Renormalization according to Wilson

Suppose we have integrated out fields with momenta $> \Lambda$. We have renormalized fields (at the scale Λ) and $g(\Lambda)$. Now we want to integrate out also fields with momenta between Λ' and Λ ($\Lambda' \ll \Lambda$), and to obtain $g(\Lambda')$. For example, if we consider interaction between a quark and an antiquark at a distance r, $\Lambda' \sim 1/r$.

Renormalization according to Wilson

Suppose we have integrated out fields with momenta $> \Lambda$. We have renormalized fields (at the scale Λ) and $g(\Lambda)$. Now we want to integrate out also fields with momenta between Λ' and Λ ($\Lambda' \ll \Lambda$), and to obtain $g(\Lambda')$. For example, if we consider interaction between a quark and an antiquark at a distance r, $\Lambda' \sim 1/r$.
Vacuum can be considered as a dielectric medium

$$g^2(\Lambda') = \frac{g^2(\Lambda)}{\varepsilon}$$

where only fields with momenta between Λ' and Λ contribute to ε. $\varepsilon > 1$ — screening, $\varepsilon < 1$ — antiscreening (asymptotic freedom).

Magnetic properties of vacuum

Lorentz invariance: $\varepsilon\mu = 1$

$\mu > 1$ (paramagnetic) — asymptotic freedom

Magnetic properties of vacuum

Lorentz invariance: $\varepsilon\mu = 1$
$\mu > 1$ (paramagnetic) — asymptotic freedom
When we switch magnetic field B on

$$\Delta E_{\text{vac}} = \left(\mu^{-1} - 1\right) \frac{B^2}{2} V$$

Magnetic properties of vacuum

Lorentz invariance: $\varepsilon\mu = 1$
$\mu > 1$ (paramagnetic) — asymptotic freedom
When we switch magnetic field B on

$$\Delta E_{\text{vac}} = \left(\mu^{-1} - 1\right) \frac{B^2}{2} V$$

We shall show

$$\Delta E_{\text{vac}} = -\beta_0 \frac{g^2}{(4\pi)^2} \log \frac{\Lambda^2}{\Lambda'^2} \cdot \frac{B^2}{2} V$$

$$\mu = \varepsilon^{-1} = 1 + \beta_0 \frac{g^2}{(4\pi)^2} \log \frac{\Lambda^2}{\Lambda'^2}$$

$$g^2(\Lambda') = \left[1 + \beta_0 \frac{g^2}{(4\pi)^2} \log \frac{\Lambda^2}{\Lambda'^2}\right] g^2(\Lambda)$$

Vacuum energy

First we consider electrodynamics.

Vacuum energy

First we consider electrodynamics.
Charged scalar field (particle + antiparticle)

$$E_{\text{vac}} = 2 \sum_i \frac{\omega_i}{2} = \sum_i \omega_i$$

Vacuum energy

First we consider electrodynamics.
Charged scalar field (particle + antiparticle)

$$E_{\text{vac}} = 2\sum_i \frac{\omega_i}{2} = \sum_i \omega_i$$

Charged fermion field — energy of the Dirac sea

$$E_{\text{vac}} = -\sum_i \omega_i$$

Vacuum energy

First we consider electrodynamics.
Charged scalar field (particle + antiparticle)

$$E_{\text{vac}} = 2\sum_i \frac{\omega_i}{2} = \sum_i \omega_i$$

Charged fermion field — energy of the Dirac sea

$$E_{\text{vac}} = -\sum_i \omega_i$$

Factor $(-1)^{2s}$

Pauli paramagnetism

Massless particle $\omega = k$

Pauli paramagnetism

Massless particle $\omega = k$
Switching magnetic field on

$$\omega = \sqrt{k^2 - g_s s_z eB}$$

g_s — giromagnetic ratio

Pauli paramagnetism

Massless particle $\omega = k$
Switching magnetic field on

$$\omega = \sqrt{k^2 - g_s s_z e B}$$

g_s — giromagnetic ratio
Massless particles only have $s_z = \pm s$

$$\Delta E_{\text{Pauli}} = (-1)^{2s} \int \frac{V\, d^3k}{(2\pi)^3} \left[\sqrt{k^2 + g_s s e B} + \sqrt{k^2 - g_s s e B} - 2k \right]$$

$$= -(-1)^{2s} V \frac{(g_s s e B)^2}{4} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^3} = -(-1)^{2s} V \frac{(g_s s e B)^2}{8\pi^2} \int \frac{dk}{k}$$

$$= -2(-1)^{2s} (g_s s)^2 \frac{e^2}{(4\pi)^2} \log \frac{\Lambda^2}{\Lambda'^2} \cdot \frac{B^2}{2} V$$

Landau levels

Massless charged scalar field $L = (D_\mu \varphi)^+ D^\mu \varphi$

Landau levels

Massless charged scalar field $L = (D_\mu \varphi)^+ D^\mu \varphi$
Magnetic field B along z: $A_y = Bx$

$$\left[\nabla^2 - e^2 B^2 x^2 - 2ieBx \frac{\partial}{\partial y} + E^2\right] \varphi = 0$$

Landau levels

Massless charged scalar field $L = (D_\mu \varphi)^+ D^\mu \varphi$
Magnetic field B along z: $A_y = Bx$

$$\left[\nabla^2 - e^2 B^2 x^2 - 2ieBx \frac{\partial}{\partial y} + E^2 \right] \varphi = 0$$

$$\varphi = e^{i(k_y y + k_z z)} \varphi(x)$$

Landau levels

Massless charged scalar field $L = (D_\mu \varphi)^+ D^\mu \varphi$
Magnetic field B along z: $A_y = Bx$

$$\left[\nabla^2 - e^2 B^2 x^2 - 2ieBx \frac{\partial}{\partial y} + E^2 \right] \varphi = 0$$

$$\varphi = e^{i(k_y y + k_z z)} \varphi(x)$$

Oscillator

$$\left[-\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\omega^2}{2} x^2 - E_n \right] \psi_n = 0 \qquad E_n = \omega \left(n + \tfrac{1}{2} \right)$$

Landau levels

Massless charged scalar field $L = (D_\mu \varphi)^+ D^\mu \varphi$
Magnetic field B along z: $A_y = Bx$

$$\left[\nabla^2 - e^2 B^2 x^2 - 2ieBx \frac{\partial}{\partial y} + E^2 \right] \varphi = 0$$

$$\varphi = e^{i(k_y y + k_z z)} \varphi(x)$$

Oscillator

$$\left[-\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\omega^2}{2} x^2 - E_n \right] \psi_n = 0 \qquad E_n = \omega \left(n + \tfrac{1}{2} \right)$$

$$E^2 = k_z^2 + 2eB \left(n + \tfrac{1}{2} \right) \qquad \varphi = e^{i(k_y y + k_z z)} \psi_n \left(x - \frac{k_y}{eB} \right)$$

Degeneracy

Box $V = L_x \times L_y \times L_z$

$$k_z = \frac{2\pi}{L_z} n_z \qquad dn_z = \frac{L_z \, dk_z}{2\pi}$$

Degeneracy

Box $V = L_x \times L_y \times L_z$

$$k_z = \frac{2\pi}{L_z} n_z \qquad dn_z = \frac{L_z\, dk_z}{2\pi}$$

$$k_y = \frac{2\pi}{L_y} n_y \qquad \frac{k_y}{eB} \in [0, L_x] \qquad n_y \in \left[0, \frac{eB\, L_x L_y}{2\pi}\right]$$

Degeneracy

Box $V = L_x \times L_y \times L_z$

$$k_z = \frac{2\pi}{L_z} n_z \qquad dn_z = \frac{L_z\, dk_z}{2\pi}$$

$$k_y = \frac{2\pi}{L_y} n_y \qquad \frac{k_y}{eB} \in [0, L_x] \qquad n_y \in \left[0, \frac{eB\, L_x L_y}{2\pi}\right]$$

Vacuum energy

$$E_{\text{vac}} = \sum_{n=0}^{\infty} f\left(n + \tfrac{1}{2}\right)$$

$$f(x) = \frac{eBV}{(2\pi)^2} \int_{-\infty}^{+\infty} \sqrt{k_z^2 + 2eBx}\, dk_z$$

Euler summation formula

Smooth function $(L \gg 1)$

$$\sum_{n=0}^{N} f\left(n + \tfrac{1}{2}\right) \approx \int_{0}^{N+1} f(x)\,dx$$

Correction?

$$\int_{0}^{N+1} f(x)\,dx = \sum_{n=0}^{N} \int_{-1/2}^{1/2} f\left(n + \tfrac{1}{2} + x\right)\,dx$$

$$\int_{0}^{N+1} f(x)\,dx = \sum_{n=0}^{N} \int_{-1/2}^{1/2} f\left(n+\tfrac{1}{2}+x\right)\,dx$$

$$= \sum_{n=0}^{N} \int_{-1/2}^{1/2} \left[f\left(n+\tfrac{1}{2}\right) + \frac{1}{2} f''\left(n+\tfrac{1}{2}\right) x^2 + \cdots \right] dx$$

$$\int_{0}^{N+1} f(x)\,dx = \sum_{n=0}^{N} \int_{-1/2}^{1/2} f\left(n+\tfrac{1}{2}+x\right)\,dx$$

$$= \sum_{n=0}^{N} \int_{-1/2}^{1/2} \left[f\left(n+\tfrac{1}{2}\right) + \frac{1}{2} f''\left(n+\tfrac{1}{2}\right) x^2 + \cdots \right] dx$$

$$= \sum_{n=0}^{N} f\left(n+\tfrac{1}{2}\right) + \frac{1}{24} \sum_{n=0}^{N} f''\left(n+\tfrac{1}{2}\right) + \cdots$$

$$\int_{0}^{N+1} f(x)\,dx = \sum_{n=0}^{N} \int_{-1/2}^{1/2} f\left(n+\tfrac{1}{2}+x\right)\,dx$$

$$= \sum_{n=0}^{N} \int_{-1/2}^{1/2} \left[f\left(n+\tfrac{1}{2}\right) + \frac{1}{2} f''\left(n+\tfrac{1}{2}\right) x^2 + \cdots \right] dx$$

$$= \sum_{n=0}^{N} f\left(n+\tfrac{1}{2}\right) + \frac{1}{24} \sum_{n=0}^{N} f''\left(n+\tfrac{1}{2}\right) + \cdots$$

$$= \sum_{n=0}^{N} f\left(n+\tfrac{1}{2}\right) + \frac{1}{24} \int_{0}^{N+1} f''(x)\,dx + \cdots$$

$$\int_0^{N+1} f(x)\,dx = \sum_{n=0}^{N} \int_{-1/2}^{1/2} f\left(n + \tfrac{1}{2} + x\right)\,dx$$

$$= \sum_{n=0}^{N} \int_{-1/2}^{1/2} \left[f\left(n + \tfrac{1}{2}\right) + \frac{1}{2} f''\left(n + \tfrac{1}{2}\right) x^2 + \cdots \right] dx$$

$$= \sum_{n=0}^{N} f\left(n + \tfrac{1}{2}\right) + \frac{1}{24} \sum_{n=0}^{N} f''\left(n + \tfrac{1}{2}\right) + \cdots$$

$$= \sum_{n=0}^{N} f\left(n + \tfrac{1}{2}\right) + \frac{1}{24} \int_0^{N+1} f''(x)\,dx + \cdots$$

$$= \sum_{n=0}^{N} f\left(n + \tfrac{1}{2}\right) + \frac{1}{24} f'(x)\Big|_0^{N+1} + \cdots$$

Euler summation formula

$$\sum_{n=0}^{N} f\left(n + \tfrac{1}{2}\right) = \int\limits_{0}^{N+1} f(x)\, dx - \frac{1}{24} f'(x)\Big|_{0}^{N+1} + \cdots$$

Landau diamagnetism

$$E_{\text{vac}} = \int_0^\infty f(x) - \frac{1}{24} f'(x) \Big|_0^\infty$$

The integral is the vacuum energy at $B = 0$

Landau diamagnetism

$$E_{\text{vac}} = \int_0^\infty f(x) - \frac{1}{24} f'(x) \Big|_0^\infty$$

The integral is the vacuum energy at $B = 0$

$$\Delta E_{\text{Landau}} = \frac{1}{24} f'(0) = \frac{e^2 B^2 V}{48\pi^2} \int \frac{dk}{k} = \frac{1}{3} \frac{e^2}{(4\pi)^2} \log \frac{\Lambda^2}{\Lambda'^2} \cdot \frac{B^2}{2} V$$

Landau diamagnetism

$$E_{\text{vac}} = \int_0^\infty f(x) - \frac{1}{24} f'(x)\Big|_0^\infty$$

The integral is the vacuum energy at $B = 0$

$$\Delta E_{\text{Landau}} = \frac{1}{24} f'(0) = \frac{e^2 B^2 V}{48\pi^2} \int \frac{dk}{k} = \frac{1}{3} \frac{e^2}{(4\pi)^2} \log \frac{\Lambda^2}{\Lambda'^2} \cdot \frac{B^2}{2} V$$

Full QED result

$$\Delta E_{\text{vac}} = -\beta_0 \frac{e^2}{(4\pi)^2} \log \frac{\Lambda^2}{\Lambda'^2} \cdot \frac{B^2}{2} V$$

$$\beta_0 = \sum_s (-1)^{2s} \left[2(g_s s)^2 - \frac{n_s}{3} \right]$$

n_s — the number of polarization states: $n_0 = 1$, $n_{s \neq 0} = 2$

$g_{1/2} = 2$

$$\not{D}\psi = 0 \qquad D_\mu = \partial\mu - ieA_\mu \qquad \not{D}^2\psi = 0$$

$g_{1/2} = 2$

$$\not{D}\psi = 0 \qquad D_\mu = \partial\mu - ieA_\mu \qquad \not{D}^2\psi = 0$$

$$\not{D}^2 = \partial^2 - e^2A^2 - 2ieA^\mu\partial_\mu - ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu$$

We suppose $\partial \cdot A = 0$

$$\not{D}^2 = D^2 - \frac{ie}{4}F_{\mu\nu}[\gamma^\mu, \gamma^\nu]$$

$g_{1/2} = 2$

$$\not{D}\psi = 0 \qquad D_\mu = \partial\mu - ieA_\mu \qquad \not{D}^2\psi = 0$$

$$\not{D}^2 = \partial^2 - e^2A^2 - 2ieA^\mu\partial_\mu - ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu$$

We suppose $\partial \cdot A = 0$

$$\not{D}^2 = D^2 - \frac{ie}{4}F_{\mu\nu}[\gamma^\mu, \gamma^\nu]$$

$A^\mu = (0, 0, Bx^1, 0)$: $F_{12} = -F_{21} = -B$

$$\not{D}^2 = D^2 + ieB\gamma^1\gamma^2 = D^2 + 2eBs_z$$

$g_{1/2} = 2$

$$\not{D}\psi = 0 \qquad D_\mu = \partial\mu - ieA_\mu \qquad \not{D}^2\psi = 0$$

$$\not{D}^2 = \partial^2 - e^2A^2 - 2ieA^\mu\partial_\mu - ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu$$

We suppose $\partial \cdot A = 0$

$$\not{D}^2 = D^2 - \frac{ie}{4}F_{\mu\nu}[\gamma^\mu, \gamma^\nu]$$

$A^\mu = (0, 0, Bx^1, 0)$: $F_{12} = -F_{21} = -B$

$$\not{D}^2 = D^2 + ieB\gamma^1\gamma^2 = D^2 + 2eBs_z$$

$$\left[\nabla^2 - e^2B^2x^2 - 2ieBx\frac{\partial}{\partial y} + 2eBs_z + E^2\right]\psi = 0$$

$E^2 = k_z^2 + 2eB\left(n + \frac{1}{2}\right) - 2eBs_z$

$g_{1/2} = 2$

Quark

Chromomagnetic field along a_0 such that t^{a_0} is diagonal (for $SU(3)$ $a_0 = 8$): the sum of squares of colour "charges" is $\operatorname{Tr} t^{a_0} t^{a_0}$ (no summation)

$$\operatorname{Tr} t^a t^b = T_F \delta^{ab}$$

Quark

Chromomagnetic field along a_0 such that t^{a_0} is diagonal (for $SU(3)$ $a_0 = 8$): the sum of squares of colour "charges" is $\mathrm{Tr}\, t^{a_0} t^{a_0}$ (no summation)

$$\mathrm{Tr}\, t^a t^b = T_F \delta^{ab}$$

Contribution to β_0 (n_f flavours)

$$-\left(1 - \frac{1}{3}\right) 2 T_F n_f$$

$g_1 = 2$

$SU(2)$ Yang–Mills equation

$$D^\nu G^a_{\mu\nu} = \left(\partial^\nu \delta^{ab} + g\varepsilon^{acb} A^{c\nu}\right) G^b_{\mu\nu} = 0$$

External field A^3_μ, linearize in $A^{1,2}_\mu$.

$g_1 = 2$

$SU(2)$ Yang–Mills equation

$$D^{\nu} G^{a}_{\mu\nu} = \left(\partial^{\nu} \delta^{ab} + g \varepsilon^{acb} A^{c\nu} \right) G^{b}_{\mu\nu} = 0$$

External field A^3_μ, linearize in $A^{1,2}_\mu$.
For $A^-_\mu = A^1_\mu - i A^2_\mu$ we get

$$D^{\nu} G^{-}_{\mu\nu} + i g G^{3}_{\mu\nu} A^{-\nu} = 0$$

$$G^{-}_{\mu\nu} = D_\mu A^-_\nu - D_\nu A^-_\mu \qquad D_\mu = \partial_\mu - i g A^3_\mu$$

$g_1 = 2$

$SU(2)$ Yang–Mills equation

$$D^\nu G^a_{\mu\nu} = \left(\partial^\nu \delta^{ab} + g\varepsilon^{acb} A^{c\nu}\right) G^b_{\mu\nu} = 0$$

External field A^3_μ, linearize in $A^{1,2}_\mu$.
For $A^-_\mu = A^1_\mu - iA^2_\mu$ we get

$$D^\nu G^-_{\mu\nu} + igG^3_{\mu\nu} A^{-\nu} = 0$$

$$G^-_{\mu\nu} = D_\mu A^-_\nu - D_\nu A^-_\mu \qquad D_\mu = \partial_\mu - igA^3_\mu$$

Gauge $D^\mu A^-_\mu = 0$ (we use $[D_\mu, D_\nu] = -igG^3_{\mu\nu}$)

$$D^2 A^-_\mu - 2igG^3_{\mu\nu} A^{-\nu} = 0$$

$g_1 = 2$

$SU(2)$ Yang–Mills equation

$$D^\nu G^a_{\mu\nu} = \left(\partial^\nu \delta^{ab} + g\varepsilon^{acb} A^{c\nu}\right) G^b_{\mu\nu} = 0$$

External field A^3_μ, linearize in $A^{1,2}_\mu$.
For $A^-_\mu = A^1_\mu - iA^2_\mu$ we get

$$D^\nu G^-_{\mu\nu} + igG^3_{\mu\nu} A^{-\nu} = 0$$

$$G^-_{\mu\nu} = D_\mu A^-_\nu - D_\nu A^-_\mu \qquad D_\mu = \partial_\mu - igA^3_\mu$$

Gauge $D^\mu A^-_\mu = 0$ (we use $[D_\mu, D_\nu] = -igG^3_{\mu\nu}$)

$$D^2 A^-_\mu - 2igG^3_{\mu\nu} A^{-\nu} = 0$$

$G^3_{12} = -G^3_{21} = -B$, $A^{-2} = is_z A^{-1}$ $(s_z = \pm 1)$

$$\left[D^2 + 2igBs_z\right] A^{-1} = 0$$

Gluon

Gluons with colour a_1 such that t^{a_1} is diagonal don't interact with our chromomagnetic field (for $SU(3)$ $a_1 = 3$). All other gluons can be arranged into pairs with positive and negative "colour charges". The sum of their squares (both signs!) is C_A: in the adjoint representation

$$\operatorname{Tr} t^a t^b = C_A \delta^{ab}$$

i.e. $2e^2 \to C_A g^2$

Gluon

Gluons with colour a_1 such that t^{a_1} is diagonal don't interact with our chromomagnetic field (for $SU(3)$ $a_1 = 3$). All other gluons can be arranged into pairs with positive and negative "colour charges". The sum of their squares (both signs!) is C_A: in the adjoint representation

$$\mathrm{Tr}\, t^a t^b = C_A \delta^{ab}$$

i.e. $2e^2 \to C_A g^2$

$$\beta_0 = \left(4 - \tfrac{1}{3}\right) C_A - \left(1 - \tfrac{1}{3}\right) 2T_F n_f$$

Pauli paramagnetism $(g_1 \cdot 1)^2 = 4$ is stronger than Landau diamagnetism $-1/3$