

Massless QCD

$$L = \sum_i \bar{q}_{0i} i \not{D} q_{0i} - \frac{1}{4} G_{0\mu\nu}^a G_0^{a\mu\nu}$$

$$D_\mu q_0 = (\partial_\mu - i g_0 A_{0\mu}) q_0 \quad A_{0\mu} = A_{0\mu}^a t^a$$

$$[D_\mu, D_\nu] q_0 = -i g_0 G_{0\mu\nu} q_0 \quad G_{0\mu\nu} = G_{0\mu\nu}^a t^a$$

$$G_{0\mu\nu}^a = \partial_\mu A_{0\nu}^a - \partial_\nu A_{0\mu}^a + g_0 f^{abc} A_{0\mu}^b A_{0\nu}^c$$

Massless QCD

$$L = \sum_i \bar{q}_{0i} i \not{D} q_{0i} - \frac{1}{4} G_{0\mu\nu}^a G_0^{a\mu\nu} - \frac{1}{2a_0} (\partial_\mu A_0^{a\mu})^2 + (\partial^\mu \bar{c}_0^a) (D_\mu c_0^a)$$

$$D_\mu q_0 = (\partial_\mu - ig_0 A_{0\mu}) q_0 \quad A_{0\mu} = A_{0\mu}^a t^a$$

$$[D_\mu, D_\nu] q_0 = -ig_0 G_{0\mu\nu} q_0 \quad G_{0\mu\nu} = G_{0\mu\nu}^a t^a$$

$$G_{0\mu\nu}^a = \partial_\mu A_{0\nu}^a - \partial_\nu A_{0\mu}^a + g_0 f^{abc} A_{0\mu}^b A_{0\nu}^c$$

$$D_\mu c_0^a = (\partial_\mu \delta^{ab} - ig_0 A_{0\mu}^{ab}) c_0^b \quad A_{0\mu}^{ab} = A_{0\mu}^c (t^c)^{ab}$$

$$[t^a, t^b] = if^{abc} t^c \quad (t^c)^{ab} = if^{acb}$$

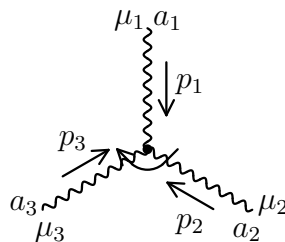
Feynman rules

$$\begin{aligned} \bullet \xrightarrow[p]{} \bullet &= iS_0(p) & S_0(p) &= \frac{1}{\not{p}} = \frac{\not{p}}{p^2} \\ \begin{matrix} a \\ \mu \end{matrix} \bullet \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \bullet \begin{matrix} b \\ \nu \end{matrix} &= -i\delta^{ab}D_{\mu\nu}^0(p) \\ \begin{matrix} a \\ \bullet \end{matrix} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \bullet \begin{matrix} b \\ \bullet \end{matrix} &= i\delta^{ab}G_0(p) & G_0(p) &= \frac{1}{p^2} \end{aligned}$$

Feynman rules

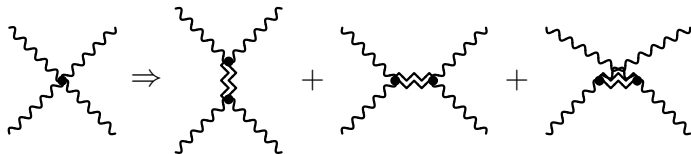
$$\begin{aligned} \text{---} \xrightarrow{p} \text{---} &= iS_0(p) & S_0(p) &= \frac{1}{\not{p}} = \frac{\not{p}}{p^2} \\ \begin{matrix} a \\ \mu \end{matrix} \text{---} \text{wavy} \text{---} \begin{matrix} b \\ \nu \end{matrix} &= -i\delta^{ab} D_{\mu\nu}^0(p) \\ \begin{matrix} a \\ \text{---} \end{matrix} \xrightarrow{p} \begin{matrix} b \\ \text{---} \end{matrix} &= i\delta^{ab} G_0(p) & G_0(p) &= \frac{1}{p^2} \\ \begin{matrix} \mu & a \\ \text{wavy} \\ \text{---} \end{matrix} &= t^a \times ig_0 \gamma^\mu \end{aligned}$$

Feynman rules


$$= i f^{a_1 a_2 a_3} \times i g_0 V^{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)$$

$$V^{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3) =$$
$$(p_3 - p_2)^{\mu_1} g^{\mu_2 \mu_3} + (p_1 - p_3)^{\mu_2} g^{\mu_3 \mu_1} + (p_2 - p_1)^{\mu_3} g^{\mu_1 \mu_2}$$

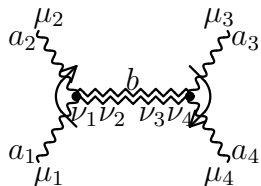
Feynman rules



$$\begin{array}{c} a \\ \bullet \\ \mu\nu \\ \bullet \\ b \\ \alpha\beta \end{array} = i\delta^{ab}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})$$

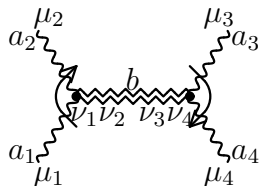
$$\begin{array}{c} \nu \\ b \\ \bullet \\ \bullet \\ a \\ \mu \\ \bullet \\ c \\ \alpha\beta \end{array} = if^{abc} \times g_0 g^{\mu\alpha} g^{\nu\beta}$$

Feynman rules



$$\begin{aligned}
 & i f^{a_1 a_2 b} i f^{a_3 a_4 b} \times g_0 g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} i (g_{\nu_1 \nu_3} g_{\nu_2 \nu_4} - g_{\nu_1 \nu_4} g_{\nu_2 \nu_3}) g_0 g^{\mu_3 \nu_3} g^{\mu_4 \nu_4} \\
 & = i f^{a_1 a_2 b} i f^{a_3 a_4 b} \times i g_0^2 (g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3})
 \end{aligned}$$

Feynman rules



$$\begin{aligned}
 & i f^{a_1 a_2 b} i f^{a_3 a_4 b} \times g_0 g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} i (g_{\nu_1 \nu_3} g_{\nu_2 \nu_4} - g_{\nu_1 \nu_4} g_{\nu_2 \nu_3}) g_0 g^{\mu_3 \nu_3} g^{\mu_4 \nu_4} \\
 & = i f^{a_1 a_2 b} i f^{a_3 a_4 b} \times i g_0^2 (g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3})
 \end{aligned}$$

$$= i f^{abc} \times i g_0 p^\mu$$

Renormalization

$$\begin{aligned}q_{i0} &= Z_q^{1/2} q_i & A_0 &= Z_A^{1/2} A & c_0 &= Z_c^{1/2} c \\ a_0 &= Z_A a & g_0 &= Z_\alpha^{1/2} g\end{aligned}$$

Renormalization

$$q_{i0} = Z_q^{1/2} q_i \quad A_0 = Z_A^{1/2} A \quad c_0 = Z_c^{1/2} c$$
$$a_0 = Z_A a \quad g_0 = Z_\alpha^{1/2} g$$

$$\frac{\alpha_s(\mu)}{4\pi} = \mu^{-2\epsilon} \frac{g^2}{(4\pi)^{d/2}} e^{-\gamma\epsilon}$$
$$\frac{g_0^2}{(4\pi)^{d/2}} = \mu^{2\epsilon} \frac{\alpha_s(\mu)}{4\pi} Z_\alpha(\alpha(\mu)) e^{\gamma\epsilon}$$

$SU(N_c)$

Fundamental representation: q^i

$$q \rightarrow Uq \quad \text{or} \quad q^i \rightarrow U^i_j q^j$$

$SU(N_c)$

Fundamental representation: q^i

$$q \rightarrow Uq \quad \text{or} \quad q^i \rightarrow U^i_j q^j$$

Conjugated fundamental representation: $q_i^+ = (q^i)^*$

$$q^+ \rightarrow q^+ U^+ \quad \text{or} \quad q_i^+ \rightarrow q_j^+ (U^+)^j_i \quad \text{where} \quad (U^+)^j_i = (U^i_j)^*$$

$SU(N_c)$

Fundamental representation: q^i

$$q \rightarrow Uq \quad \text{or} \quad q^i \rightarrow U^i_j q^j$$

Conjugated fundamental representation: $q_i^+ = (q^i)^*$

$$q^+ \rightarrow q^+ U^+ \quad \text{or} \quad q_i^+ \rightarrow q_j^+ (U^+)^j_i \quad \text{where} \quad (U^+)^j_i = (U^i_j)^*$$

The scalar product is invariant: $q^+ q' \rightarrow q^+ U^+ U q' = q^+ q'$
(mesons)

$$\delta_j^i \rightarrow \delta_l^k U^i_k (U^+)^l_j = U^i_k (U^+)^k_j = \delta_j^i$$

$SU(N_c)$

Fundamental representation: q^i

$$q \rightarrow Uq \quad \text{or} \quad q^i \rightarrow U^i_j q^j$$

Conjugated fundamental representation: $q_i^+ = (q^i)^*$

$$q^+ \rightarrow q^+ U^+ \quad \text{or} \quad q_i^+ \rightarrow q_j^+ (U^+)^j_i \quad \text{where} \quad (U^+)^j_i = (U^i_j)^*$$

The scalar product is invariant: $q^+ q' \rightarrow q^+ U^+ U q' = q^+ q'$
(mesons)

$$\delta_j^i \rightarrow \delta_l^k U^i_k (U^+)^l_j = U^i_k (U^+)^k_j = \delta_j^i$$

$\varepsilon^{i_1 \dots i_{N_c}}$ and $\varepsilon_{i_1 \dots i_{N_c}}$ are also invariant (baryons)

$$\varepsilon^{i_1 \dots i_{N_c}} \rightarrow \varepsilon^{j_1 \dots j_{N_c}} U^{i_1}_{j_1} \dots U^{i_{N_c}}_{j_{N_c}} = \det U \cdot \varepsilon^{i_1 \dots i_{N_c}} = \varepsilon^{i_1 \dots i_{N_c}}$$

Infinitesimal transformations

$$U = 1 + i\alpha^a t^a$$

Infinitesimal transformations

$$U = 1 + i\alpha^a t^a$$

$$U^\dagger U = 1 + i\alpha^a (t^a - (t^a)^\dagger) = 1 \quad \Rightarrow \quad (t^a)^\dagger = t^a$$

$$\det U = 1 + i\alpha^a \text{Tr} t^a = 1 \quad \Rightarrow \quad \text{Tr} t^a = 0$$

$N_c^2 - 1$ generators t^a

Infinitesimal transformations

$$U = 1 + i\alpha^a t^a$$

$$U^+ U = 1 + i\alpha^a (t^a - (t^a)^+) = 1 \quad \Rightarrow \quad (t^a)^+ = t^a$$

$$\det U = 1 + i\alpha^a \text{Tr } t^a = 1 \quad \Rightarrow \quad \text{Tr } t^a = 0$$

$N_c^2 - 1$ generators t^a

$$\text{Tr } t^a t^b = T_F \delta^{ab}$$

Infinitesimal transformations

$$U = 1 + i\alpha^a t^a$$

$$U^+ U = 1 + i\alpha^a (t^a - (t^a)^+) = 1 \Rightarrow (t^a)^+ = t^a$$

$$\det U = 1 + i\alpha^a \text{Tr} t^a = 1 \Rightarrow \text{Tr} t^a = 0$$

$N_c^2 - 1$ generators t^a

$$\text{Tr} t^a t^b = T_F \delta^{ab}$$

$$[t^a, t^b] = i f^{abc} t^c \quad i f^{abc} = \frac{1}{T_F} \text{Tr} [t^a, t^b] t^c$$

Infinitesimal transformations

$$U = 1 + i\alpha^a t^a$$

$$U^+ U = 1 + i\alpha^a (t^a - (t^a)^+) = 1 \quad \Rightarrow \quad (t^a)^+ = t^a$$

$$\det U = 1 + i\alpha^a \text{Tr} t^a = 1 \quad \Rightarrow \quad \text{Tr} t^a = 0$$

$N_c^2 - 1$ generators t^a

$$\text{Tr} t^a t^b = T_F \delta^{ab}$$

$$[t^a, t^b] = i f^{abc} t^c \quad i f^{abc} = \frac{1}{T_F} \text{Tr} [t^a, t^b] t^c$$

$$[t^a, t^b]_+ = 2 \frac{T_F}{N_c} \delta^{ab} + d^{abc} t^c \quad d^{abc} = \frac{1}{T_F} \text{Tr} [t^a, t^b]_+ t^c$$

Adjoint representation

$$A^a = q^+ t^a q' \quad A^a \rightarrow q^+ U^+ t^a U q' = U^{ab} A^b$$

$$U^+ t^a U = U^{ab} t^b \quad U^{ab} = \frac{1}{T_F} \text{Tr} U^+ t^a U t^b$$

Adjoint representation

$$A^a = q^+ t^a q' \quad A^a \rightarrow q^+ U^+ t^a U q' = U^{ab} A^b$$

$$U^+ t^a U = U^{ab} t^b \quad U^{ab} = \frac{1}{T_F} \text{Tr} U^+ t^a U t^b$$

$(t^a)^i_j$ — fixed numbers (invariant tensor)

$$(t^a)^i_j \rightarrow U^{ab} U^i_k (t^b)^k_l (U^+)^l_j = (t^a)^i_j$$

Adjoint representation

$$A^a = q^+ t^a q' \quad A^a \rightarrow q^+ U^+ t^a U q' = U^{ab} A^b$$

$$U^+ t^a U = U^{ab} t^b \quad U^{ab} = \frac{1}{T_F} \text{Tr} U^+ t^a U t^b$$

$(t^a)^i_j$ — fixed numbers (invariant tensor)

$$(t^a)^i_j \rightarrow U^{ab} U^i_k (t^b)^k_l (U^+)^l_j = (t^a)^i_j$$

Infinitesimal transformations

$$A^a \rightarrow q^+ (1 - i\alpha^c t^c) t^a (1 + i\alpha^c t^c) q' = q^+ (t^a + i\alpha^c i f^{acb} t^b) q'$$

$$U^{ab} = \delta^{ab} + i\alpha^c (t^c)^{ab} \quad (t^c)^{ab} = i f^{acb}$$

Adjoint representation

$$A^a = q^+ t^a q' \quad A^a \rightarrow q^+ U^+ t^a U q' = U^{ab} A^b$$

$$U^+ t^a U = U^{ab} t^b \quad U^{ab} = \frac{1}{T_F} \text{Tr} U^+ t^a U t^b$$

$(t^a)^i_j$ — fixed numbers (invariant tensor)

$$(t^a)^i_j \rightarrow U^{ab} U^i_k (t^b)^k_l (U^+)^l_j = (t^a)^i_j$$

Infinitesimal transformations

$$A^a \rightarrow q^+ (1 - i\alpha^c t^c) t^a (1 + i\alpha^c t^c) q' = q^+ (t^a + i\alpha^c i f^{acb} t^b) q'$$

$$U^{ab} = \delta^{ab} + i\alpha^c (t^c)^{ab} \quad (t^c)^{ab} = i f^{acb}$$

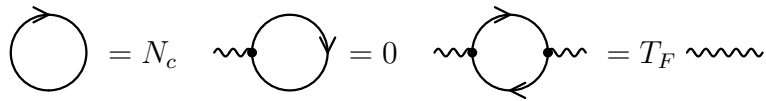
Commutation relation

$$(t^a)^{dc} (t^b)^{ce} - (t^b)^{dc} (t^a)^{ce} = i f^{abc} (t^c)^{de}$$

follows from the Jacobi identity

$$[t^a, [t^b, t^d]] + [t^b, [t^d, t^a]] + [t^d, [t^a, t^b]] = 0$$

Graphical form

$$\begin{array}{ccc} \text{Tr } 1 = N_c & \text{Tr } t^a = 0 & \text{Tr } t^a t^b = T_F \delta^{ab} \end{array}$$


The image shows three Feynman diagrams representing trace identities. The first diagram is a circle with a single arrow pointing clockwise, representing the trace of the identity matrix, $\text{Tr } 1 = N_c$. The second diagram is a circle with a wavy line entering from the left and an arrow pointing clockwise, representing the trace of a generator t^a , $\text{Tr } t^a = 0$. The third diagram is a circle with two wavy lines entering from the left and two arrows pointing clockwise, representing the trace of the product of two generators $t^a t^b$, $\text{Tr } t^a t^b = T_F \delta^{ab}$.

$$T_F = \frac{1}{2}$$

Cvitanović algorithm 1

Invariant tensor — via δ_j^i

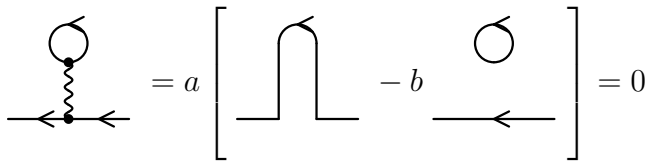
$$(t^a)^i_j (t^a)^k_l = a [\delta_l^i \delta_j^k - b \delta_j^i \delta_l^k]$$

$= a \left[\begin{array}{c} \text{---} \downarrow \text{---} \\ \text{---} \end{array} - b \begin{array}{c} \text{---} \uparrow \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \rightarrow \text{---} \\ \text{---} \leftarrow \text{---} \end{array} \right]$

Cvitanović algorithm 2

Multiply by δ_i^j

$$(t^a)^i_i (t^a)^k_l = 0 = a [\delta_l^k - b N_c \delta_l^k]$$



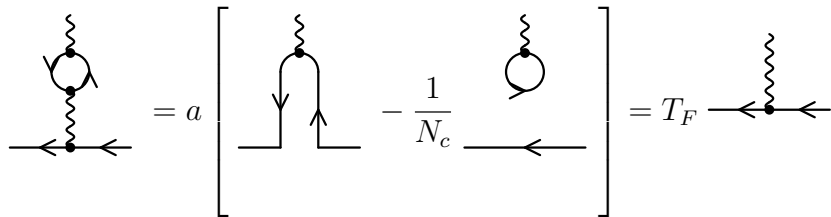
The diagram shows an equation between Feynman diagrams. On the left is a diagram with a horizontal line and two arrows pointing left, a vertex, a wavy line, and a loop. This is equal to a times a bracketed sum of two diagrams: a diagram with a horizontal line and two arrows pointing left, a vertex, and a loop, and a diagram with a horizontal line and one arrow pointing left, a vertex, and a loop. The second diagram in the bracket is multiplied by $-b$. The entire expression is equal to 0.

$$b = \frac{1}{N_c}$$

Cvitanović algorithm 3

Multiply by $(t^b)^{j_i}$

$$(t^b)^{j_i}(t^a)^i_j(t^a)^k_l = T_F(t^b)^k_l = a \left[(t^b)^k_l - \frac{1}{N_c}(t^b)^i_i\delta_l^k \right]$$



$$a = T_F$$

Cvitanović algorithm

$$(t^a)^i_j (t^a)^k_l = T_F \left[\delta_l^i \delta_j^k - \frac{1}{N_c} \delta_j^i \delta_l^k \right]$$

$$= T_F \left[\begin{array}{c} \text{---} \downarrow \text{---} \\ \text{---} \end{array} - \frac{1}{N_c} \begin{array}{c} \text{---} \uparrow \text{---} \\ \text{---} \end{array} \right]$$

Cvitanović algorithm

$$(t^a)^i_j (t^a)^k_l = T_F \left[\delta_l^i \delta_j^k - \frac{1}{N_c} \delta_j^i \delta_l^k \right]$$

$$\begin{array}{c} \rightarrow \bullet \rightarrow \\ | \text{wavy} \\ \leftarrow \bullet \leftarrow \end{array} = T_F \left[\begin{array}{c} \square \downarrow \\ \square \uparrow \end{array} - \frac{1}{N_c} \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \right]$$

$$\begin{array}{c} \square \downarrow \\ \square \uparrow \end{array} = \frac{1}{T_F} \left[\begin{array}{c} \rightarrow \bullet \rightarrow \\ | \text{wavy} \\ \leftarrow \bullet \leftarrow \end{array} + \frac{1}{N_c} \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \right]$$

$$q^i q_j^+ = \frac{1}{T_F} \left[(q^+ t^a q') (t^a)^i_j + \frac{1}{N_c} (q^+ q') \delta_j^i \right]$$

The product of the fundamental representation and its conjugate is the sum of the adjoint representation and the trivial one. The state of a quark–antiquark pair with some fixed colours is a superposition of the colour-singlet and the colour-adjoint states.

Counting gluons

$$\begin{aligned} N_g &= \text{[Diagram: A circle with a wavy (gluon) boundary]} \\ &= \frac{1}{T_F} \text{[Diagram: A circle with a wavy boundary and an internal fermion loop]} \\ &= \frac{1}{T_F} \text{[Diagram: A circle with a wavy internal line and a fermion loop]} \\ &= \text{[Diagram: A circle with two vertical lines and a fermion loop]} - \frac{1}{N_c} \text{[Diagram: A circle with a fermion loop]} = N_c^2 - 1 \end{aligned}$$

C_F

$$\begin{aligned} & \text{Diagram with wavy loop} = T_F \left[\text{Diagram with fermion loop} - \frac{1}{N_c} \text{Diagram with straight line} \right] \\ & = T_F \left(N_c - \frac{1}{N_c} \right) \longrightarrow \end{aligned}$$

C_F

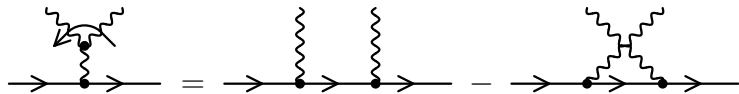
$$\text{Diagram} = T_F \left[\text{Diagram} - \frac{1}{N_c} \text{Diagram} \right]$$

$$= T_F \left(N_c - \frac{1}{N_c} \right) \longrightarrow$$

$$\text{Diagram} = C_F \longrightarrow \quad \text{or} \quad t^a t^a = C_F$$

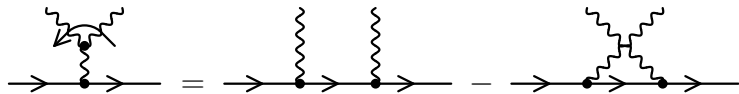
$$C_F = T_F \left(N_c - \frac{1}{N_c} \right)$$

3-gluon vertex

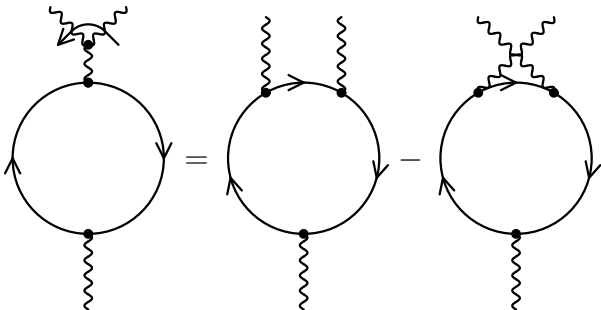


$$[t^a, t^b] = i f^{abc} t^c$$

3-gluon vertex



$$[t^a, t^b] = if^{abc}t^c$$



3-gluon vertex



The diagram on the left shows a three-gluon vertex. A central red dot is connected to three wavy red lines. Two lines enter from the top, and one line exits downwards. The lines are connected by a small loop with arrows indicating a clockwise direction.

$$= \frac{1}{T_F} \left[\text{Diagram 1} - \text{Diagram 2} \right]$$

The diagram on the right shows the decomposition of the vertex into two loop diagrams. The first diagram is a circle with three vertices (red dots) and three wavy lines (two entering from the top, one exiting from the bottom). The second diagram is identical but with the wavy lines permuted: one enters from the top left, one enters from the top right, and one exits from the bottom.

3-gluon vertex

$$= \frac{1}{T_F} \left[\text{Diagram 1} - \text{Diagram 2} \right]$$

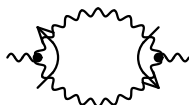
The diagram shows the 3-gluon vertex in red. The left side is a tree-level vertex with two incoming gluons and one outgoing gluon, with a loop of two gluons on the top vertex. The right side is the difference of two loop diagrams, each with a gluon loop and a ghost loop, enclosed in large square brackets. The first loop diagram has the gluon loop on top and the ghost loop on the bottom. The second loop diagram has the gluon loop on the left and the ghost loop on the right.

$$= \text{Diagram 3} - \text{Diagram 4}$$

The diagram shows the 3-gluon vertex in black. The left side is a tree-level vertex with two incoming gluons and one outgoing gluon, with a loop of two gluons on the top vertex. The right side is the difference of two loop diagrams, each with a gluon loop and a ghost loop. The first loop diagram has the gluon loop on top and the ghost loop on the bottom. The second loop diagram has the gluon loop on the left and the ghost loop on the right.

$$(t^a)^{dc} (t^b)^{ce} - (t^b)^{dc} (t^a)^{ce} = i f^{abc} (t^c)^{de}$$

C_A


$$= \frac{2}{T_F^2} \left[\text{Diagram 1} - \text{Diagram 2} \right]$$
$$= \frac{2}{T_F} \left[\text{Diagram 3} - \frac{1}{N_c} \text{Diagram 4} \right]$$
$$- \left[\text{Diagram 5} + \frac{1}{N_c} \text{Diagram 6} \right]$$

The diagrams are:
Diagram 1: Two fermion loops connected by two wavy lines.
Diagram 2: Two fermion loops connected by two wavy lines, with a different fermion flow.
Diagram 3: A fermion loop with a wavy line and a fermion line crossing.
Diagram 4: Two fermion loops connected by a wavy line.
Diagram 5: A fermion loop with a wavy line and a fermion line crossing, with a different fermion flow.
Diagram 6: Two fermion loops connected by a wavy line, with a different fermion flow.

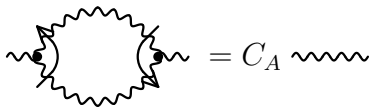
C_A

$$\begin{aligned} &= \frac{2}{T_F} \left[\text{Diagram 1} - \text{Diagram 2} \right] \\ &= 2 \left[\text{Diagram 3} - \frac{1}{N_c} \text{Diagram 4} \right. \\ &\quad \left. - \text{Diagram 5} - \text{Diagram 6} + \frac{1}{N_c} \text{Diagram 7} \right] \\ &= 2T_F N_c \text{Diagram 8} \end{aligned}$$

The diagrams are as follows:

- Diagram 1: A circle with two wavy lines on the left and two on the right. A wavy line connects the two bottom vertices.
- Diagram 2: A circle with two wavy lines on the left and two on the right. A vertical wavy line connects the two top vertices.
- Diagram 3: A circle with two wavy lines on the left and two on the right. A vertical line with an arrow connects the two top vertices.
- Diagram 4: A circle with two wavy lines on the left and two on the right. A vertical line with an arrow connects the two bottom vertices.
- Diagram 5: A circle with two wavy lines on the left and two on the right. A vertical line with an arrow connects the two top vertices.
- Diagram 6: A circle with two wavy lines on the left and two on the right. A vertical line with an arrow connects the two bottom vertices.
- Diagram 7: A circle with two wavy lines on the left and two on the right. A vertical line with an arrow connects the two top vertices.
- Diagram 8: A single wavy line.

C_A



A Feynman diagram showing a gluon loop (represented by a sun-like shape) with two external wavy lines. This is equated to a single wavy line multiplied by the constant C_A .

$$i f^{acd} i f^{bdc} = C_A \delta^{ab} \quad C_A = 2T_F N_c$$

Example 1

$$\text{Diagram} = T_F \left[\text{Diagram} - \frac{1}{N_c} \text{Diagram} \right]$$

$$= -\frac{T_F}{N_c} \text{Diagram}$$

$$t^a t^b t^a = -\frac{T_F}{N_c} t^b \quad -\frac{T_F}{N_c} = C_F - \frac{C_A}{2}$$

Example 2

$$\begin{aligned}
 & \text{Diagram 1} = \frac{1}{T_F} \left[\text{Diagram 2} - \text{Diagram 3} \right] \\
 & = \text{Diagram 4} - \frac{1}{N_c} \text{Diagram 5} - \text{Diagram 6} + \frac{1}{N_c} \text{Diagram 7} \\
 & = \text{Diagram 8} - \text{Diagram 9}
 \end{aligned}$$

The diagram shows the following steps:

- Step 1:** A triangle diagram with a wavy line on the top edge and a straight line on the bottom edge. It is equal to $\frac{1}{T_F}$ times the difference of two bubble diagrams. Each bubble diagram has a wavy line on the top edge and a straight line on the bottom edge.
- Step 2:** The difference of two bubble diagrams is further decomposed into four terms: a diagram with a wavy line on the top edge and a straight line on the bottom edge, a term $-\frac{1}{N_c}$ times a bubble diagram with a wavy line on the top edge and a straight line on the bottom edge, a diagram with a wavy line on the top edge and a straight line on the bottom edge, and a term $+\frac{1}{N_c}$ times a bubble diagram with a wavy line on the top edge and a straight line on the bottom edge.
- Step 3:** The four terms are simplified into two diagrams: a diagram with a wavy line on the top edge and a straight line on the bottom edge, and a diagram with a wavy line on the top edge and a straight line on the bottom edge.

Example 2

$$\begin{aligned}
 &= T_F \left[\begin{array}{c} \text{Diagram 1} - \frac{1}{N_c} \text{Diagram 2} \\ - \text{Diagram 3} + \frac{1}{N_c} \text{Diagram 4} \end{array} \right] \\
 &= T_F N_c \text{Diagram 5} \\
 i f^{abc} t^b t^a &= \frac{C_A}{2} t^c
 \end{aligned}$$

The diagrams are:

- Diagram 1: A fermion line with a wavy line and a loop attached to the top vertex.
- Diagram 2: A fermion line with a wavy line attached to the top vertex.
- Diagram 3: A fermion line with a loop attached to the top vertex.
- Diagram 4: A fermion line with a wavy line attached to the bottom vertex.
- Diagram 5: A fermion line with a wavy line attached to the top vertex.

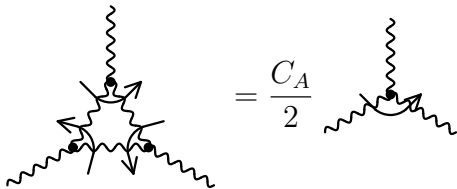
Shorter solution

$$\begin{aligned} & \text{Diagram 1} = \frac{1}{2} \left[\text{Diagram 2} - \text{Diagram 3} \right] \\ & = \frac{1}{2} \text{Diagram 4} = \frac{C_A}{2} \text{Diagram 5} \end{aligned}$$

The diagrams are as follows:

- Diagram 1:** A fermion line with two vertices. The upper vertex is connected to a wavy line that ends in a fermion line with an arrow pointing up and to the right.
- Diagram 2:** Similar to Diagram 1, but the wavy line is a straight line.
- Diagram 3:** Similar to Diagram 1, but the wavy line forms a loop that connects the two vertices.
- Diagram 4:** Similar to Diagram 1, but the wavy line forms a loop that connects the two vertices, with a fermion line segment inside the loop.
- Diagram 5:** A fermion line with two vertices. The upper vertex is connected to a wavy line that ends in a fermion line with an arrow pointing up and to the right.

Example 3



The diagram shows a reduction of a complex Feynman diagram. On the left, a central vertex is connected to three external wavy lines. The top line is a single wavy line. The bottom-left and bottom-right lines are each connected to a sub-diagram consisting of a wavy line loop with a fermion line (solid line with arrows) passing through it. On the right, the same diagram is shown as a single vertex connected to three external wavy lines, with a coefficient $\frac{C_A}{2}$ in front of it.

$$= \frac{C_A}{2}$$

$$i f^{adf} i f^{bed} i f^{cfe} = \frac{C_A}{2} i f^{abc}$$