Massless QCD

$$L = \sum_i \bar{q}_{0i} i \not{D} q_{0i} - \frac{1}{4} G^a_{0\mu\nu} G_0^{a\mu\nu}$$

$$D_\mu q_0 = \left(\partial_\mu - i g_0 A_{0\mu}\right) q_0 \qquad A_{0\mu} = A^a_{0\mu} t^a$$

$$[D_\mu, D_\nu] q_0 = -i g_0 G_{0\mu\nu} q_0 \qquad G_{0\mu\nu} = G^a_{0\mu\nu} t^a$$

$$G^a_{0\mu\nu} = \partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu} + g_0 f^{abc} A^b_{0\mu} A^c_{0\nu}$$

Massless QCD

$$L = \sum_i \bar{q}_{0i} i \not{D} q_{0i} - \frac{1}{4} G^a_{0\mu\nu} G^{a\mu\nu}_0 - \frac{1}{2a_0} \left(\partial_\mu A^{a\mu}_0\right)^2 + (\partial^\mu \bar{c}^a_0)(D_\mu c^a_0)$$

$$D_\mu q_0 = \left(\partial_\mu - i g_0 A_{0\mu}\right) q_0 \qquad A_{0\mu} = A^a_{0\mu} t^a$$

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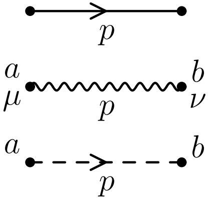
$$G^a_{0\mu\nu} = \partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu} + g_0 f^{abc} A^b_{0\mu} A^c_{0\nu}$$

$$D_\mu c^a_0 = (\partial_\mu \delta^{ab} - i g_0 A^{ab}_{0\mu}) c^b_0 \qquad A^{ab}_{0\mu} = A^c_{0\mu} (t^c)^{ab}$$

$$[t^a, t^b] = i f^{abc} t^c \qquad (t^c)^{ab} = i f^{acb}$$

Feynman rules

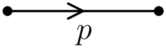


$$= iS_0(p) \qquad S_0(p) = \frac{1}{\not{p}} = \frac{\not{p}}{p^2}$$

$$= -i\delta^{ab} D^0_{\mu\nu}(p)$$

$$= i\delta^{ab} G_0(p) \qquad G_0(p) = \frac{1}{p^2}$$

Feynman rules



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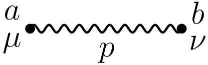

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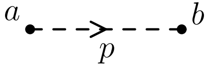
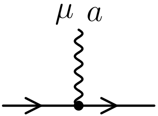
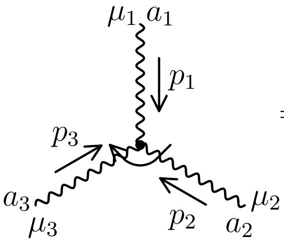


$$= i\delta^{ab} G_0(p) \qquad G_0(p) = \frac{1}{p^2}$$



$$= t^a \times ig_0\gamma^\mu$$

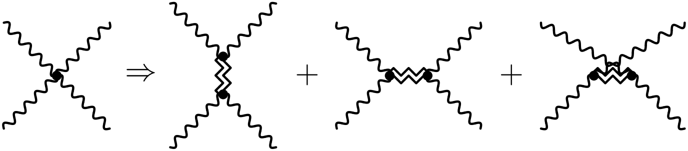
Feynman rules



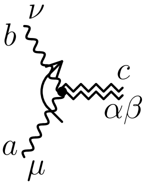
$$= if^{a_1 a_2 a_3} \times i g_0 V^{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)$$

$$V^{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3) =$$
$$(p_3 - p_2)^{\mu_1} g^{\mu_2 \mu_3} + (p_1 - p_3)^{\mu_2} g^{\mu_3 \mu_1} + (p_2 - p_1)^{\mu_3} g^{\mu_1 \mu_2}$$

Feynman rules

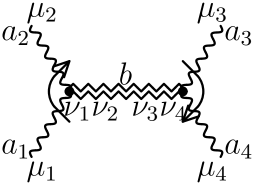


$$= i\delta^{ab}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})$$



$$= if^{abc} \times g_0 g^{\mu\alpha}g^{\nu\beta}$$

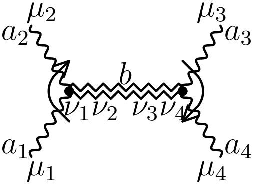
Feynman rules



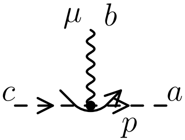
$$if^{a_1 a_2 b} i f^{a_3 a_4 b} \times g_0 g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} i \left(g_{\nu_1 \nu_3} g_{\nu_2 \nu_4} - g_{\nu_1 \nu_4} g_{\nu_2 \nu_3} \right) g_0 g^{\mu_3 \nu_3} g^{\mu_4 \nu_4}$$

$$= if^{a_1 a_2 b} i f^{a_3 a_4 b} \times i g_0^2 \left(g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} \right)$$

Feynman rules



$$if^{a_1 a_2 b} if^{a_3 a_4 b} \times g_0 g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} i \left(g_{\nu_1 \nu_3} g_{\nu_2 \nu_4} - g_{\nu_1 \nu_4} g_{\nu_2 \nu_3} \right) g_0 g^{\mu_3 \nu_3} g^{\mu_4 \nu_4}$$
$$= if^{a_1 a_2 b} if^{a_3 a_4 b} \times i g_0^2 \left(g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} \right)$$



$$= if^{abc} \times i g_0 p^\mu$$

Renormalization

$$q_{i0} = Z_q^{1/2} q_i \qquad A_0 = Z_A^{1/2} A \qquad c_0 = Z_c^{1/2} c$$
$$a_0 = Z_A a \qquad g_0 = Z_\alpha^{1/2} g$$

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$$\frac{\alpha_s(\mu)}{4\pi} = \mu^{-2\varepsilon} \frac{g^2}{(4\pi)^{d/2}} e^{-\gamma\varepsilon}$$
$$\frac{g_0^2}{(4\pi)^{d/2}} = \mu^{2\varepsilon} \frac{\alpha_s(\mu)}{4\pi} Z_\alpha(\alpha(\mu)) e^{\gamma\varepsilon}$$

$SU(N_c)$

Fundamental representation: q^i

$$q \to Uq \quad \text{or} \quad q^i \to U^i{}_j q^j$$

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$$q^+ \to q^+ U^+ \quad \text{or} \quad q_i^+ \to q_j^+ (U^+)^j{}_i \quad \text{where} \quad (U^+)^j{}_i = \left(U^i{}_j\right)^*$$

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$\varepsilon^{i_1 \ldots i_{N_c}}$ and $\varepsilon_{i_1 \ldots i_{N_c}}$ are also invariant (baryons)

$$\varepsilon^{i_1 \ldots i_{N_c}} \to \varepsilon^{j_1 \ldots j_{N_c}} \, U^{i_1}{}_{j_1} \cdots U^{i_{N_c}}{}_{j_{N_c}} = \det U \cdot \varepsilon^{i_1 \ldots i_{N_c}} = \varepsilon^{i_1 \ldots i_{N_c}}$$

Infinitesimal transformations

$$U = 1 + i\alpha^a t^a$$

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$$\det U = 1 + i\alpha^a \, \mathrm{Tr}\, t^a = 1 \quad \Rightarrow \quad \mathrm{Tr}\, t^a = 0$$

$N_c^2 - 1$ generators t^a

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$$[t^a, t^b]_+ = 2\frac{T_F}{N_c}\delta^{ab} + d^{abc} t^c \qquad d^{abc} = \frac{1}{T_F} \operatorname{Tr}[t^a, t^b]_+ t^c$$

Adjoint representation

$$A^a = q^+ t^a q' \qquad A^a \to q^+ U^+ t^a U q' = U^{ab} A^b$$

$$U^+ t^a U = U^{ab} t^b \qquad U^{ab} = \frac{1}{T_F} \operatorname{Tr} U^+ t^a U t^b$$

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$(t^a)^i{}_j$ — fixed numbers (invariant tensor)

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Infinitesimal transformations

$$A^a \to q^+(1 - i\alpha^c t^c) t^a (1 + i\alpha^c t^c) q' = q^+(t^a + i\alpha^c i f^{acb} t^b) q'$$

$$U^{ab} = \delta^{ab} + i\alpha^c (t^c)^{ab} \qquad (t^c)^{ab} = i f^{acb}$$

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$$U^{ab} = \delta^{ab} + i\alpha^c (t^c)^{ab} \qquad (t^c)^{ab} = i f^{acb}$$

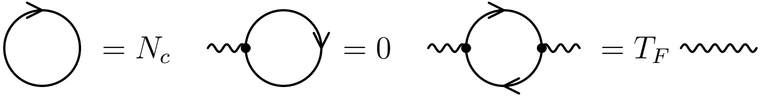
Commutation relation

$$(t^a)^{dc}(t^b)^{ce} - (t^b)^{dc}(t^a)^{ce} = i f^{abc} (t^c)^{de}$$

follows from the Jacobi identity

$$[t^a, [t^b, t^d]] + [t^b, [t^d, t^a]] + [t^d, [t^a, t^b]] = 0$$

Graphical form



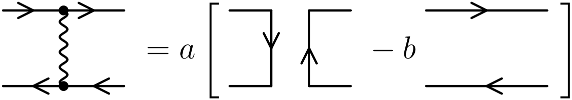
$\mathrm{Tr}\, 1 = N_c$ $\quad$ $\mathrm{Tr}\, t^a = 0$ $\quad$ $\mathrm{Tr}\, t^a t^b = T_F \delta^{ab}$

$$T_F = \frac{1}{2}$$

Cvitanović algorithm 1

Invariant tensor — via δ^i_j

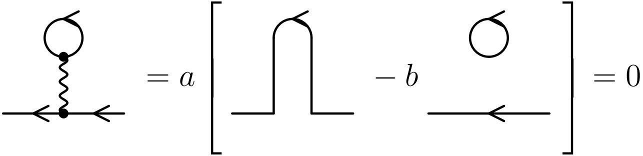
$$(t^a)^i{}_j (t^a)^k{}_l = a\left[\delta^i_l \delta^k_j - b\delta^i_j \delta^k_l\right]$$



Cvitanović algorithm 2

Multiply by δ_i^j

$$(t^a)^i{}_i (t^a)^k{}_l = 0 = a \left[\delta_l^k - b N_c \delta_l^k \right]$$

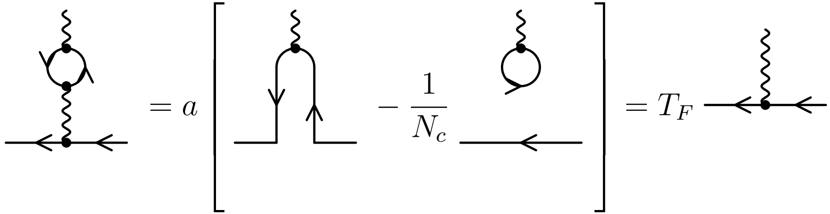


$$b = \frac{1}{N_c}$$

Cvitanović algorithm 3

Multiply by $(t^b)^j{}_i$

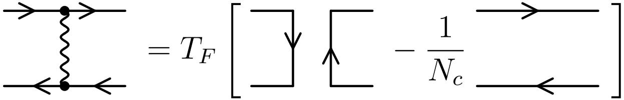
$$(t^b)^j{}_i (t^a)^i{}_j (t^a)^k{}_l = T_F (t^b)^k{}_l = a \left[(t^b)^k{}_l - \frac{1}{N_c} (t^b)^i{}_i \delta^k_l \right]$$



$$a = T_F$$

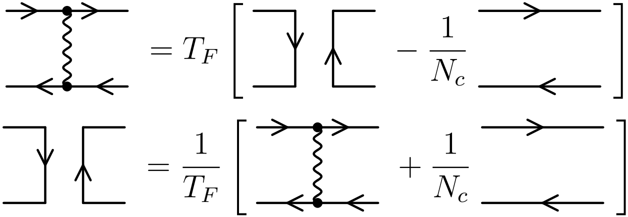
Cvitanović algorithm

$$(t^a)^i{}_j (t^a)^k{}_l = T_F \left[\delta^i_l \delta^k_j - \frac{1}{N_c} \delta^i_j \delta^k_l \right]$$



Cvitanović algorithm

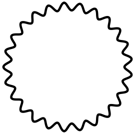
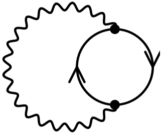
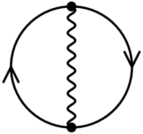
$$(t^a)^i{}_j (t^a)^k{}_l = T_F \left[\delta^i_l \delta^k_j - \frac{1}{N_c} \delta^i_j \delta^k_l \right]$$

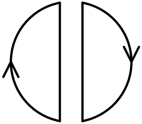
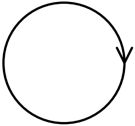


$$q'^i q^+_j = \frac{1}{T_F} \left[(q^+ t^a q')(t^a)^i{}_j + \frac{1}{N_c} (q^+ q') \delta^i_j \right]$$

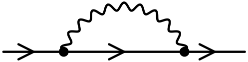
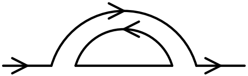
The product of the fundamental representation and its conjugate is the sum of the adjoint representation and the trivial one. The state of a quark–antiquark pair with some fixed colours is a superposition of the colour-singlet and the colour-adjoint states.

Counting gluons

$N_g =$  $= \frac{1}{T_F}$  $= \frac{1}{T_F}$ 

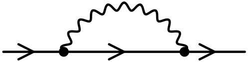
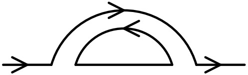
$=$  $- \frac{1}{N_c}$  $= N_c^2 - 1$

C_F

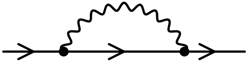
 $= T_F$ [ $- \frac{1}{N_c}$ ]

$= T_F \left(N_c - \frac{1}{N_c} \right)$ 

C_F

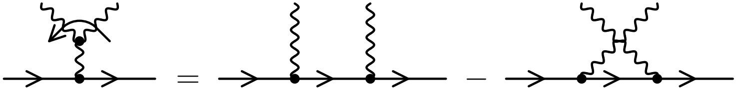
 $= T_F \Big[$  $- \frac{1}{N_c}$  $\Big]$

$= T_F \left(N_c - \frac{1}{N_c} \right)$ 

 $= C_F$  or $t^a t^a = C_F$

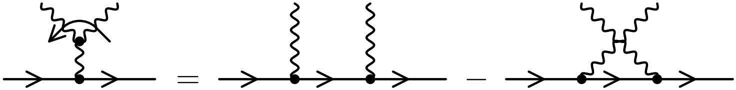
$$C_F = T_F \left(N_c - \frac{1}{N_c} \right)$$

3-gluon vertex

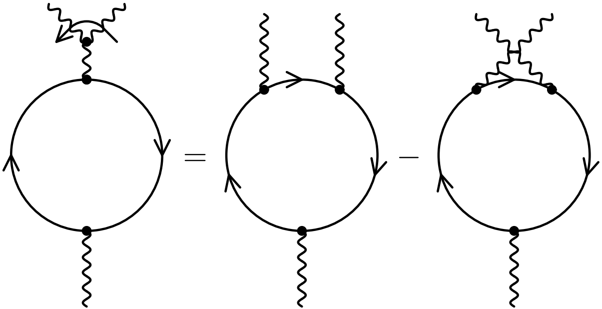


$$[t^a, t^b] = i f^{abc} t^c$$

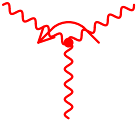
3-gluon vertex



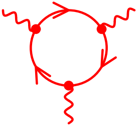
$$[t^a, t^b] = i f^{abc} t^c$$



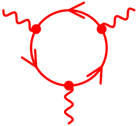
3-gluon vertex



$$= \frac{1}{T_F} \Bigg[$$

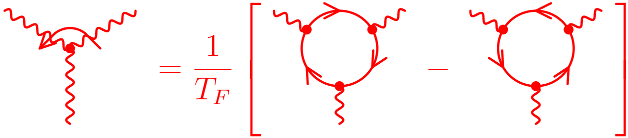


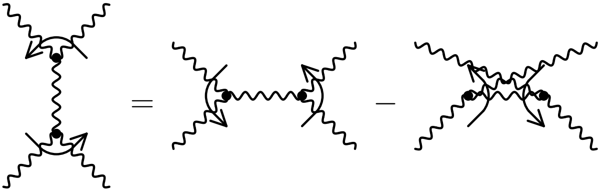
$$-$$



$$\Bigg]$$

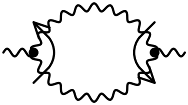
3-gluon vertex

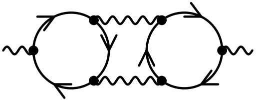
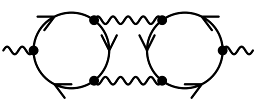


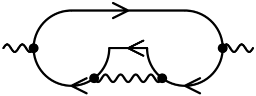
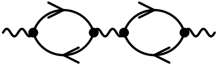


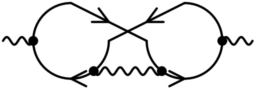
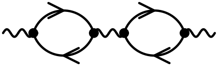
$$(t^a)^{dc}(t^b)^{ce} - (t^b)^{dc}(t^a)^{ce} = if^{abc}(t^c)^{de}$$

C_A

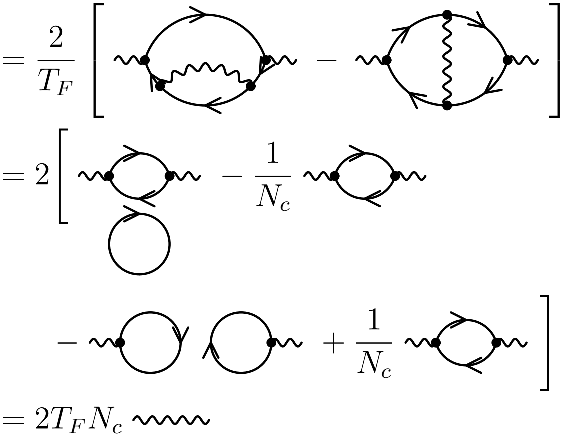


$= \frac{2}{T_F^2} \Bigg[$  $-$  $\Bigg]$

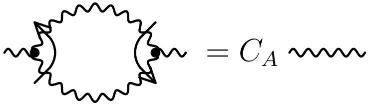
$= \frac{2}{T_F} \Bigg[$  $- \frac{1}{N_c}$ 

$-$  $+ \frac{1}{N_c}$  $\Bigg]$

C_A

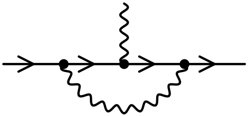
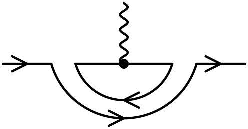
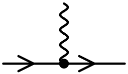


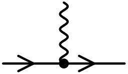
C_A



$$if^{acd}if^{bdc} = C_A \delta^{ab} \qquad C_A = 2T_F N_c$$

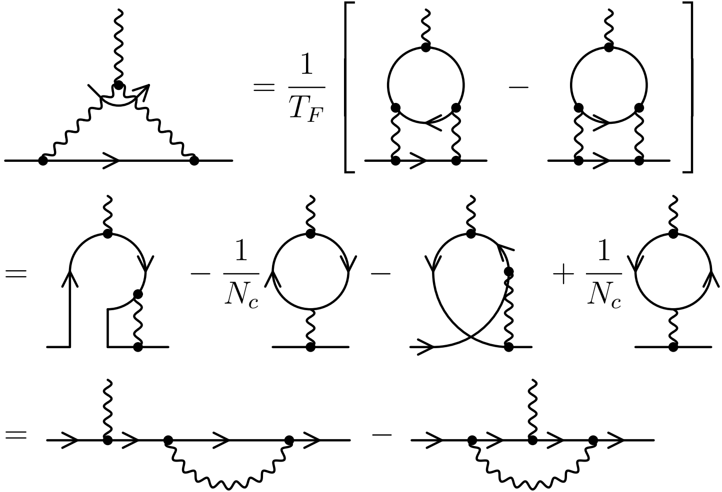
Example 1

 $= T_F \Bigg[$  $- \frac{1}{N_c}$  $\Bigg]$

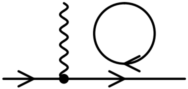
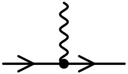
$= -\frac{T_F}{N_c}$ 

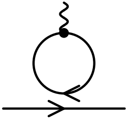
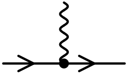
$$t^a t^b t^a = -\frac{T_F}{N_c} t^b \qquad -\frac{T_F}{N_c} = C_F - \frac{C_A}{2}$$

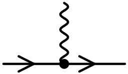
Example 2



Example 2

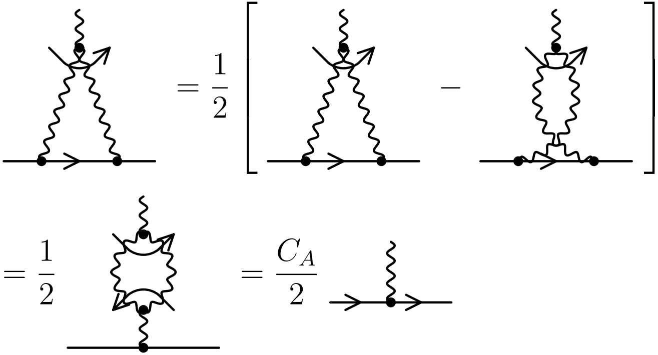
$= T_F \Big[$  $- \frac{1}{N_c}$ 

$-$  $+ \frac{1}{N_c}$  $\Big]$

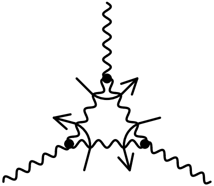
$= T_F N_c$ 

$$i f^{abc} t^b t^a = \frac{C_A}{2} t^c$$

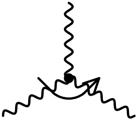
Shorter solution



Example 3



$$= \frac{C_A}{2}$$



$$if^{adf}\,if^{bed}\,if^{cfe} = \frac{C_A}{2}\,if^{abc}$$