $\mathbb{F}_K^3$, $K \in \mathbb{T}^3$ is three resp. six-dimensional manifolds isomorphic to $\mathbb{T}^3$ resp. $(\mathbb{T}^3)^2$.

Lemma

The mapping $\pi_k^{(2)}$, $k \in \mathbb{T}^3$ *resp.* $\pi_K^{(3)}$, $K \in \mathbb{T}^3$ *are bijective from* $\mathbb{F}_k^2 \subset (\mathbb{T}^3)^2$ *resp.* $\mathbb{F}_K^3 \subset (\mathbb{T}^3)^3$ *onto* $\mathbb{T}^3$ *resp.* $(\mathbb{T}^3)^2$ *with the inverse mapping given by*

$$(\pi_k^{(2)})^{-1}(k_\beta) = (k_\beta, k - k_\beta)$$

resp.

$$(\pi_K^{(3)})^{-1}(k_\alpha, k_\beta) = (k_\alpha, k_\beta, K - k_\alpha - k_\beta).$$

Let $L_2^e(\mathbb{T}^3) \subset L_2(\mathbb{T}^3)$. The fiber operators $\tilde{h}(k)$, $k \in \mathbb{T}^3$ are unitarily equivalent to the operators $h(k)$, $k \in \mathbb{T}^3$, of the form

$$h(k) = h_0(k) - v. \tag{3}$$

The operators $h_0(k)$ and v acts on the Hilbert space $L_2^e(\mathbb{T}^3)$:

$$(h_0(k)f)(p) = \mathcal{E}_k(p)f(p), \quad f \in L_2^e(\mathbb{T}^3),$$

where

$$\mathcal{E}_k(q) = \varepsilon(\frac{k}{2} + q) + \varepsilon(\frac{k}{2} - q) = 2\sum_{i=1}^{d}[1 - \cos(\frac{K_j}{2})\cos q_j]$$

and

$$(vf)(q) = \frac{\mu}{(2\pi)^3}\int_{\mathbb{T}^3} f(q')dq', \quad f \in L_2^e(\mathbb{T}^3).$$

The fiber operators $\widetilde{H}(K),\ K \in \mathbb{T}^3$ from the direct integral decomposition are unitarily equivalent to the operators $H(K)$:

$$H(K) = H_0(K) - V_1 - V_2 - V_3.$$

The operators $H_0(K)$ and $V_\alpha \equiv V$, $\alpha = 1, 2, 3$, acts on the Hilbert space

$$L_2^e((\mathbb{T}^3)^2) \cong L_2(\mathbb{T}^3) \otimes L_2^e(\mathbb{T}^3)$$

and in the coordinates $(k_\alpha, k_\beta) \in (\mathbb{T}^3)^2$ have form

$$(H_0(K)f)(k_\alpha, k_\beta) = E(K; k_\alpha, k_\beta)f(k_\alpha, k_\beta), \quad f \in L_2^e((\mathbb{T}^3)^2),$$

The fiber operators.

$$E(K;k_\alpha,k_\beta)=\varepsilon(K-k_\alpha)+\varepsilon(\frac{k_\alpha}{2}-k_\beta)+\varepsilon(\frac{k_\alpha}{2}+k_\beta)$$

and

$$V=I\otimes v,$$

where $\otimes-$ is the tensor product.

Since the particles are identical we have only one channel operator $H_{ch}(K), K \in \mathbb{T}^3$ acting in the Hilbert space $L_2^e((\mathbb{T}^3)^2) \cong L_2(\mathbb{T}^3) \otimes L_2^e(\mathbb{T}^3)$ as

$$H_{ch}(K) = H_0(K) - V,$$

where $H_0(K)$ resp. V is mult. resp. part. int.oper.
The decomposition of the space $L_2^e((\mathbb{T}^3)^2)$ into the direct integral

$$L_2^e((\mathbb{T}^3)^2) = \int\limits_{k \in \mathbb{T}^3} \oplus L_2^e(\mathbb{T}^3) dk$$

yields for the operator $H_{ch}(K)$ the decomposition into the direct integral

$$H_{ch}(K) = \int\limits_{k \in \mathbb{T}^3} \oplus H_{ch}(K, k) dk.$$

The fiber operator $H_{ch}(K,k)$ has the form

$$H_{ch}(K,k) = \varepsilon(K-k)I + h_\mu(k),$$

where I is identity op-r and $h_\mu(k)$ is the two-particle op-r.
Denote by

$$\tau_\mu(K,k) = \varepsilon(K-k) + z_\mu(k),$$

where $z_\mu(k)$ is the unique eigenvalue of $h_\mu(k)$
The representation of the $H_{ch}(K,k)$ implies

$$\sigma(H_{ch}(K,k)) = \tau_\mu(K,k) \cup \left[E_{\min}(K), E_{\max}(K)\right].$$

Lemma

The following equality holds

$$\sigma_{ess}(H_\mu(K)) = \cup_k \tau_\mu(K,k) \cup [E_{min}(K), E_{max}(K)].$$

Theorem

The following equality holds

$$\sigma(H_{ch}(K)) = \sigma_{ess}(H_\mu(K)).$$

By Weyl's theorem the ess.spectrum coincides with $\sigma(h_0(k))$ of $h_0(k)$, i.e.,

$$\sigma_{\text{ess}}(h(k)) = [\mathcal{E}_{\min}(k), \mathcal{E}_{\max}(k)],$$

where

$$\mathcal{E}_{\min}(k) \equiv \min_{p\in\mathbb{T}^3} \mathcal{E}_k(p), \quad \mathcal{E}_{\max}(k) \equiv \max_{p\in\mathbb{T}^3} \mathcal{E}_k(p)$$

Let $r_0(k,z)$ be resolvent of $h_0(k)$.
For any $k \in \mathbb{T}^3, z < \varepsilon_{\min}(k)$ Fredholm's determinant of $h_\mu(k)$

$$\Delta_\mu(k,z) = 1 - \frac{\mu}{(2\pi)^3} \int_{\mathbb{T}^3} (\mathcal{E}_k(q) - z)^{-1} dq. \tag{4}$$

Lemma

Let $k \in \mathbb{T}^3$. The number $z < \varepsilon_{\min}(k)$ is an eigenvalue of the operator $h_\mu(k)$ if and only if

$$\Delta_\mu(k, z) = 0.$$

Let $d \geq 3$. We introduce the parameter $0 < \eta(\mathcal{E}_0) < \infty$ as

$$\eta(\mathcal{E}_0) = [\frac{1}{(2\pi)^d} \int\limits_{\mathbb{T}^d} \frac{dq}{\mathcal{E}_0(q)})]^{-1}.$$

Lemma

The following statements are equivalent:
(i) the operator $h_\mu(0)$ *has a zero energy resonance;*
(ii) $\Delta_\mu(0,0)=0$;
(iii) $\mu=\eta(\mathcal{E}_0)$.

Remark

Remark that if $\Delta_{\eta(\mathcal{E}_0)}(0,0)=0$, *then the equation* $h_{\eta(\mathcal{E}_0)}(0)f=0$ *has a solution*

$$f(p)=\frac{const}{\mathcal{E}_0(p)}\in L_e^1(\mathbb{T}^d)\setminus L_e^2(\mathbb{T}^d).$$

where $L_e^1(\mathbb{T}^d)$ *is the Banach space of integrable functions.*

Definition

The operator $h_{\eta(\mathcal{E}_{min}(0))}(0)$ is said to have a virtual level (zero energy resonance) if $\Delta_{\eta(\mathcal{E}_{min}(0))}(0,0)=0$. We call that the point $z=0$ is regular point of the essential spectrum of $h_{\eta(\mathcal{E}_{min}(0))}(0)$ if $\Delta_{\eta(\mathcal{E}_{min}(0))}(0,0)\neq 0$.

Theorem

(i)Let $\mu < \eta(\mathcal{E}_0)$*. Then the operator* $h_\mu(0)$ *has non eigenvalue lying below the essential spectrum and* $z = 0$ *is regular point of the essential spectrum of* $h_{\eta(\mathcal{E}_{min}(0))}(0)$.

(ii)Let $\mu = \eta(\mathcal{E}_0)$*. Then the operator* $h_{\eta(\mathcal{E}_0)}(0)$ *has a zero energy resonance* $z_{\eta(\mathcal{E}_0)}(0) = 0$ *and for all* $k \in \mathbb{T}^3_0 = \mathbb{T}^3 \setminus \{0\}$ *the* $h_{\eta(\mathcal{E}_0)}(k)$ *has a unique eigenvalue* $z_{\eta(\mathcal{E}_0)}(k)$*, such that* $0 < z_{\eta(\mathcal{E}_0)}(k) < \mathcal{E}_{\min}(k)$.

(iii) For any $\mu > \eta(\mathcal{E}_0)$ the point $z = 0$ is regular point of the essential spectrum of $h_{\eta(\mathcal{E}_{min}(0))}(0)$ and for all $k \in \mathbb{T}^3$ the $h_\mu(k)$ has a unique eigenvalue $z_\mu(k)$ lying below the essential spectrum. Moreover $z_\mu(k)$ is even, analytic in $\mathbb{T}^3$,

$$z_\mu(k) < z_{\eta(\mathcal{E}_0)}(k),$$
$$z_\mu(k) = -\mu + O(1), \mu \to +\infty.$$

Remark

In the case (i) it may exists a region $G \subset \mathbb{T}^3$ and for $k \in G$ the operator $h_\mu(k)$ has an eigenvalue below the bottom of the essential spectrum.

Set.

$$E_{\min}(K) = \min_{p,q\in\mathbb{T}^3} E(K,p,q),$$

$$E_{max}(K) = \max_{p,q\in\mathbb{T}^3} E(K,p,q).$$

$$\tau_{\mu,\inf}(K) = \inf_{k\in\mathbb{T}^3}[z_\mu(k) + \varepsilon(K-k)].$$

$$\tau_{\mu,\sup}(K) = \sup_{k\in\mathbb{T}^3}[z_\mu(k) + \varepsilon(K-k)]$$

The essential spectrum of $H_\mu(K), K \in \mathbb{T}^3$ described by

Theorem

For the essential spectrum $\sigma_{ess}(H_\mu(K))$ of $H_\mu(K)$ the equality holds

$$\sigma_{ess}(H_\mu(K)) = \cup_{k\in\mathbb{T}^3}\tau_\mu(K,k) \cup [E_{min}(K), E_{max}(K)].$$

Remark

The ess. spec. of $H_{\eta(\mathcal{E}_0)}(K)$ coincides with the segment

$$\sigma_{ess}(H_{\eta(\mathcal{E}_0)}(K)) = [\tau_{\eta(\mathcal{E}_0),\mathrm{inf}}(K), E_{max}(K)].$$

Moreover

$$E_{\min}(0) = \tau_{\eta(\mathcal{E}_0),\mathrm{inf}}(0) = 0$$

and for any $0 \neq K \in \mathbb{T}_0^3$ the relations

$$E_{min}(K) > \tau_{\eta(\mathcal{E}_0),\mathrm{inf}}(K) > 0$$

are hold.
Thus the two-part. ess. spec. of $H_{\eta(\mathcal{E}_0)}(0)$ below the bottom of the three-part. ess. spec. is empty set and for any $K \neq 0$ the operator $H_{\eta(\mathcal{E}_0)}(K)$ has nonempty two-particle negative ess.

Remark

Since

$$z_{\mu}(k) = -\mu + O(1), \mu \to +\infty,$$
$$\tau_{\mu,\mathrm{inf}}(K) = -\mu + O(1), \mu \to +\infty,$$
$$\tau_{\mu,\mathrm{sup}}(K) = -\mu + O(1), \mu \to +\infty.$$

For sufficiently large $\mu > 0$ the essential spectrum of the operator $H_{\mu}(K)$ consists of two different segments

$$\sigma_{ess,two} = [\tau_{\mu,\mathrm{inf}}(K), \tau_{\mu,\mathrm{sup}}(K)] = \cup_{k \in \mathbb{T}^3} \tau_{\mu}(K,k),$$
$$\sigma_{ess,three}(H_{\mu}(K)) = [E_{min}(K), E_{max}(K)].$$

We denote by $N_\mu(K,z)$ the number of eigenvalues of $H_\mu(K)$ below $z \le \tau_\mu(K)$.

Theorem

The operator $H_{\eta(\mathcal{E}_0)}(0)$ has infinitely many eigenvalues lying below the bottom $\tau_{\eta(\mathcal{E}_0)}(0) = 0$ of the ess. spec.and the func. $N_{\eta(\mathcal{E}_0)}(0,z)$ obeys the relation

$$\lim_{z\to -0} \frac{N_{\eta(\mathcal{E}_0)}(0,z)}{|\log|z||} = \frac{\lambda_0}{2\pi}, \tag{5}$$

where λ_0 is the unique positive solution of the equation

$$\lambda = \frac{8\sinh \pi\lambda/6}{\sqrt{3}\cosh \pi\lambda/2}. \tag{6}$$

Theorem

For all $K \in U_\delta^0(0)$ the number $N_{\eta(\mathcal{E}_0)}(K,0)$ is finite and the following asymptotics holds

$$\lim_{|K|\to 0} \frac{N_{\eta(\mathcal{E}_0)}(K,0)}{|\log|K||} = 2(\frac{\lambda_0}{2\pi}). \tag{7}$$

Remark

For any $\mu < \eta(\mathcal{E}_0)$ the equality holds

$$\sigma_{ess}(H_\mu(K)) = [E_{min}(K), E_{max}(K)]$$

and the operator $H_\mu(K)$ has finitely many eigenvalues outside of ess.spec.

Remark

For any $\mu > \eta(\mathcal{E}_0)$ the equality

$$\sigma_{ess}(H_\mu(K)) = [\tau_{\mu,\inf}(K), \tau_{\mu,\sup}(K)] \cup [E_{min}(K), E_{max}(K)].$$

holds, where $\tau_\mu(K) < E_{min}(K)$. In this case the three-particle operator has non-empty two-particle essential. spec. and

$$N_\mu(K, \tau(K)) < \infty.$$

But in the gap $(\tau_{\mu,\sup}(K), E_{min}(K))$ the operator $H_\mu(K)$ may have infinitely many eigenvalues, which cannot be in the continuous operators.

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