The Big Bang Cosmology:
Lecture #3
The Universe: from Today back to BBN

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Outline

1. The present Universe
   - MD/Λ transition
   - RD/MD transition
   - The age and the horizon of the Universe
   - Brightness–redshift dependence
2. Recombination
   - temperature
   - time
   - last scattering
   - horizon
   - angular size

Friedmann equaiton for the present Universe

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G \left(\rho_{\mathrm{M}} + \rho_{rad} + \rho_{\Lambda} + \rho_{\mathrm{curv}}\right)$$

$$\frac{8\pi}{3} G \rho_{\mathrm{curv}} = -\frac{\varkappa}{a^2}\,, \quad \rho_c \equiv \frac{3}{8\pi G} H_0^2$$

$$\rho_c = \rho_{\mathrm{M},0} + \rho_{rad,0} + \rho_{\Lambda,0} = \rho_c = 0.53 \cdot 10^{-5} \frac{\mathrm{GeV}}{\mathrm{cm}^3}\,, \qquad \text{for } h = 0.7$$

$$\Omega_X \equiv \frac{\rho_{X,0}}{\rho_c}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G \rho_c \left[\Omega_{\mathrm{M}} \left(\frac{a_0}{a}\right)^3 + \Omega_{rad} \left(\frac{a_0}{a}\right)^4 + \Omega_{\Lambda} + \Omega_{curv} \left(\frac{a_0}{a}\right)^2\right]$$

Deceleration–acceleration transition

$$\dot{a}^2 = \frac{8\pi}{3} G\rho_c \left(\frac{\Omega_{\rm M} a_0^3}{a} + \Omega_\Lambda a^2 \right) \qquad \ddot{a} = a \frac{4\pi}{3} G\rho_c \left(2\Omega_\Lambda - \Omega_{\rm M} \left(\frac{a_0}{a} \right)^3 \right)$$

$$z_{acc} = \left(\frac{2\Omega_\Lambda}{\Omega_{\rm M}} \right)^{1/3} - 1 \approx 0.85$$

$z \equiv a_0/a - 1$ at $z \gg 1$ dust-dominated

matter-radiation (RD/MD) transition

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho_c \left[\Omega_{\rm M}\left(\frac{a_0}{a}\right)^3 + \Omega_{rad}\left(\frac{a_0}{a}\right)^4\right]$$

RD: $a(t) \propto \sqrt{t}$ MD: $a(t) \propto t^{2/3}$

$$z_{eq} + 1 = \frac{a_0}{a_{eq}} \sim \frac{\Omega_{\rm M}}{\Omega_\gamma} \sim 10^4\,, \qquad T_{eq} = T_0(1+z_{eq}) \sim 10^4\ \text{K} \sim 1\ \text{eV}\,.$$

More accurately: $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$

$$\rho_\nu = 3\cdot 2\cdot \frac{7}{8}\frac{\pi^2}{30} T_\nu^4\,, \quad \rho_{rad} = \rho_\gamma + \rho_\nu = \left[2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\right]\frac{\pi^2}{30} T^4$$

matter-radiation (RD/MD) transition

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho_c \left[\Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_{rad} \left(\frac{a_0}{a}\right)^4\right]$$

RD: $a(t) \propto \sqrt{t}$ smooth transition MD: $a(t) \propto t^{2/3}$

$$z_{eq} + 1 = \frac{a_0}{a_{eq}} = 0.6 \frac{\Omega_M}{\Omega_\gamma} = 3.0 \cdot 10^3 \,, \qquad T_{eq} = T_0(1 + z_{eq}) = 0.7 \text{ eV}$$

More accurately: $t_{eq} = \frac{1}{2H_{eq}} = \frac{M_{Pl}^*}{2T_{eq}^2} = 80$ kyr $\quad T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$

$$\rho_\nu = 3 \cdot 2 \cdot \frac{7}{8} \frac{\pi^2}{30} T_\nu^4 \,, \quad \rho_{rad} = \rho_\gamma + \rho_\nu = \left[2 + \frac{21}{4} \left(\frac{4}{11}\right)^{4/3}\right] \frac{\pi^2}{30} T^4$$

The age of the Universe

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\Omega_{\rm M}\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda\right],$$

$$\Omega_{\rm M} + \Omega_\Lambda = 1\,, \quad a(t) = a_0\left(\frac{\Omega_{\rm M}}{\Omega_\Lambda}\right)^{1/3}\left[\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right]^{2/3}$$

$$t_0 = \frac{2}{3\sqrt{\Omega_\Lambda}}\frac{1}{H_0}{\rm Arsinh}\sqrt{\frac{\Omega_\Lambda}{\Omega_{\rm M}}} = 1.38\cdot 10^{10}\ {\rm yr}$$

$$\Omega_{\rm M} = 0.24\,,\ \ \Omega_\Lambda = 0.76\,,\ \ h = 0.73$$

The size of the Horizon: visible part of the Universe

$$a(t) = a_0 \left(\frac{\Omega_{\rm M}}{\Omega_\Lambda}\right)^{1/3} \left[\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda} H_0 t\right)\right]^{2/3}$$

$$l_{\rm H,0} = a_0 \int_0^{t_0} \frac{dt}{a(t)} = \frac{2}{H_0} \cdot 1.8 = 14.8 \text{ Gpc}$$

Brightness–redshift dependence in the Universe

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

coordinate distance $\chi = \int_{t_i}^{t_0} \frac{dt}{a(t)}$ $\qquad z(t) = \frac{a_0}{a(t)} - 1$

$$\chi(z) = \int_0^z \frac{dz'}{a_0 H_0} \frac{1}{\sqrt{\Omega_{\mathrm{M}}(z'+1)^3 + \Omega_\Lambda + \Omega_{curv}(z'+1)^2}}$$

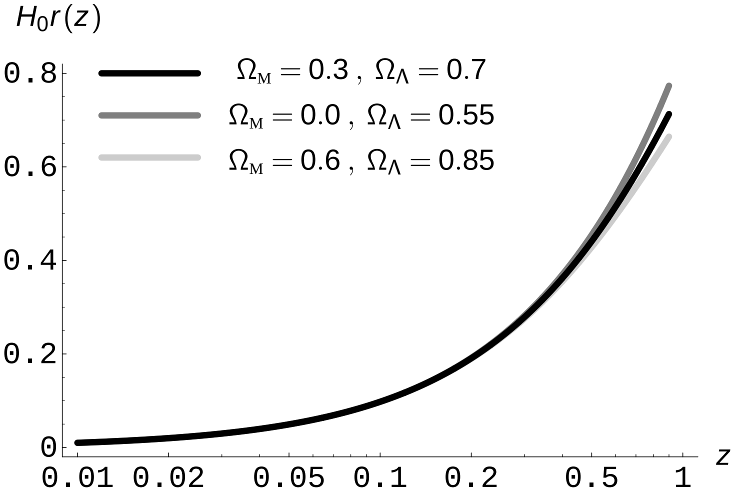
$$a_0^2 H_0^2 \Omega_{curv} = 1\,, \quad \Omega_{\mathrm{M}} + \Omega_\Lambda + \Omega_{curv} = 1$$

$$S(z) = 4\pi r^2(z)\,, \qquad r(z) = a_0 \sinh\chi(z)$$

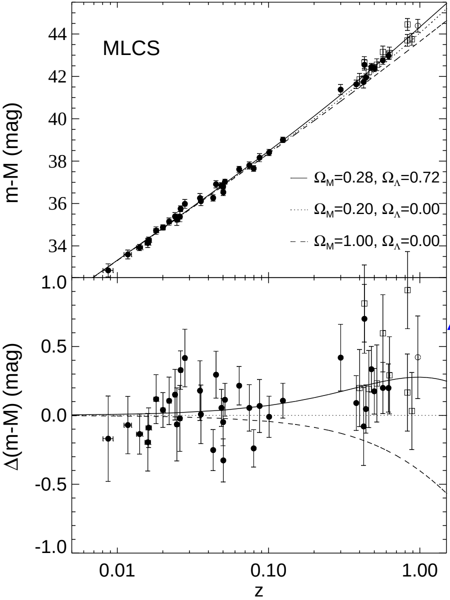
detector: $N_\gamma \propto S^{-1}$, $\omega = \omega_i/(1+z)$, $dt_0 = (1+z)dt_i$
hence, for the brightness (the flux as mesuared by a detector) one has

$$J = \frac{L}{(1+z)^2 S(z)} \equiv \frac{L}{4\pi r_{ph}^2}\,, \quad r_{ph} = (1+z)\cdot r(z)$$

Degeneracies in brightness–redshift dependence



Brightness–redshift dependence: SN Ia

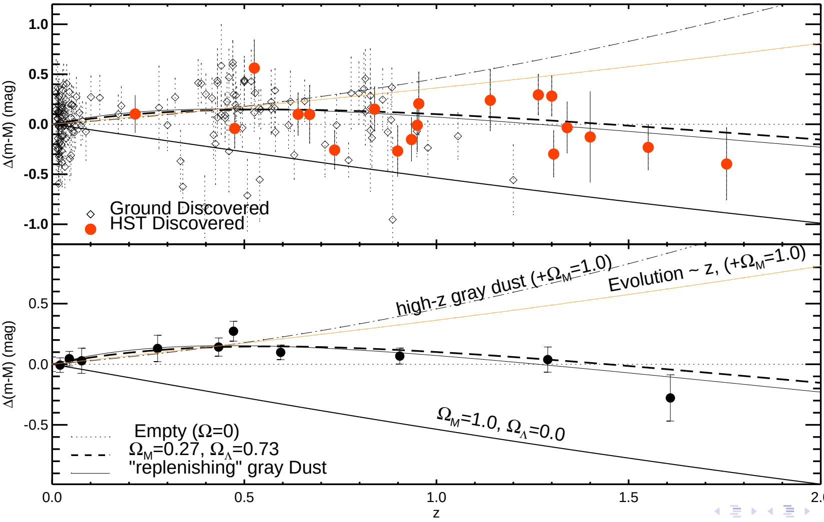


$$(m - M) = 5\log\frac{r_{ph}}{1\ \mathrm{Mpc}} + 25$$

$$\Delta(m - M) = 5\log\frac{r_{ph}}{r_{ph}(\Omega_c{=}0.8, \Omega_{\mathrm{M}}{=}0.2)}$$

Brightness–redshift dependence: SN Ia

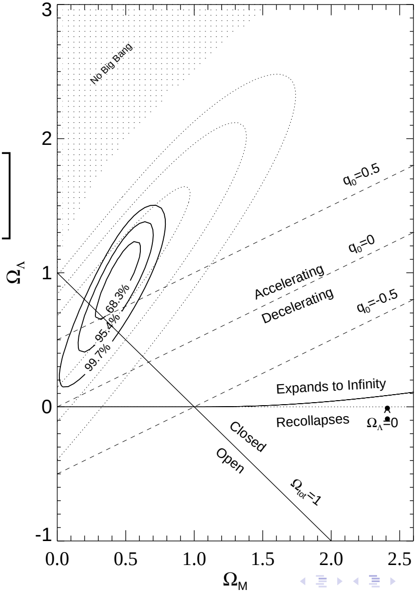
$$\Delta(m-M) = 5\log \frac{r_{ph}}{r_{ph}(\Omega_c = 0.8, \Omega_{\mathrm{M}} = 0.2)}$$



Cosmological parameters from SN Ia

$$\chi(z) = \frac{1}{a_0 H_0}\left[z - \frac{z^2}{4}(\Omega_M - 2\Omega_\Lambda) + C_1(\Omega_M - 2\Omega_\Lambda)\cdot z^3 + C_2(2\Omega_M - \Omega_\Lambda)\cdot z^3\right]$$

$$q_0 = \frac{1}{H_0^2}\left(\frac{\ddot{a}}{a}\right)_0$$



Temperature of the recombination

$$n_e = g_e \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{(\mu_e - m_e)/T} \,, \quad g_e = 2 \,,$$

$$n_p = g_p \left(\frac{m_p T}{2\pi}\right)^{3/2} e^{(\mu_p - m_p)/T} \,, \quad g_p = 2 \,,$$

$$n_{\rm H} = g_{\rm H} \left(\frac{m_{\rm H} T}{2\pi}\right)^{3/2} e^{(\mu_{\rm H} - m_{\rm H})/T} \,, \quad g_{\rm H} = 4$$

T_r: $\quad n_p(T_r) \simeq n_{\rm H}(T_r)$

baryon number conservation

$$n_p + n_{\rm H} = n_{\rm B} \,, \quad n_{\rm B}(T) = \eta_{\rm B} n_\gamma(T) \,,$$

chemical equilibrium

$$p + e \leftrightarrow H + \gamma : \quad \mu_p + \mu_e = \mu_{\rm H}$$

electroneutrality

$$n_p = n_e$$

Temperature of the recombination

$$X_p \equiv \frac{n_p}{n_{\rm B}}\,, \quad X_{\rm H} \equiv \frac{n_{\rm H}}{n_{\rm B}}\,, \quad \Delta_{\rm H} \equiv m_p + m_e - m_{\rm H} = 13.6\ \text{eV}$$

$$X_{\rm H} = \frac{2\zeta(3)}{\pi^2}\eta_{\rm B}\left(\frac{2\pi T}{m_e}\right)^{3/2} X_p^2 \mathrm{e}^{\frac{\Delta_{\rm H}}{T}}$$

Saha equation: $X_{\rm H} + X_p = 1$

$$X_p + \frac{2\zeta(3)}{\pi^2}\eta_{\rm B}\left(\frac{2\pi T}{m_e}\right)^{3/2} X_p^2 \mathrm{e}^{\frac{\Delta_{\rm H}}{T}} = 1$$

recombination: $X_p \sim 1$, $X_{\rm H} \sim 1$ $\qquad T_r = 0.33$ eV

$$T_r \approx \frac{\Delta_{\rm H}}{\ln\left(\frac{\sqrt{\pi}}{4\sqrt{2}\zeta(3)}\left(\frac{m_e}{\Delta_{\rm H}}\right)^{3/2}\eta_{\rm B}^{-1}\right)} \approx 0.38\ \text{eV}$$

Recombination: time

recombination: $n_p \sim n_{\mathrm{H}} \sim 1$ $T_r = 0.33$ eV

$z_r \sim 1400$

$$t_r = \frac{2}{3H(t_r)} = \left[\frac{M_{Pl}^2}{6\pi\rho_{\mathrm{M}}(T_r)}\right]^{1/2} .$$

$$\rho_{\mathrm{M}}(T) = \frac{\Omega_{\mathrm{M}}}{\Omega_{\mathrm{B}}} \cdot \rho_{\mathrm{B}}(T) = \frac{\Omega_{\mathrm{M}}}{\Omega_{\mathrm{B}}} \cdot m_p \cdot n_{\mathrm{B}}(T) = \frac{\Omega_{\mathrm{M}}}{\Omega_{\mathrm{B}}} \cdot m_p \eta_{\mathrm{B}} \cdot n_\gamma(T) .$$

$$t_r = \left[\frac{\pi}{12\zeta(3)} \frac{\Omega_{\mathrm{B}}}{\Omega_{\mathrm{M}}} \frac{M_{Pl}^2}{\eta_{\mathrm{B}} T_r^3 m_p}\right]^{1/2} .$$

$$t_r = 200\,000 \text{ years}$$

Last scattering: $\gamma e \to \gamma e$

$$\sigma_{\mathrm{T}} = \frac{8\pi}{3}\frac{\alpha^2}{m_e^2} \approx 0.67\cdot 10^{-24}\ \mathrm{cm}^2\,, \qquad \tau_\gamma = \frac{1}{\sigma_{\mathrm{T}}\cdot n_e(T)}$$

$$n_e^2 = n_p^2 = \left(\frac{m_e T}{2\pi}\right)^{3/2} \frac{2\zeta(3)}{\pi^2} T^3 \eta_{\mathrm{B}} \mathrm{e}^{-\Delta_{\mathrm{H}}/T}$$

last scattering: $\tau_\gamma(T_f) \simeq H^{-1}(T_f) \simeq t_f$

$$\frac{\Delta_{\mathrm{H}}}{T_f} = \ln\left[\sigma_{\mathrm{T}}^2 t_r^2 \cdot \left(\frac{m_e T_r}{2\pi}\right)^{3/2} \eta_{\mathrm{B}} \frac{2\zeta(3)}{\pi^2} T_r^3\right]$$

$$T_f = 0.27\ \mathrm{eV}\,, \quad z = 1100\,, \quad t_f = 2.7\cdot 10^5\ \mathrm{yr}$$

for general processes one should solve kinetic equations

$$\frac{dn_{X_i}}{dt} + 3Hn_{X_i} = \int (production - destruction)$$

Recombination: horizon

matter domination: $l_{\mathrm{H},r} = 2H_r^{-1}$

$$H_r^2 = \frac{8\pi}{3} G\rho_{\mathrm{M}}(t_r) = \frac{8\pi}{3} G\rho_{\mathrm{M},0}\left(\frac{a_0}{a_r}\right)^3 = \frac{8\pi}{3} G\rho_c\Omega_{\mathrm{M},0}(1+z_r)^3 \, .$$

at recombination: $l_{\mathrm{H},r} = \frac{2}{H_0\sqrt{\Omega_{\mathrm{M}}}} \frac{1}{(1+z_r)^{3/2}}$

today: $l_{\mathrm{H},r}(t_0) = \frac{2}{H_0\sqrt{\Omega_{\mathrm{M}}}} \frac{1}{\sqrt{1+z_r}}$

$$\frac{l_{H_0}}{l_{\mathrm{H},r}(t_0)} \sim \sqrt{1+z_r} \simeq 30$$

Recombination: angle

$$\chi_r = \int_{t_r}^{t_0} \frac{dt}{a(t)}\,, \quad \Delta\theta_r = \frac{l_{H,r}}{r_a(z_r)}$$

$$r_a(z_r) = (1+z_r)^{-1} \cdot a_0 \cdot \mathrm{sh}\chi_r$$

$$\Delta\theta_r = \frac{1}{\sqrt{z_r+1}}\,, \quad \Omega_{curv} = \Omega_\Lambda = 0\,.$$

$$\Delta\theta_r = \frac{1}{\sqrt{z_r+1}} \frac{2\sqrt{\Omega_{curv}/\Omega_{\mathrm{M}}}}{\sinh\left(2\sqrt{\Omega_{curv}/\Omega_{\mathrm{M}}}I\right)}\,.$$

$$I = \int_0^1 \frac{dy}{\sqrt{1+\frac{\Omega_\Lambda}{\Omega_{\mathrm{M}}}y^6}}$$