INTRODUCTION TO MODIFIED GRAVITY AND GRAVITATIONAL ALTERNATIVE FOR DARK ENERGY

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I. Introduction

The modified gravity approach is extremely attractive in the applications for late accelerating universe and dark energy. Indeed,
1. Modified gravity provides the very natural gravitational alternative for dark energy. The cosmic speed-up is explained simply by the fact of the universe expansion where some sub-dominant terms (like $1/R$) may become essential at small curvature.
2. Modified gravity presents very natural unification of the early-time inflation and late-time acceleration thanks to different role of gravitational terms relevant at small and at large curvature. Moreover, some models of modified gravity are predicted by string/M-theory considerations.
3. It may serve as the basis for unified explanation of dark energy and dark matter. Some cosmological effects (like galaxies rotation curves) may be explained in frames of modified gravity.

4. Assuming that universe is entering the phantom phase, modified gravity may naturally describe the transition from non-phantom phase to phantom one without necessity to introduce the exotic matter (like the scalar with wrong sign kinetic term or ideal fluid with EoS parameter less than -1). In addition, often the phantom phase in modified gravity is transient. Hence, no future Big Rip is usually expected there.
5. Modified gravity quite naturally describes the transition from decceleration to acceleration in the universe evolution.
6. The effective dark energy dominance may be assisted by the modification of gravity. Hence, the coincidence problem is solved there simply by the fact of the universe expansion.
7. Modified gravity is expected to be useful in high energy physics (for instance, for the explanation of hierarchy problem or unification of GUTs with gravity).
8. Despite quite stringent constraints from Solar System tests, there are versions of modified gravity which may be viable theories competing with General Relativity at current epoch.

II. The modified $f(R)$ gravity

Let us start from the rather general 4-dimensional action:

$$S = \int d^4x \sqrt{-g} \left\{ f(R) + L_m \right\} \ . \tag{1}$$

Here R is the scalar curvature, $f(R)$ is an arbitrary function and L_m is a matter Lagrangian density. The equation of the motion is given by

$$0 = \frac{1}{2} g_{\mu\nu} f(R) - R_{\mu\nu} f'(R) - \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \nabla^2 f'(R) + \frac{1}{2} T_{\mu\nu} \ . \tag{2}$$

With no matter and for the Ricci tensor $R_{\mu\nu}$ being covariantly constant, the equation of motion corresponding to the action (1) is:

$$0 = 2f(R) - Rf'(R) \ , \tag{3}$$

which is the algebraic equation with respect to R. If the solution of Eq.(3) is positive, it expresses deSitter universe and if negative, anti-deSitter universe.

In the following, the metric is assumed to be in the FRW form:

$$ds^2 = -dt^2 + \hat{a}(t)^2 \sum_{i=1}^{3} \left(dx^i\right)^2 \ . \tag{4}$$

Here we assume that the spatial part is flat as suggested by the observation of the Cosmic Microwave Background (CMB) radiation. Without the matter and in FRW background, Eq.(2) gives

$$0 = -\frac{1}{2}f(R) + 3\left(H^2 + \dot{H}\right) f'(R) - 6\frac{\dot{H}}{H} f''(R) - 18H^2 \frac{d}{dt}\left(\frac{\dot{H}}{H^2}\right) f''(R) \, . \tag{5}$$

Here R is given by $R = 12H^2 + 6\dot{H}$. Our main purpose is to look for accelerating cosmological solutions of the following form: de Sitter (dS) space, where H is constant and $a(t) \propto \mathrm{e}^{Ht}$, quintessence and phantom like cosmologies:

$$a = \left\{ \begin{array}{ll} a_0 t^{h_0}, & \text{when } h_0 > 0 \text{ (quintessence)} \\ a_0 \left(t_s - t\right)^{h_0}, & \text{when } h_0 < 0 \text{ (phantom)} \end{array} \right. \ . \tag{6}$$

Introducing the auxiliary fields, A and B, one can rewrite the action (1) as follows:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{\kappa^2} \left\{ B \left(R - A \right) + f(A) \right\} + \mathcal{L}_{\mathrm{matter}} \right] . \qquad (7)$$

One is able to eliminate B, and to obtain

$$S = \int d^4x \sqrt{-g} \Big[\frac{1}{\kappa^2} \left\{ f'(A) \left(R - A \right) + f(A) \right\} + \mathcal{L}_{\mathrm{matter}} \Big] , \qquad (8)$$

and by using the conformal transformation $g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu}$ $(\sigma = -\ln f'(A))$, the action (8) is rewritten as the Einstein-frame action:

$$S_E = \int d^4x \sqrt{-g} \Big[\frac{1}{\kappa^2} \left(R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) \right) + \mathcal{L}^{\sigma}_{\mathrm{matter}} \Big] . \qquad (9)$$

Here,

$$V(\sigma) = \mathrm{e}^{\sigma} G \left(\mathrm{e}^{-\sigma} \right) - \mathrm{e}^{2\sigma} f \left(G \left(\mathrm{e}^{-\sigma} \right) \right) = \frac{A}{f'(A)} - \frac{f(A)}{f'(A)^2} . \qquad (10)$$

The action (8) is called the Jordan-frame action. In the Einstein-frame action, the matter couples with the scalar field σ. One may define the effective EoS parameter w_{eff} in Jordan frame as

$$w_{\mathrm{eff}} = \frac{p}{\rho} = -1 - \frac{2\dot{H}}{3H^2} \ , \tag{11}$$

The scale factor in Einstein frame (when the two frames appear) is denoted as $a(t)$.

A. Modified gravity with negative and positive powers of the curvature

As the first gravitational alternative for dark energy we consider the following action

$$f(R) = R - \frac{c}{(R-\Lambda_1)^n} + b\,(R-\Lambda_2)^m \ . \qquad (12)$$

Here we assume the coefficients n, m, c, $b > 0$ but n, m may be fractional.

For the action (12), Eq.(3) has the following form:

$$0 = -R + \frac{(n+2)c}{(R-\Lambda_1)^n} + (m-2)b\,(R-\Lambda_2)^m \ . \qquad (13)$$

Especially when $n = 1$ and $m = 2$, one gets

$$R = R_\pm = \frac{\Lambda_1 \pm \sqrt{\Lambda_1^2 + 12c}}{2} \ . \qquad (14)$$

If $c > 0$, one solution corresponds to deSitter space and another to anti-deSitter. If $-\frac{\Lambda_1^2}{12} < c < 0$ and $\Lambda_1 > 0$, both of solutions express the deSitter space. Hence, the natural possibility for the unification of early-time inflation with late-time acceleration appears.
By assuming the FRW universe metric (4), one may define the Hubble rate by $H = \dot{\hat{a}}/\hat{a}$. The contribution from matter may be neglected. Especially when $n = 1$, $m = 2$, and $\Lambda_1 = \Lambda_2 = 0$ in (12) and the curvature is small, we obtain
$\hat{a} \propto t^2$. We now consider the more general case that $f(R)$ is given by (12) when the curvature is small. Neglecting the contribution from the matter again, solving (2), we obtain $\hat{a} \propto t^{\frac{(n+1)(2n+1)}{n+2}}$.

B. ln R gravity

Other gravitational alternatives for dark energy may be suggested along the same line. As an extension of the theory of the previous section, one may consider the model containing the logarithm of the scalar curvature R:

$$f(R) = R + \alpha' \ln \frac{R}{\mu^2} + \beta R^m \ . \tag{15}$$

We should note that $m = 2$ choice simplifies the model.
We can consider late FRW cosmology when the scalar curvature R is small. Solving (2), it follows that the power law inflation could occur: $\hat{a} \propto t^{\frac{1}{2}}$. Since $\dot{\hat{a}} > 0$ but $\ddot{\hat{a}} < 0$, the deccelerated expansion occurs.
One may discuss further generalizations

$$f(R) = R + \gamma R^{-n} \left(\ln \frac{R}{\mu^2} \right)^m \ . \tag{16}$$

Here n is rectricted by $n > -1$ (m is an arbitrary) in order that the second term could be more dominant than the Einstein term when R is small.

For this model, we find

$$\hat{a} \sim t^{\frac{(n+1)(2n+1)}{n+2}} \ . \tag{17}$$

This does not depend on m. The effective $w_{\rm eff}$ is given by

$$w_{\rm eff} = -\frac{6n^2 + 7n - 1}{3(n+1)(2n+1)} \ . \tag{18}$$

Then $w_{\rm eff}$ can be negative if

$$-1 < n < -\frac{1}{2} \text{ or } n > \frac{-7 + \sqrt{73}}{12} = 0.1287\cdots \ . \tag{19}$$

From (17), the condition that the universe could accelerate is $\frac{(n+1)(2n+1)}{n+2} > 1$, that is:

$$n > \frac{-1 + \sqrt{3}}{2} = 0.366\cdots \ . \tag{20}$$

Clearly, the effective EoS parameter w may be within the existing bounds.

C. Modified gravity coupled with matter

The ideal fluid is taken as the matter with the constant w: $p = w\rho$. Then from the energy conservation law it follows $\rho = \rho_0 a^{-3(1+w)}$. In a some limit, strong cuvature or weak one, $f(R)$ may behave as $f(R) \sim f_0 R^\alpha$, with constant f_0 and α. An exact solution of the equation of motion is found to be

$$a = a_0 t^{h_0} \ , \quad h_0 \equiv \frac{2\alpha}{3(1+w)} \ ,$$

$$a_0 \equiv \left[-\frac{6f_0 h_0}{\rho_0} \left(-6h_0 + 12h_0^2\right)^{\alpha-1} \left\{(1-2\alpha)(1-\alpha) - (2-\alpha)h_0\right\} \right]^{-3}$$

When $\alpha = 1$, the result $h_0 = \frac{2}{3(1+w)}$ in the Einstein gravity is reproduced. The effective $w_{\rm eff}$ may be defined by $h_0 = \frac{2}{3(1+w_{\rm eff})}$. By using (21), one finds the effective $w_{\rm eff}$ (11) is given by

$$w_{\rm eff} = -1 + \frac{1+w}{\alpha} \ . \tag{22}$$

Hence, if w is greater than -1 (effective quintessence or even usual ideal fluid with positive w), when α is negative, we obtain the effective phantom phase where $w_{\rm eff}$ is less than -1.
One may now take $f(R)$ as

$$f(R) = \frac{1}{\kappa^2}\left(R - \gamma R^{-n} + \eta R^2\right) \ . \tag{23}$$

When the cuvature is small, the second term becomes dominant and one may identify $f_0 = -\frac{\gamma}{\kappa^2}$ and $\alpha = -n$. Then from (22), it follows $w_{\rm eff} = -1 - \frac{1+w}{n}$. Hence, if $n > 0$, an effective phantom era occurs even if $w > -1$.

D. The equivalence with scalar-tensor theory

It is very interesting that $f(R)$ gravity is in some sense equivalent to the scalar-tensor theory with the action:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi)\right\}\ ,$$

$$\omega(\phi) = -\frac{2}{\kappa^2}h'(\phi)\ , \quad V(\phi) = \frac{1}{\kappa^2}\left(3h(\phi)^2 + h'(\phi)\right)\ . \qquad (24)$$

Here $h(\phi)$ is a proper function of the scalar field ϕ. Imagine the following FRW cosmology is constructed:

$$\phi = t\ , \quad H = h(t)\ . \qquad (25)$$

Then *any* cosmology defined by $H = h(t)$ in (25) can be realized by (24).

Indeed, if one defines a new field φ as

$$\varphi = \int d\phi \sqrt{|\omega(\phi)|} \, , \tag{26}$$

the action (24) can be rewritten as

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R \mp \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \tilde{V}(\varphi) \right\} \, . \tag{27}$$

In case the sign in front of the kinetic term of φ in (27) is $-$, we can use the conformal transformation $g_{\mu\nu} \to \mathrm{e}^{\pm\kappa\varphi\sqrt{\frac{2}{3}}} g_{\mu\nu}$, and make the kinetic term of φ vanish. Hence, one obtains

$$S = \int d^4x \sqrt{-g} \left\{ \frac{\mathrm{e}^{\pm\kappa\varphi\sqrt{\frac{2}{3}}}}{2\kappa^2} R - \mathrm{e}^{\pm 2\kappa\varphi\sqrt{\frac{2}{3}}} \tilde{V}(\varphi) \right\} \, . \tag{28}$$

The action (28) may be called as Jordan frame action and the action (27) as the Einstein frame action.

Since φ becomes the auxiliary field, one may delete φ by using an equation of motion:

$$R = \mathrm{e}^{\pm\kappa\varphi\sqrt{\frac{2}{3}}} \left(4\kappa^2 \tilde{V}(\varphi) \pm 2\kappa\sqrt{\frac{3}{2}}\tilde{V}'(\varphi) \right) \ , \qquad (29)$$

which may be solved with respect to R as $\varphi = \varphi(R)$. One can rewrite the action (28) in the form of $f(R)$ gravity :

$$\begin{aligned} S &= \int d^4x\sqrt{-g} f(R) \ , \\ f(R) &\equiv \frac{\mathrm{e}^{\pm\kappa\varphi(R)\sqrt{\frac{2}{3}}}}{2\kappa^2} R - \mathrm{e}^{\pm 2\kappa\varphi(R)\sqrt{\frac{2}{3}}}\tilde{V}\left(\varphi(R)\right) \ . \end{aligned} \qquad (30)$$

III. String-inspired Gauss-Bonnet gravity as dark energy

We consider a model of the scalar field ϕ coupled with gravity. As a stringy correction, the term proportional to the GB invariant $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is added. The starting action is given by

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{\gamma}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + f(\phi)G\right\} ,$$
$$V = V_0\mathrm{e}^{-\frac{2\phi}{\phi_0}} , \quad f(\phi) = f_0\mathrm{e}^{\frac{2\phi}{\phi_0}} . \tag{31}$$

Here $\gamma = \pm 1$.

For the canonical scalar, $\gamma = 1$ but at least when GB term is not included, the scalar behaves as phantom only when $\gamma = -1$. Starting with FRW universe metric (11) in sect.4 and assuming (6) in sect.4, the following solutions may be obtained

$$\begin{aligned} V_0 t_1^2 &= -\frac{1}{\kappa^2 (1 + h_0)} \left\{ 3h_0^2 (1 - h_0) + \frac{\gamma \phi_0^2 \kappa^2 (1 - 5h_0)}{2} \right\} , \\ \frac{48 f_0 h_0^2}{t_1^2} &= -\frac{6}{\kappa^2 (1 + h_0)} \left(h_0 - \frac{\gamma \phi_0^2 \kappa^2}{2} \right) . \end{aligned} \tag{32}$$

Even if $\gamma = -1$, there appear the solutions describing non-phantom cosmology coresponding the quintessence or matter.

As an example, we consider the case that $h_0 = -\frac{80}{3} < -1$, which gives $w_{\rm eff} = -1.025$. Simple tuning gives other acceptable values of the effective w in the range close to -1. This is consistent with the observational bounds for effective w Then from (32), one obtains

$$V_0 t_1^2 = \frac{1}{\kappa^2}\left(\frac{531200}{231} + \frac{403}{154}\gamma\phi_0\kappa^2\right) \ ,$$

$$\frac{f_0}{t_1^2} = -\frac{1}{\kappa^2}\left(\frac{9}{49280} + \frac{27}{7884800}\gamma\phi_0\kappa^2\right) \ . \tag{33}$$

Therefore even starting from the canonical scalar theory with positive potential, we may obtain a solution which reproduces the observed value of w.

If ϕ and H are constants: $\phi = \varphi_0$, $H = H_0$, this corresponds to deSitter space. Then the solution of equations of motion gives:

$$H_0^2 = -\frac{\mathrm{e}^{-\frac{2\varphi_0}{\phi_0}}}{8 f_0 \kappa^2} \ . \tag{34}$$

Therefore in order for the solution to exist, the condition is $f_0 < 0$. In (34), φ_0 can be arbitrary.

IV. Modified gravity: non-linear coupling, cosmic acceleration
A. Gravitational solution of coincidence problem

As an example of such theory, the following action is considered:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{\kappa^2} R + \left(\frac{R}{\mu^2} \right)^{\alpha} L_d \right\} . \tag{35}$$

Here L_d is matter-like action (dark energy). The choice of parameter μ may keep away the unwanted instabilities which often occur in higher derivative theories.
By the variation over $g_{\mu\nu}$, the equation of motion follows:

$$0 = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{\kappa^2} \left\{ \frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} \right\} + \tilde{T}^{\mu\nu} . \tag{36}$$

Here the effective EMT tensor $\tilde{T}_{\mu\nu}$ is defined by

$$\begin{aligned}
\tilde{T}^{\mu\nu} &\equiv \frac{1}{\mu^{2\alpha}} \left\{ -\alpha R^{\alpha-1} R^{\mu\nu} L_d + \alpha \left(\nabla^\mu \nabla^\nu - g^{\mu\nu} \nabla^2 \right) \left(R^{\alpha-1} L_d \right) + R^\alpha T^{\cdot} \right. \\
T^{\mu\nu} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left(\int d^4x \sqrt{-g} L_d \right)
\end{aligned}$$

Let free massless scalar be a matter

$$L_d = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \ . \tag{38}$$

Then the equation given by the variation over ϕ has the following form:

$$0 = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \phi} = \frac{1}{\sqrt{-g}} \partial_\mu \left(R^\alpha \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) \ . \tag{39}$$

The metric again corresponds to FRW universe with flat 3-space. If we assume ϕ only depends on t ($\phi = \phi(t)$), the solution of scalar field equation (39) is given by

$$\dot{\phi} = qa^{-3}R^{-\alpha} \ . \tag{40}$$

Here q is a constant of the integration. Hence $R^{\alpha}L_d = \frac{q^2}{2a^6 R^{\alpha}}$, which becomes dominant when R is small (large) compared with the Einstein term $\frac{1}{\kappa^2}R$ if $\alpha > -1$ ($\alpha < -1$). Thus, one arrives at the remarkable possibility that dark energy grows to asymptotic dominance over the usual matter with decrease of the curvature. At current universe, this solves the coincidence problem (the equality of the energy density for dark energy and for matter) simply by the fact of the universe expansion.

Substituting (40) into (36), the $(\mu, \nu) = (t, t)$ component of equation of motion has the following form:

$$
\begin{aligned}
0 \quad = \quad & -\frac{3}{\kappa^2} H^2 + \frac{36 q^2}{\mu^{2\alpha} a^6 \left(6\dot{H} + 12 H^2\right)^{\alpha+2}} \left\{ \frac{\alpha(\alpha+1)}{4} \ddot{H} H + \frac{\alpha+1}{4} \dot{H}^2 \right. \\
& \left. + \left(1 + \frac{13}{4}\alpha + \alpha^2\right) \dot{H} H^2 + \left(1 + \frac{7}{2}\alpha\right) H^4 \right\} \ . \qquad (41)
\end{aligned}
$$

The accelerated FRW solution of (41) exists

$$
a = a_0 t^{\frac{\alpha+1}{3}} \quad \left(H = \frac{\alpha+1}{3t}\right) , \quad a_0^6 \equiv \frac{\kappa^2 q^2 \left(2\alpha - 1\right)\left(\alpha - 1\right)}{\mu^{2\alpha} 3 \left(\alpha+1\right)^{\alpha+1} \left(\frac{2}{3}\left(2\alpha - 1\right)\right)^{\alpha+2}} \ . \qquad (42)
$$

Eq.(42) tells that the universe accelerates, that is, $\ddot{a} > 0$ if $\alpha > 2$. If $\alpha < -1$, the solution (42) describes shrinking universe if $t > 0$. If the time is shifted as $t \to t - t_s$ (t_s is a constant), the accelerating and expanding universe occurs when $t < t_s$. In the solution with $\alpha < -1$ there appears a Big Rip singularity at $t = t_s$. For the matter with the relation $p = w\rho$, where p is the pressure and ρ is the energy density, from the usual FRW equation, one has $a \propto t^{\frac{2}{3(w+1)}}$. For $a \propto t^{h_0}$ it follows $w = -1 + \frac{2}{3h_0}$, and the accelerating expansion ($h_0 > 1$) of the universe occurs if $-1 < w < -\frac{1}{3}$. For the case of (42), one finds

$$w = \frac{1-\alpha}{1+\alpha} \ . \tag{43}$$

Then if $\alpha < -1$, we have $w < -1$, which is an effective phantom. For the general matter with the relation $p = w\rho$ with constant w, the energy E and the energy density ρ behave as $E \sim a^{-3w}$ and $\rho \sim a^{-3(w+1)}$. Thus, for the standard phantom with $w < -1$, the density becomes large with time and might generate the Big Rip.

B. Dynamical cosmological constant theory: an exact example

the following action similar to the one under consideration has been proposed:

$$I = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} + \alpha_0 R^2 + \frac{\left(\kappa^4\partial_\mu\varphi\partial^\mu\varphi\right)^q}{2q\kappa^4 f(R)^{2q-1}} - V(\varphi)\right] , \quad (44)$$

where $f(R)$ is a proper function. When the curvature is small, it is assumed $f(R)$ behaves as

$$f(R) \sim \left(\kappa^2 R^2\right)^m . \quad (45)$$

Here m is positive. When the curvature is small, the vacuum energy, and therefore the value of the potential becomes small. Then one may assume, for the small curvature, $V(\varphi)$ behaves as

$$V(\varphi) \sim V_0\left(\varphi - \varphi_c\right) . \quad (46)$$

Here V_0 and φ_c are constants. If $q > 1/2$, the factor in front of the kinetic term of φ in (44) becomes large.

There is an exactly solvable model which realizes the above scenario. Let us choose

$$f(R) = \beta R^2 \ , \quad V(\varphi) = V_0 \left(\varphi - \varphi_c\right) \ . \tag{47}$$

Here β is a constant. R^2 term is neglected by putting $\alpha_0 = 0$ in (44) since the curvature is small. Searching for the solution (6) in sect.4 and choosing $\varphi = \varphi_c + \varphi_0/t^2$ or $\varphi = \varphi_c + \varphi_0/\left(t_s - t\right)^2$, the following restrictions are obtained

$$\varphi_0^2 = \frac{54\beta\left(-1 + 2h_0\right)^3 h_0^4}{\kappa^2\left(12h_0^2 - 2h_0 - 1\right)} \, , \quad V_0 = \pm\frac{3h_0 + 1}{\sqrt{6\kappa^2\left(12h_0^2 - 2h_0 - 1\right)\left(-1 + 2h_0\right)}} \tag{48}$$

Since φ_0^2 should be positive, one finds

$$\begin{aligned} &\text{when } \beta > 0 \ , \qquad \frac{1-\sqrt{13}}{12} < h_0 < \frac{1+\sqrt{13}}{12} \text{ or } h_0 \geq \frac{1}{2} \ , \\ &\text{when } \beta < 0 \ , \qquad h_0 < \frac{1-\sqrt{13}}{12} \text{ or } \frac{1+\sqrt{13}}{12} < h_0 \leq \frac{1}{2} \ . \end{aligned} \tag{49}$$

For example, if $h_0 = -1/60$, which gives $w_{\rm eff} = -1.025$, we find

$$\kappa V_0 = \pm \frac{19}{34}\sqrt{\frac{15}{31}} = \pm 0.388722... \ . \tag{50}$$

For $h_0 > 0$ case, since $R = 6\dot{H} + 12H^2$, the curvature R decreases as t^{-2} with time t and φ approaches to φ_c but does not arrive at φ_c in a finite time, as expected .
As H behaves as h_0/t or $h_0/(t_s - t)$ for (6) in sect.4, if we substitute the value of the age of the present universe 10^{10}years$\sim (10^{-33}\mathrm{eV})^{-1}$ into t or $t_s - t$, the observed value of H could be reproduced, which could explain the smallness of the effective cosmological constant $\Lambda \sim H^2$. Note that even if there is no potential term, that is, $V_0 = 0$, when $\beta < 0$, there is a solution

$$h_0 = -\frac{1}{3} < \frac{1-\sqrt{13}}{12} = -0.2171... \ , \tag{51}$$

which gives the EoS parameter : $w = -3$, although w is not realistic. Playing with different choices of the potential and non-linear coupling more realistic predictions may be obtained.

V. Late-time cosmology in modified Gauss-Bonnet gravity

A. $f(G)$ gravity

Our next example is modified Gauss-Bonnet gravity. Let us start from the action :

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + f(G) + \mathcal{L}_m\right) \ . \qquad (52)$$

Here $\mathcal{L}_m$ is the matter Lagrangian density and G is the GB invariant: $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}$. By variation over $g_{\mu\nu}$ one gets:

$$\begin{aligned}
0 = & \frac{1}{2\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + T^{\mu\nu} + \frac{1}{2}g^{\mu\nu}f(G) - 2f'(G)RR^{\mu\nu} \\
& +4f'(G)R^\mu{}_\rho R^{\nu\rho} - 2f'(G)R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau} - 4f'(G)R^{\mu\rho\sigma\nu}R_{\rho\sigma} + 2\,(\nabla^\mu\nabla^\nu \\
& -2g^{\mu\nu}\left(\nabla^2 f'(G)\right)R - 4\left(\nabla_\rho\nabla^\mu f'(G)\right)R^{\nu\rho} - 4\left(\nabla_\rho\nabla^\nu f'(G)\right)R^{\mu\rho} \\
& +4\left(\nabla^2 f'(G)\right)R^{\mu\nu} + 4g^{\mu\nu}\left(\nabla_\rho\nabla_\sigma f'(G)\right)R^{\rho\sigma} - 4\left(\nabla_\rho\nabla_\sigma f'(G)\right)R^{\mu\rho\nu}
\end{aligned}$$

where $T^{\mu\nu}$ is the matter EM tensor.

By choosing the spatially-flat FRW universe metric (4) in sect.4, the equation corresponding to the first FRW equation has the following form:

$$0 = -\frac{3}{\kappa^2} H^2 + G f'(G) - f(G) - 24 \dot{G} f''(G) H^3 + \rho_m \ , \qquad (54)$$

where ρ_m is the matter energy density. When $\rho_m = 0$, Eq. (54) has a deSitter universe solution where H, and therefore G, are constant. For $H = H_0$, with constant H_0, Eq. (54) turns into

$$0 = -\frac{3}{\kappa^2} H_0^2 + 24 H_0^4 f'\left(24 H_0^4\right) - f\left(24 H_0^4\right) \ . \qquad (55)$$

For a large number of choices of the function $f(G)$, Eq. (55) has a non-trivial ($H_0 \neq 0$) real solution for H_0 (deSitter universe).

We now consider the case $\rho_m \neq 0$. Assuming that the EoS parameter $w \equiv p_m/\rho_m$ for matter (p_m is the pressure of matter) is a constant then, by using the conservation of energy: $\dot{\rho}_m + 3H\left(\rho_m + p_m\right) = 0$, we find $\rho = \rho_0 a^{-3(1+w)}$. The function $f(G)$ is chosen as

$$f(G) = f_0 \left|G\right|^{\beta} \ , \tag{56}$$

with constant f_0 and β. If $\beta < 1/2$, $f(G)$ term becomes dominant compared with the Einstein term when the curvature is small. If we neglect the contribution from the Einstein term in (54), the following solution may be found

$$h_0 = \frac{4\beta}{3(1+w)} \, , \quad a_0 = \Big[-\frac{f_0(\beta-1)}{\left(h_0-1\right)\rho_0}\left\{24\left|h_0^3\left(-1+h_0\right)\right|\right\}^{\beta}\left(h_0-1+4\beta \tag{57}$$

Then the effective EoS parameter w_{eff} (11) in sect.4 is less than -1 if $\beta < 0$, and for $w > -1$ is

$$w_{\text{eff}} = -1 + \frac{2}{3h_0} = -1 + \frac{1+w}{2\beta} \ , \tag{58}$$

which is again less than -1 for $\beta < 0$. Thus, if $\beta < 0$, we obtain an effective phantom with negative h_0 even in the case when $w > -1$. In the phantom phase, there might seem to occur the Big Rip at $t = t_s$ [**?**]. Near this Big Rip , however, the curvature becomes dominant and then the Einstein term dominates, so that the $f(G)$ term can be neglected. Therefore, the universe behaves as $a = a_0 t^{2/3(w+1)}$ and as a consequence the Big Rip does not eventually occur. The phantom era is transient.

B. $f(G,R)$ gravity

It is interesting to study late-time cosmology in generalized theories, which include both the functional dependence from curvature as well as from the Gauss-Bonnet term :

$$S = \int d^4x\sqrt{-g}\,(f(G,R) + \mathcal{L}_m) \ . \tag{59}$$

The following solvable model is considered:

$$f(G,R) = R\tilde{f}\left(\frac{G}{R^2}\right) \ , \quad \tilde{f}\left(\frac{G}{R^2}\right) = \frac{1}{2\kappa^2} + f_0\left(\frac{G}{R^2}\right) \ . \tag{60}$$

The FRW solution may be found again:

$$H = \frac{h_0}{t} \ , \quad h_0 = \frac{\frac{3}{\kappa^2} - 2f_0 \pm \sqrt{8f_0\left(f_0 - \frac{3}{8\kappa^2}\right)}}{\frac{6}{\kappa^2} + 2f_0} \ . \tag{61}$$

Then, for example, if $\kappa^2 f_0 < -3$, there is a solution describing a phantom with $h_0 < -1 - \sqrt{2}$ and a solution describing the effective matter with $h_0 > -1 + \sqrt{2}$. Late-time cosmology in other versions of such theory may be constructed.

Inhomogeneous equation of state of the universe dark fluid

Let us remind several simple facts about the universe filled with ideal fluid. By using the energy conservation law $0 = \dot{\rho} + 3H\left(p + \rho\right)$, when ρ and p satisfy the following simple EOS $p = w\rho$ with constant w, we find $\rho = \rho_0 a^{-3(1+w)}$. Then by using the first FRW equation $(3/\kappa^2)H^2 = \rho$, the well-known solution follows $a = a_0\left(t - t_1\right)^{\frac{2}{3(w+1)}}$ $(w > -1)$ or $a_0\left(t_2 - t\right)^{\frac{2}{3(w+1)}}$ $w \neq -1$ $(w < -1)$ and $a = a_0 \mathrm{e}^{\kappa t\sqrt{\frac{\rho_0}{3}}}$ when $w = -1$, which is the deSitter universe. Here t_1 and t_2 are constants of the integration. When $w < -1$, there appears a Big Rip singularity in a finite time at $t = t_2$.

In general, the singularities in dark energy universe may behave in a different way. Type I ("Big Rip") : For $t \to t_s$, $a \to \infty$, $\rho \to \infty$ and $|p| \to \infty$ Type II ("sudden") : For $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ or 0 and $|p| \to \infty$ Type III : For $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|p| \to \infty$ Type IV : For $t \to t_s$, $a \to a_s$, $\rho \to 0$, $|p| \to 0$ and higher derivatives of H diverge. This also includes the case when ρ (p) or both of them tend to some finite values while higher derivatives of H diverge. Here t_s, a_s and ρ_s are constants with $a_s \neq 0$.

The singularities in the inhomogeneous EoS dark fluid universe

One may start from the dark fluid with the following EOS:

$$p = -\rho - f(\rho) \ , \tag{62}$$

where $f(\rho)$ can be an arbitrary function in general. The choice $f(\rho) \propto \rho^\alpha$ with a constant α was proposed. Then the scale factor is given by

$$a = a_0 \exp\left(\frac{1}{3}\int \frac{d\rho}{f(\rho)}\right) \ , \tag{63}$$

and the cosmological time may be found

$$t = \int \frac{d\rho}{\kappa\sqrt{3\rho}f(\rho)} \ , \tag{64}$$

As an example we may consider the case that

$$f(\rho) = A\rho^\alpha \ . \tag{65}$$

Then we find :

- In case $\alpha = 1/2$ or $\alpha = 0$, there does not appear any singularity.
- In case $\alpha > 1$, when $t \to t_0$, the energy density behaves as $\rho \to \infty$ and therefore $|p| \to \infty$. Then the scale factor a is finite even if $\rho \to \infty$. Therefore $\alpha > 1$ case corresponds to type III singularity.
- In $\alpha = 1$ case, if $A > 0$, there occurs the Big Rip or type I singularity but if $A \leq 0$, there does not appear future singularity.
- In case $1/2 < \alpha < 1$, when $t \to t_0$, all of ρ, $|p|$, and a diverge if $A > 0$ then this corresponds to type I singularity.

- In case $0 < \alpha < 1/2$, when $t \to t_0$, we find ρ, $|p| \to 0$ and $a \to a_0$ but
$$\ln a \sim |t - t_0|^{\frac{\alpha-1}{\alpha-1/2}} \ . \qquad (66)$$
Since the exponent $(\alpha - 1)/(\alpha - 1/2)$ is not always an integer, even if a is finite, the higher derivatives of H diverge in general. Therefore this case corresponds to type IV singularity.
- In case $\alpha < 0$, when $t \to t_0$, we find $\rho \to 0$, $a \to a_0$ but $|p| \to \infty$. Therefore this case corresponds to type II singularity.

At the next step, we consider the inhomogeneous EOS for dark fluid, so that the dependence from Hubble parameter is included in EOS. This new terms may origin from string/M-theory, braneworld or modified gravity

$$p = -\rho + f(\rho) + G(H) \ . \tag{67}$$

where $G(H)$ is some function.
In general, EOS needs to be double-valued in order for the transition (crossing of phantom divide) to occur between the region $w < -1$ and the region $w > -1$. Then there could not be one-to-one correspondence between p and ρ. In such a case, we may suggest the implicit, inhomogeneous equation of the state

$$F(p, \rho, H) = 0 \ . \tag{68}$$

The following example may be of interest:

$$(p+\rho)^2 - C_0\rho^2\left(1-\frac{H_0}{H}\right) = 0 \ . \tag{69}$$

Here C_0 and H_0 are positive constants. Hence, the Hubble rate looks as

$$H = \frac{16}{9C_0^2 H_0 \left(t-t_-\right)\left(t_+-t\right)} \ . \tag{70}$$

and

$$p = -\rho\left\{1+\frac{3C_0^2}{4H_0}\left(t-t_0\right)\right\} \ , \quad \rho = \frac{2^8}{3^3 C_0^4 H_0^2 \kappa^2 \left(t-t_-\right)^2\left(t_+-t\right)^2} \ . \tag{71}$$

In (70), since $t_- < t_0 < t_+$, as long as $t_- < t < t_+$, the Hubble rate H is positive. The Hubble rate H has a minimum $H = H_0$ when $t = t_0 = (t_- + t_+)/2$ and diverges when $t \to t_\pm$. Then one may regard $t \to t_-$ as a Big Bang singularity and $t \to t_+$ as a Big Rip one. As clear from (71), the parameter $w = p/\rho$ is larger than -1 when $t_- < t < t_0$ and smaller than -1 when $t_0 < t < t_+$. Therefore there occurs the crossing of phantom divide $w = -1$ when $t = t_0$ thanks to the effect of inhomogeneous term in EOS. In principle, the more general EOS may contain the derivatives of H, like $\dot{H}$, $\ddot{H}$, ... More general EOS than (68) may have the following form:

$$F\left(p, \rho, H, \dot{H}, \ddot{H}, \cdots\right) = 0 \ . \tag{72}$$