Lecture 3. Polyakov formulation of the string theory.

The basic disadvantidge of the Nambu-Goto action is its nonlinearity in terms of $x^\mu(\tau,\sigma)$.

This problem is solved by using the following action

$$S = -\frac{T}{2}\iint du^0 du^1 \sqrt{|g|}\, g^{\alpha\beta}\,\partial_\alpha x^\mu \partial_\beta x_\mu ,$$

$$u = (u^0, u^1),\quad u^0 = \tau,\ u^1 = \sigma,\quad \mu = 0,1,\dots,D-1;$$

$$\alpha,\beta = 0,1,\qquad g_{\alpha\beta}\, g^{\beta\gamma} = \delta_\alpha^\gamma .$$

$g_{\alpha\beta}(u)$ is an auxiliary field (metric);
$x^\mu(u^0, u^1)$ is the string position vector.

Symmetry properties:

$$\delta u^\alpha = \xi^\alpha(u^0, u^1),\qquad \delta x^\mu = -\xi^\alpha \partial_\alpha x^\mu ,$$

$$\delta g_{\alpha\beta} = \xi^\gamma \partial_\gamma g_{\alpha\beta} + g_{\gamma\beta}\,\partial_\alpha \xi^\gamma + g_{\alpha\gamma}\,\partial_\beta \xi^\gamma ;$$

$+$

Weyl transformations

$$g_{\alpha\beta}(u) \rightarrow \exp(\varphi(u))\cdot g_{\alpha\beta}(u)$$

global Poincaré transformations

$$\delta x^\mu = \omega^{\mu\nu} x_\nu + \varepsilon^\mu,\quad \omega^{\mu\nu} + \omega^{\nu\mu} = 0,\quad \delta g_{\alpha\beta} = 0.$$

At the classical level, new action is completely equivalent to the Nambu-Goto action.

$$\frac{\delta S}{\delta x^\mu} = 0 \mapsto \Delta_g x^\mu = \frac{1}{\sqrt{|g|}}\left(\partial_\alpha g^{\alpha\beta}\sqrt{|g|}\,\partial_\beta x^\mu\right) = 0,$$

Laplace-Beltrami operator for metric field $g_{\alpha\beta}(u)$

$$\frac{\delta S}{\delta g^{\alpha\beta}} = 0 \mapsto \frac{1}{2}\sqrt{|g|}\,T_{\alpha\beta} = \qquad (*)$$

$$= -\frac{T}{2}\sqrt{|g|}\left(\partial_\alpha x^\mu \partial_\beta x_\mu - \frac{1}{2} g_{\alpha\beta}\partial_\gamma x^\mu \partial_\delta x_\mu g^{\gamma\delta}\right) = 0$$

$$dg = dg_{\alpha\beta}\cdot g^{\alpha\beta}\cdot g = -g_{\alpha\beta}\,dg^{\alpha\beta} g$$

$$\frac{\partial\sqrt{|g|}}{\partial g^{\alpha\beta}} = -\frac{1}{2}\sqrt{|g|}\,g_{\alpha\beta}$$

$T_{\alpha\beta}(u)$ is a "metric" symmetric energy-momentum tensor of string coordinates $x^\mu(u)$.

Solution to eq. (*) is

$$g_{\alpha\beta}(u) = f(u)\cdot\partial_\alpha x^\mu \partial_\beta x_\mu,$$ with $f(u)$ being arbitrary function

$$g_{\alpha\beta}\,g^{\beta\alpha} = 2.$$

$$\Delta_g x^\mu \Rightarrow \Delta_{ind.\,met.} x^\mu = 0$$

$x^\mu(u)$ should be a minimal surface.

$$S \to S_{N-G} = -T \iint d^2u \sqrt{|g|_{ind.}}$$

Orthonormal gauge

$$g_{\alpha\beta}(u) = e^{\varphi(u)} \eta_{\alpha\beta}, \quad \eta_{\alpha\beta} = \mathrm{diag}\,(1, -1)$$

$$T_{\alpha\beta} = 0 \to \begin{pmatrix} \dot{x}^2 - \frac{1}{2}(\dot{x}^2 - x'^2) & \dot{x}x' \\ \frac{1}{2}(\dot{x}^2 + x'^2) & \\ \dot{x}x' & \frac{1}{2}(\dot{x}^2 - x'^2) \end{pmatrix} = 0$$

$$\downarrow$$

$$\dot{x}^2 + x'^2 = 0, \quad x'\dot{x} = 0$$

$$(\dot{x} \pm x')^2 = 0$$

Quantization:

⊗ Conformal invariance is broken due to the quantum anomaly (or conformal anomaly).

Partition function

$$Z = \int Dg_{\alpha\beta}\, Dx^\mu \exp\left\{-\frac{T}{2}\iint d^2u \sqrt{g}\, g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\mu - \mu_0^2 \iint d^2u \sqrt{g}\right\}$$

integration by parts

$$= \int Dg_{\alpha\beta}\, Dx^\mu \exp\left\{-\frac{T}{2}\iint d^2u \sqrt{g}\, x^\mu \Delta_g x^\mu - \mu_0^2 \iint d^2u \sqrt{g}\right\}$$

Let us put

$$e^{-F} = \int Dx^\mu \exp\left\{-\frac{T}{2}\iint d^2u \sqrt{g}\, x^\nu \Delta x^\nu - \mu_0^2 \iint d^2u \sqrt{g}\right\}$$

$$\frac{\delta}{\delta g^{\alpha\beta}} \qquad -\frac{2}{\sqrt{g}}\frac{\delta F}{\delta g^{\alpha\beta}} = \langle T_{\alpha\beta} \rangle$$

In the conformal flat metric

$$g_{\alpha\beta}(u) = \rho(u)\,\delta_{\alpha\beta}, \quad \rho(u) = e^{\varphi(u)}$$

$$\langle T_\alpha{}^\alpha \rangle = -2\frac{\delta F}{\delta \rho} = D\, Y(u,u';t)$$

$$\left.\begin{aligned} &\frac{d}{dt} Y(u,u';t) = -\Delta Y(u,u';t) \\ &Y(u,u';0) = \frac{1}{\sqrt{g}}\,\delta(u,u') \end{aligned}\right\} \text{"heat" equation}$$

$$\Upsilon(u,u';t) = -\frac{i}{4\pi t} + \frac{R}{24\pi} + O(t)$$

$$R = -\frac{1}{\rho}\partial^2 \ln\rho$$

$$\langle T_\alpha{}^\alpha \rangle = \frac{D}{24\pi} R + \text{const}$$

$$-F = -\frac{D}{48\pi}\iint d^2u \left[\frac{1}{2}(\partial_i \ln\rho)^2 + \mu_0^2 \rho\right]$$

Faddeev-Popov determinant

$$\Delta^{1/2}(\varphi) = \frac{1}{2}\ln\det(L) = \frac{26}{48\pi}\iint d^2u\left[\frac{1}{2}(\partial_i\varphi)^2 + \mu_0^2 e^{\varphi}\right]$$

where

$$L_{\alpha\beta} = \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha + g_{\gamma\delta}\nabla^\gamma \nabla^\delta \delta_{\alpha\beta}$$

Finally we obtain

$$Z = \int [D\varphi] \exp\left\{-\frac{26-D}{48\pi}\iint d^2u\left[\frac{1}{2}(\partial_i\varphi)^2 + \mu_0^2 e^{\varphi}\right]\right.$$

$$\partial_i^2\varphi + \mu_0^2 e^{\varphi} = 0$$ Liouville equation.

Spinning string

$x^\mu(\tau,\sigma) + S^\mu_A(\tau,\sigma)$ $\mu = 0,1,2,\dots,D-1.$

$A = 1,2.$ ← spinor index

anticommuting Grassmann spinors in two-dimensional space-time (Majorana spinors)

Action for spinning string in the orthonormal gauge:

$$S = -\frac{T}{2}\iint d\tau\, d\sigma \left(\eta^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x_\mu + i\bar{S}^\mu \rho^\alpha \partial_\alpha S_\mu\right)$$

ρ^α, $\alpha = 0,1$ are two-dim. Dirac γ matrices

$$\rho^0 = -i\sigma_2,\quad \rho^1 = \sigma_1,\quad \rho^5 = \sigma_3,\quad \rho^\alpha\rho^\beta = \eta_{\alpha\beta} + \rho^5 \varepsilon_{\alpha\beta}$$

$$\eta_{\alpha\beta} = \mathrm{diag}(1,-1) \quad \varepsilon_{01} = 1$$

$$S^*_\mu = S_\mu,\quad \bar{S}_\mu = S^+\rho^0 = S^T_\mu \rho^0.$$

Orthonormal gauge conditions should be emposed by "hand"

$$(\dot{x}+x')^2 + iS^\mu_2(\dot{S}_{2\mu} + S'_{2\mu}) = 0,$$

$$(\dot{x}-x')^2 + iS^\mu_1(\dot{S}_{1\mu} + S'_{1\mu}) = 0;$$

$$(\dot{x}_\mu - x'_\mu)S^\mu_1 = 0, \quad (\dot{x}_\mu + x'_\mu)S^\mu_2 = 0.$$

These conditions can be rewritten as

$$T_{\alpha\beta}=0\,,\quad J_\alpha=0$$

$$T_{\alpha\beta}=\partial_\alpha x^\mu \partial_\beta x_\mu-\frac{1}{2}\eta_{\alpha\beta}(\partial x)^2+\frac{i}{4}\bar{S}(\rho_\alpha\partial_\beta+\rho_\beta\partial_\alpha)S$$

↑ energy-momentum tensor

$$J_\alpha=(\partial_\beta x^\mu)\rho^\beta\rho_\alpha S_\mu \leftarrow \text{supercurrent density}$$

Equations of motion:

$$\ddot{x}^\mu-x''^\mu=0,\quad \dot{S}_1^\mu=-S_1'^\mu,\quad \dot{S}_2^\mu=S_2'^\mu$$

boundary conditions

$$x'^\mu(\tau,0)=x'^\mu(\tau,\pi)=0,$$

$$S_1^\mu(\tau,0)=S_2^\mu(\tau,0),\quad S_1^\mu(\tau,\pi)=\varepsilon S_2^\mu(\tau,\pi)$$

$$\varepsilon=\begin{cases}-1 & \text{Neveu-Schwarz model (mesons with integer spin)}\\ +1 & \text{Ramond dual model (fermions with half-integer spin)}\end{cases}$$

Solutions to the equations of motion

$$x_\mu(\tau,\sigma)=\frac{i}{\sqrt{\pi\gamma}}\sum_{n\neq0}e^{-in\tau}\frac{\alpha_{n\mu}}{n}\cos(n\sigma)+Q_\mu+P_\mu\frac{\tau}{\pi\gamma}$$

$$\gamma=T$$

$$S_1^\mu(\tau,\sigma)=\frac{1}{\sqrt{2\pi T}}\sum_k b_k^\mu\exp[-ik(\tau-\sigma)],$$

$$S_2^\mu(\tau,\sigma)=\frac{1}{\sqrt{2\pi T}}\sum_k b_k^\mu\exp[-ik(\tau+\sigma)]$$

$$[b^\mu_m, b^\nu_n]_+ = -\eta^{\mu\nu}\delta_{n,-m}, \quad \eta^{\mu\nu} = diag(1,-1,-1,),$$

$$b^\mu_k = b^{*\mu}_{-k}$$

$$k = \begin{cases} \varepsilon = -1, & (2n+1)/2 \\ \varepsilon = +1, & \text{integer values} \end{cases}$$

Virasoro operators are

$$G_n = -\frac{1}{2}\sum_m :\alpha_{n-m}\alpha_m: - \frac{1}{2}\sum_r \left(r - \frac{n}{2}\right) :b_{n-r} b_r:,$$

$$H_r = -\sum b_{r-m}\alpha_m, \quad \alpha_0^\mu = \frac{p^\mu}{\sqrt{T\pi}}$$

$$G_n = 0, \quad H_r = 0$$

$$r = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots, \text{ when } \varepsilon = -1$$

$$r = 0, \pm 1, \pm 2, \ldots, \text{ when } \varepsilon = +1$$

The constraint algebra is

$$\varepsilon = +1$$

$$[G_n, G_m]_- = (n-m)G_{n+m} + \frac{D}{8}n^3\delta_{n,-m},$$

$$[H_n, H_m]_+ = 2G_{n+m} + \frac{D}{2}n^2\delta_{n,-m},$$

anomaly terms

$$[G_n, H_m]_- = \left(\frac{n}{2} - m\right)H_{n+m}$$

Conditions on the physical state vectors ($\varepsilon=+1$)

$$G_n|\psi\rangle = H_n|\psi\rangle = 0, \quad n \geq 1,$$

$$(H_0 - \sqrt{C_+})|\psi\rangle = 0,$$

$$(G_0 - C_+)|\psi\rangle = 0$$

The mass operators are

$$\varepsilon=+1: \alpha' M^2 = \sum_{n=1}^{\infty} n\alpha_n^{+\mu}\alpha_{n\mu} + \sum_{n=1}^{\infty} n\, b_n^{+\mu} b_{n\mu} - C_+;$$

$$\varepsilon=-1: \alpha' M^2 = \sum_{n=1}^{\infty} n\alpha_n^{+\mu}\alpha_{n\mu} + \sum_{r=1/2}^{\infty} r\, b_r^{+\mu} b_{r\mu} - C_-;$$

No ghost theorem gives:

$$D=10, \quad C_+ = 0, \quad C_- = -\frac{1}{2}$$

tachyon absent ground state tachyonic

Lightlike gauge conditions can be imposed in the same way as in Nambu-Goto string model.

$$C_+ = \frac{D-2}{2}\Big(\sum_{n=1}^{\infty} n - \sum_{n=1} n\Big) = 0$$

$$C_- = -\frac{D-2}{2}\Big(\sum_{n=1}^{\infty} n - \sum_{r=1/2}^{\infty} r\Big) = -\frac{D-2}{2}\Big[\sum_{n=0}^{\infty} n - \sum_{n=0}^{\infty}\Big(\frac{1}{2}+n\Big)\Big] =$$

$$= -\frac{D-2}{2}\Big[\zeta(-1,0) - \zeta\Big(-1,\frac{1}{2}\Big)\Big] = \frac{D-2}{16}$$

Covariant action for spinning string

$$S = -\frac{T}{2}\iint d^2u\, V \big[g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x_\mu + i V_a^{\alpha} \bar{S}^\mu \rho^a \partial_\alpha S_\mu$$

$$+ 2 V_\alpha^{a} V_b^{\beta} \bar{\psi}_\alpha \rho^b \rho^a S^\mu \big(\partial_\beta x_\mu + \frac{1}{2}\bar{S}_\mu \psi_\beta\big)\big]$$

$V = \det \| V_a^{\alpha} \|$ $\quad\psi_\alpha(u)$ is a spin 3/2 field

$$g_{\alpha\beta} = V_\alpha^{a} V_\beta^{b} \eta_{ab}$$

$$\eta_{ab} = \mathrm{diag}(1, -1)$$

Gauge fixing: (superconformal gauge)

$$V_\alpha^{a}(u) = h(u)\,\delta_\alpha^{a}, \quad \psi_\alpha(u) = 0.$$

Symmetry properties of this action:

i) local Lorentz transformations;

ii) Weyl transformations: $x \to x$, $S \to \lambda^{-1/2} S$, $V_\alpha^{a} \to \lambda V_\alpha^{a}$, $\psi_\alpha \to \lambda^{1/2}\psi_\alpha$

iii) local supersymmetric transformations:

$$\delta x^\mu = i\bar{\varepsilon}(u) S^\mu, \quad \delta S = (\partial_\alpha x + i\bar{S}\psi_\alpha)\rho^a V_a^{\alpha} \varepsilon(u)$$

$$\delta V_a^{\alpha} = -2i\bar{\varepsilon}(u) V_b^{\alpha} V_a^{\beta} \rho^b \psi_\beta, \quad \delta\psi_\alpha = -\overset{\alpha}{D}\,\varepsilon(u)$$

covariant diff.