Equation

$$\alpha_0^+ = \frac{\sqrt{\pi\gamma}}{p^-} L_{0\perp}$$

gives the string mass squared

$$M^2 = P^2 = 2P^+P^- - \vec{P}_\perp^{\,2} = \pi\gamma \sum_{n\neq 0} \sum_{i=2}^{D-1} \alpha_{-n}^i \alpha_n^i$$

M^2 is positive definite at the classical level.

Lecture 2.

Hamiltonian description and quantization

$$P_\mu(\tau,\sigma) = -\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \gamma \frac{(\dot{x}x')x'_\mu - x'^2 \dot{x}_\mu}{\sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}}$$

There is two promary first class constraints

$$\left.\begin{aligned} \varphi_1 &= \gamma^2 x'^2 + p^2 \approx 0, \\ \varphi_2 &= x'p \approx 0 \end{aligned}\right\} (\gamma x'^\mu \pm p^\mu)^2 \approx 0$$

$$\mathcal{H}_c = -\dot{x}_\mu p^\mu - \mathcal{L} \equiv 0$$

Dynamics in phase space is determined by H_T

$$H_T = \int_0^\pi d\sigma\, \mathcal{H}_T = \int_0^\pi d\sigma \left[\lambda_1(\tau,\sigma)\varphi_1(\tau,\sigma) + \lambda_2(\tau,\sigma)\varphi_2(\tau,\sigma)\right]$$

Hamiltonian equations of motion and boundary conditions:

$$\dot{x}_\mu = -\frac{\delta H_T}{\delta p^\mu} = \{x_\mu, H_T\} = 2\lambda_1 p_\mu - \lambda_2 x'_\mu,$$

$$\dot{p}_\mu = \{p_\mu, H_T\} = \frac{\delta H_T}{\delta x^\mu} = \frac{\partial \mathcal{H}_T}{\partial x^\mu} - \frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{H}_T}{\partial x'^\mu}\right)$$

$$= -2\gamma^2 \frac{\partial}{\partial \sigma}(\lambda_1 x'_\mu) - \frac{\partial}{\partial \sigma}(\lambda_2 p_\mu)$$

$$\frac{\partial \mathcal{H}_T}{\partial x'^\mu}\bigg|_{\sigma=0,\pi} = 0 = 2\lambda_1\gamma^2 x'_\mu + \lambda_2 p_\mu \bigg|_{\sigma=0,\pi} = 0$$

Definition of Poisson brackets

$$F = \int_0^\pi d\sigma\, \mathcal{F}(x,p)\, d\sigma, \quad G = \int_0^\pi d\sigma\, \mathcal{G}(x,p)$$

$$x = x(\tau,\sigma), \quad p = p(\tau,\sigma)$$

$$\{F,G\} = \int_0^\pi d\sigma \left(\frac{\delta F}{\delta p^\mu}\cdot\frac{\delta G}{\delta x_\mu} - \frac{\delta F}{\delta x^\mu}\frac{\delta G}{\delta p_\mu}\right)$$

$$\Downarrow$$

$$\{x_\mu(\tau,\sigma), p_\nu(\tau,\sigma')\} = -\eta_{\mu\nu}\,\Delta(\sigma,\sigma'),$$

$\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,\ldots,-1)$.

$\Delta(\sigma,\sigma')$ is the δ-function for region $[0,\pi]$

Covariant quantization

If we put $\lambda_2=0$ and $\dot{\lambda}_1=0$ then
From Hamiltonian equations it follows that

$$\ddot{x}_\mu - 4\gamma^2\lambda_1\lambda_1' x_\mu' - 4\gamma^2\lambda_1^2 x_\mu'' = 0$$

Now we put $\lambda_1=(-2\gamma)^{-1}$ then

$$P_\mu = \gamma\dot{x}_\mu\,, \quad \ddot{x}_\mu - x_\mu'' = 0 \qquad (*)$$

and the Hamiltonian constraints reduce to

$$\dot{x}^2 + x'^2 = 0, \quad \dot{x}x' = 0$$

(Lagrangian orthonormal gauge conditions)

Hence, the functional arbitrariness in Hamiltonian theory is removed by fixing the Lagrange multipliers!

To obey eqs. (*) we can put

$$x_\mu(\tau,\sigma) = \frac{i}{\sqrt{\pi\gamma}}\sum_{n\neq 0} e^{-in\tau}\frac{\alpha_{n\mu}}{n}\cos(n\sigma) + Q_\mu + P_\mu\frac{\tau}{\pi\gamma},$$

$$p_\mu(\tau,\sigma) = \sqrt{\frac{\gamma}{\pi}}\sum_{n\neq 0} e^{-in\tau}\alpha_{n\mu}\cos(n\sigma) + \frac{P_\mu}{\pi}$$

Quantization

$$[\ldots,\ldots] = i\{\ldots,\ldots\}$$

$$[\alpha_{m,\mu}, \alpha_{n\nu}] = -m\,\eta_{\mu\nu}\,\delta_{n+m,0}, \quad n, m \neq 0$$

$$[Q_\mu, P_\nu] = -i\,\eta_{\mu\nu}$$

$\alpha^+_{n0}|0\rangle$ negative norm

To remove these states the Virasoro conditions

$$[L_n - \delta_{n,0}\,\alpha(0)]\,|\phi\rangle = 0, \quad \underline{n = 0, 1, 2, \ldots}$$

where $L_n = -\frac{1}{2}\sum_m :\alpha_{n-m}\alpha_m:\,, \quad L^+_n = L_{-n}$

should be imposed

$$M^2 = P^2 = -\pi\gamma \sum_{m\neq 0} :\alpha^\nu_{-m}\alpha_{m\nu}: - 2\pi\gamma\,\alpha(0)$$

Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \underbrace{\frac{D}{12}n(n^2-1)\delta_{n+m,0}}_{\text{quantum anomaly}}$$

$$\{L_n, L_m\} = -i(n-m)L_{n+m} \quad \leftarrow \text{Poisson brackets}$$

Wick theorem enables one to calculate this anomaly. It should be taken into account that the Wick pairing of the operators $\alpha^\rho_k \alpha^\sigma_j$ is equal to $\eta^{\rho\sigma}\theta(k)\theta(-j)k\,\delta_{k+j,0}$, $\theta(k)$ is a step func

The physical space of state vectors with positive norm can be constructed only in the space-time with dimension

D=26 and $\alpha(0)=1$ (tachyon state)

Noncovariant quantization

Light cone gauge conditions in the Hamiltonian formalism. There are two first-class constraints

$$\varphi_{\substack{1\\2}} = (\gamma x'^{\mu} \pm p^{\mu})^2 \approx 0$$

hence two gauge fixing conditions should be imposed

$$\chi_1 = \gamma n_\mu \left(x^\mu - \frac{P^\mu}{\gamma\pi}\tau - Q^\mu\right) = 0$$

$$\chi_2 = n^\mu \left(p_\mu - \frac{P_\mu}{\pi}\right) = 0$$

Algebra of constraints and gauge fixing conditions

$$\{\varphi_1(\sigma), \varphi_1(\sigma')\} = 4\gamma^2 (\varphi_2(\sigma) + \varphi_2(\sigma'))\frac{\partial}{\partial\sigma'}\Delta(\sigma,\sigma'),$$

$$\{\varphi_2(\sigma), \varphi_2(\sigma')\} = (\varphi_2(\sigma) + \varphi_2(\sigma'))\frac{\partial}{\partial\sigma'}\Delta(\sigma,\sigma'),$$

$$\{\varphi_1(\sigma), \varphi_2(\sigma')\} = (\varphi_1(\sigma) + \varphi_1(\sigma'))\frac{\partial}{\partial\sigma'}\Delta(\sigma,\sigma')$$

$$\{\varphi_1(\sigma),\chi_1(\sigma')\} = 2\gamma\,\underset{nP/\pi}{\underbrace{(np(\sigma))}}\,\Delta(\sigma,\sigma'), \quad !$$

$$\{\varphi_2(\sigma),\chi_1(\sigma')\} = \gamma(nx'(\sigma))\,\Delta(\sigma,\sigma') = 0$$

$$\{\varphi_1(\sigma),\chi_2(\sigma')\} = -2\gamma^2(nx'(\sigma))\,\Delta(\sigma,\sigma') = 0$$

$$\{\varphi_2(\sigma),\chi_2(\sigma')\} = \frac{nP}{\pi}\frac{\partial}{\partial\sigma'}\Delta(\sigma,\sigma'), \quad !$$

$$\{\chi_1(\sigma),\chi_2(\sigma')\} = \gamma \underset{0}{\underbrace{n^2}}\left(\frac{1}{\pi} - \Delta(\sigma,\sigma')\right).$$

Requirement of stationarity of the gauge conditions

$$\frac{d\chi_i}{d\tau} = \{\chi_i, H_T\} \approx 0$$

gives

$$\lambda_1 = -\frac{1}{2\gamma}, \qquad \lambda_2 = f(\tau) = 0$$

due to the boundary conditions

Finally we get

$$\dot{x}_\mu = \gamma^{-1}p_\mu, \quad \dot{p}_\mu = \gamma x''_\mu \longmapsto \ddot{x}^\mu - x''^\mu = 0$$

$$x'_\mu = 0, \quad \sigma = 0, \pi$$

Seperation of variables

$$x^\mu = (\underbrace{x^+, x^-}_{\text{dependent}}, \vec{x}_\perp), \qquad p^\mu = (\underbrace{p^+, p^-}_{\text{dependent}}, \vec{p}_\perp)$$

$$p^+ = \pi[\vec{P}_\perp^2 + \gamma^2 \vec{x}_\perp'^2]/(2P^-), \quad x'^+ = \pi \vec{P}_\perp \vec{x}_\perp'/P^-$$

$$p^- = P^-/\pi, \quad x'^- = 0$$

Hamiltonian for independent canonical variables $\vec{x}_\perp(\tau,\sigma)$ and $\vec{p}_\perp(\tau,\sigma)$ can be derived by making use of the canonical transformations defined by constraints and gauge conditions. It is given by

$$H = \frac{\gamma}{2}\int_0^\pi d\sigma\,[(\gamma^{-1}\vec{p}_\perp)^2 + \vec{x}_\perp'^2]$$

Hamiltonian equations are

$$\dot{\vec{x}}_\perp = \frac{\partial \mathcal{H}}{\partial \vec{p}_\perp} = \gamma^{-1}\vec{p}_\perp, \quad \dot{\vec{p}}_\perp = \frac{\partial}{\partial\sigma}\left(\frac{\partial \mathcal{H}}{\partial \vec{x}_\perp'}\right) = \gamma \vec{x}_\perp''$$

Solutions to these equations are

$$x^j(\tau,\sigma) = \frac{i}{\sqrt{\pi\gamma}}\sum_{n\neq 0} e^{-in\tau}\frac{\alpha_n^j}{n}\cos(n\sigma) + Q^j + \frac{P^j}{\pi\gamma}\tau,$$

$$p^j(\tau,\sigma) = \sqrt{\frac{\gamma}{\pi}}\sum_{n\neq 0} e^{-in\tau}\alpha_n^j\cos(n\sigma) + \frac{P^j}{\pi},$$

where $\alpha_{-n}^j = \alpha_n^{j*}, \quad j = 2, 3, \ldots, D-1.$

$$H = L_{0\perp} = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\sum_{j=2}^{D-1}\alpha_{-n}^j\alpha_n^j = \frac{\vec{P}_\perp^2}{2\pi\gamma} + \frac{1}{2}\sum_{n\neq 0}\sum_{j=2}^{D-1}\alpha_{-n}^j\alpha_n^j$$

Quantization

$$[\alpha_n^i, \alpha_m^j] = n\delta_{ij}\delta_{n+m,0}\,, \quad [Q^i, P^j] = \delta_{ij}\,,$$

$$n, m = \pm 1, \pm 2, \dots, \quad i,j = 2, 3, \dots, D-1$$

$$\alpha \to a$$

$$\alpha_n^j = \sqrt{n}\,a_n^j, \quad \alpha_{-n}^j \equiv \alpha_n^{+j} = \sqrt{n}\,a_n^{+j},$$

$$n = 1, 2, \dots$$

Relativistic invariance

The generators of the Poincaré ~~algebra~~ group should be constructed in terms of the dynamical variables of the model under consideration and the Poincaré algebra should be checked

Generators of the Poincaré group are

$$P^\mu, \; M^{\mu\nu},$$

where

$$M_{\mu\nu} = \int_0^\pi d\sigma\, (x_\mu p_\nu - x_\nu p_\mu) =$$

$$= Q_\mu P_\nu - Q_\nu P_\mu - \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}(\alpha_{-n\mu}\alpha_{n\nu} - \alpha_{-n\nu}\alpha_{n\mu})$$

$$[M^{+i}, M^{+j}] = \frac{2}{(P^-)^2}\sum_{m=1}^{\infty}\Big[m\Big(1 - \frac{1}{24}(D-2)\Big) +$$

$$+ \frac{1}{m}\Big(\frac{1}{24}(D-2) - \alpha(0)\Big)\Big](\alpha_{-m}^i\alpha_m^j - \alpha_{-m}^j\alpha_m^i)$$

$$[M^{+i}, M^{+j}] = 0 \text{ if } D=26 \text{ and } \alpha(0)=1.$$

The constant $\alpha(0)$ in L_0 and M^2 is due to the zero-point oscillations of the string

$$\sum_{n=1}^{\infty} n \sum_{i=1}^{D-1} a_n^{+i} a_n^i \xrightarrow{\text{quantization}} \sum_{n=1}^{\infty} n \sum_{i=1}^{D-1} \left(a_n^{+i} a_n^i + \frac{1}{2}\right)$$

Therefore

$$\alpha(0) = -\frac{D-2}{2} \sum_{n=1}^{\infty} n \rightarrow \frac{D-2}{24}$$

The Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \mathrm{Re}\, s > 1$$

$$\zeta(-1) = \frac{-1}{12}$$

Another regularization

$$\sum_{n=0}^{\infty} n = -\lim_{\varepsilon \to 0} \frac{\partial}{\partial \varepsilon} \sum_{n=0}^{\infty} e^{-\varepsilon n} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\left(1 - \frac{\varepsilon^2}{12} + O(\varepsilon^4)\right)$$

∞ should be ommeted

$\alpha' M^2$	State vectors	The number of states
-1	$\|p\rangle$	1
0	$a_1^i \|p\rangle$	24
1	$a_2^i \|p\rangle$	24
	$a_1^i a_1^j \|p\rangle$	300
2	$a_3^i \|p\rangle$	24
	$a_2^i a_1^j \|p\rangle$	576
	$a_1^i a_1^j a_1^k \|p\rangle$	2600

State density function for string spectrum

$$\rho(m) = A m^{-B} \underline{e^{\beta_0 m}},$$

where $\beta_0 = 2\pi[(D-2)\alpha'/6]^{1/2} = 4\pi\sqrt{\alpha'}$, $B = (D-1)/2 = 25/2$

Partition function converges only if

$$T < \frac{1}{\beta_0} = T_0 \text{ (Hagedorn temperature)}$$

$$Z = \sum_n e^{-\beta E_n} \cdot \rho(m_n)$$

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Application of the Nambu-Goto string for calculating interquark potential

String with fixed ends

$$\vec{x}(t,0)=0, \qquad \vec{x}(t,\pi)=\vec{R},$$

$$\vec{p}(t,0)=\vec{p}(t,\sigma)=0$$

Gauge $t=\tau$

$$S=-\gamma\int_{t_1}^{t_2}dt\int_0^{\pi}d\sigma\sqrt{\vec{x}'^2(1-\dot{\vec{x}}^2)+(\dot{\vec{x}}\vec{x}')^2}$$

the remaining gauge freedom

$$\tilde{\sigma}=f(t,\sigma)$$

$\downarrow$

constraint

$$\varphi(\sigma)=\vec{x}'(t,\sigma)\vec{p}(t,\sigma)\approx 0$$

gauge fixing cond.

$$\chi(\sigma)=\vec{p}^2+\gamma^2\vec{x}'^2-\gamma\approx 0$$

$$\{\varphi,\chi\}\Big|_{\Gamma^*}\neq 0$$

$$\ddot{\vec{x}}-\vec{x}''=0 \qquad \vec{p}=\sqrt{\gamma}\,\dot{\vec{x}}$$

$$\vec{x}(t,\sigma)=\vec{R}\frac{\sigma}{\pi}+\sum\frac{\vec{\alpha}_n}{\sqrt{\pi\gamma}\,n}\sin(n\sigma)e^{-in\sqrt{\gamma}t}$$

$$\vec{\alpha}_n^{\,*}=\vec{\alpha}_{-n}$$

$$E(R) = H = \sqrt{\gamma^2 R^2 + \pi\gamma \sum_{m\neq 0} \alpha_{-m}\alpha_m}$$

$$V(R) = \langle 0 | H | 0 \rangle =$$

$$= \sqrt{\gamma^2 R^2 - 2\pi\gamma\,\alpha(0)}\,, \qquad \underline{\text{instability}}$$

where $\alpha(0) = -\frac{D-2}{2}\sum_{m=1}^{\infty} m = \frac{D-2}{24}$