preamble

Quantum cosmology

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Bubble creation

Consider spherically-symmetric bubble of true vacuum in a metastable false vacuum.
- only radial degree of freedom,
- thin wall.

Two opposite forces:
Energy of vacuum decay in the bubble volume $\sim R^3$ causing expansion,
Surface tension $\sim R^2$ causing contraction. The larger bubble is — the faster it grows.
Forbidden area: $R < R_0$ when $E_{decay} < E_{surface}$.

The lagrangian can be written as

$$L = -4\pi\sigma R(1-\dot{R}^2)^{1/2} + \frac{4\pi}{3}\epsilon R^3.$$

Conjugate momenta to R is

$$p_R = 4\pi\sigma R^2\dot{R}(1-\dot{R}^2)^{-1/2},$$

and the hamiltonian is

$$\mathcal{H} = [p_R^2 + (4\pi\sigma R^2)^2]^{1/2} - \frac{4\pi}{3}\epsilon R^3$$

Energy conserves:

$$\mathcal{H} = \mathrm{const} = 0.$$

One can express

$$p_R^2 + U(R) = 0$$

with

$$U(R) = (4\pi\sigma R^2)^2(1 - R^2/R_0^2),$$

where $R_0 = 3\sigma/\epsilon$.

The solution will be

$$R(t) = (R_0^2 + t^2)^{1/2}.$$

The worldsheet of the bubble has metrics

$$ds^2 = (1 - \dot{R}^2)dt^2 - R^2(t)d\Omega^2.$$

On solution it will be a de Sitter 2+1:

$$ds^2 = d\tau^2 - R^2(\tau)d\Omega^2, \; R = R_0 \cosh(\tau/R_0).$$

Thus there will be an inflation.

In quantum mechanics one has

$$\hat{H}\psi = 0,$$

where ψ is a 'wave function of the universe'.
In the hamiltonian the quantum operator is standard:

$$p_R = -i\frac{\partial}{\partial R}.$$

The hamiltonian without square root:

$$[-\partial_R^2 + U(R)]\psi = 0.$$

The commutator of R and p_R is significant at the turning point $p_R \approx 0$. One can estimate this region as

$$\delta R/R_0 \sim (\sigma R_0^3)^{-2} \ll 1.$$

Next we will work in the WKB approximation.
The formulation of the initial conditions:
- one for multiplicative constant (not so important)
- one to distinguish the modes.

WKB solution for $R > R_0$ is

$$\psi_\pm(R) = [p(R)]^{-1/2} \exp\left(\pm i \int_{R_0}^{R} p(R')dR' \mp i\pi/4\right),$$

where p is a classical momentum

$$p(R) = [-U(R)]^{1/2}.$$

In the leading order

$$\hat{p}_R \psi_\pm(R) \approx \pm p(R) \psi_\pm(R).$$

$\psi_+(R)$ — expanding; $\psi_-(R)$ — contracting.

In the classically forbidden range $R < R_0$:

$$\tilde{\psi}_{\pm}(R) = |p(R)|^{-1/2} \exp\left(\pm \int_R^{R_0} |p(R')| dR' \right),$$

Boundary condition is for $\psi(R)$ — outgoing mode. It can be obtained by matching at $R \approx R_0$:

$$\psi(R < R_0) = \tilde{\psi}_+(R) + \frac{i}{2}\tilde{\psi}_-(R).$$

In the forbidden range $R < R_0$ the term $\tilde{\psi}_+$ dominates (except the border $R \approx R_0$).

Tunneling probability

$$\left|\frac{\psi(R_0)}{\psi(0)}\right|^2 \sim \exp\left(-2\int_0^{R_0} |p(R)| dR\right).$$

So the probability of the bubble creation is proportional to

$$\exp(-\pi^2 \sigma R_0^3/2).$$

The bubble may be not exactly spherical.
In perturbative approach one obtains scalar field living on a bubble worldsheet.

In nonperturbative approach one has to deal with parametrization of bubble worldsheet by $x^\mu(\xi^a)$, $a = 0..2$. Then

$$\psi = \int [dx^\mu] e^{iS}.$$

The conditions are: Integration over compact worldsheets, No bubble at $x^0 = 0$, Boundary at $x^0 = T$ is given, $0 < x^0(\xi) < T$.
For a bubble worldsheet observer ξ^0 is 'time' and x^0 is just one of scalar fields. So he expects the conditions by ξ^0, not x^0.

Creation of the closed universe

Consider homogeneous isotropic closed universe described by the simple action

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G} - \rho_v \right),$$

where ρ_v is a constant vacuum energy, with the metrics

$$ds^2 = \sigma^2 [N^2(t) dt^2 - a^2(t) d\Omega_3^2].$$

$\sigma^2 = 2G/(3\pi)$ is a normalizing factor.

The lagrangian is

$$L = \frac{N}{2}\left(a(1 - \dot{a}^2/N^2) - \Lambda a^3\right),$$

where $\Lambda = (4G/3)^2\rho_v$. One can find the momentum $p_a = -a\dot{a}/N$.
Then

$$L = p_a\dot{a} - N\mathcal{H},$$

where the hamiltonian is

$$\mathcal{H} = -\frac{1}{2}\left(\frac{p_a^2}{a^2} + a - \Lambda a^3\right).$$

Variation w.r.t. N gives the constraint

$$\mathcal{H} = 0$$

In the gauge $N = 1$ one has the equation of motion

$$\dot{a}^2 + 1 - \Lambda a^2 = 0.$$

The solution to it will be

$$a(t) = H^{-1} cosh(Ht),$$

where $H = \Lambda^{1/2}$.

Quantization implies the replacing

$$\partial_a \to -i\frac{\partial}{\partial a}$$

and imposing Wheeler-De Witt equation

$$\left(\frac{d^2}{da^2}+\frac{\gamma}{a}\frac{d}{da}-U(a)\right)\psi(a)=0,$$

with the potential term

$$U(a)=a^2(1-\Lambda a^2).$$

γ-term describes the non-commutation of $\hat{a}$ and $\hat{p}_a$. If we ignore it, the equation describe the particle with zero energy in the potential U.
Classically allowed region is $a \geq H^{-1}$. WKB:

$$\psi_{\pm}(a) = [p(a)]^{-1/2} \exp\left(\pm i \int_{H^{-1}}^{a} p(a')da' \pm i\pi/4\right),$$

where again $p(a) = [-U(a)]^{1/2}$. The under-barrier solution for $a < H^{-1}$:

$$\tilde{\psi}_{\pm}(a) = |p(a)|^{-1/2} \exp\left(\pm \int_{a}^{H^{-1}} |p(a')|da'\right).$$

One can find outgoing mode from the relation

$$\hat{p}_a \psi_{\pm}(a) \approx \pm p(a))\psi_{\pm}(a).$$

Now ψ_- is the expanding mode since $p_a < 0$ when $\dot{a} > 0$. (In the gauge $N = 1$).
We assume that the universe expanded from very small size to very large:

$$\psi(a > H^{-1}) = \psi_-(a).$$

That is no contraction.

To connect under-barrier solution with the outgoing mode we obtain

$$\psi(a < H^{-1}) = \tilde{\psi}_+(a) - \frac{i}{2}\tilde{\psi}_-(a).$$

This allows us to calculate the tunneling probability:

$$\left|\frac{\psi(H^{-1})}{\psi(0)}\right|^2 \sim \exp\left(-2\int_0^{H^{-1}} |p(a)|da\right) =$$

$$= \exp\left(-\frac{3}{8G^2\rho_v}\right).$$

From vacuum energy ρ_v to inflation with $V(\phi)$:

$$\left(\frac{\partial^2}{\partial a^2} - \frac{1}{a^2}\frac{\partial^2}{\partial \phi^2} - U(a,\phi)\right)\psi(a,\phi) = 0.$$

The effective potential term:

$$U(a) = a^2(1 - a^2 V(\phi)).$$

The probability of tunneling will be of the order

$$P(\phi) \propto \exp\left(-\frac{3}{8G^2 V(\phi)}\right).$$

Universe with vacuum energy and radiation:

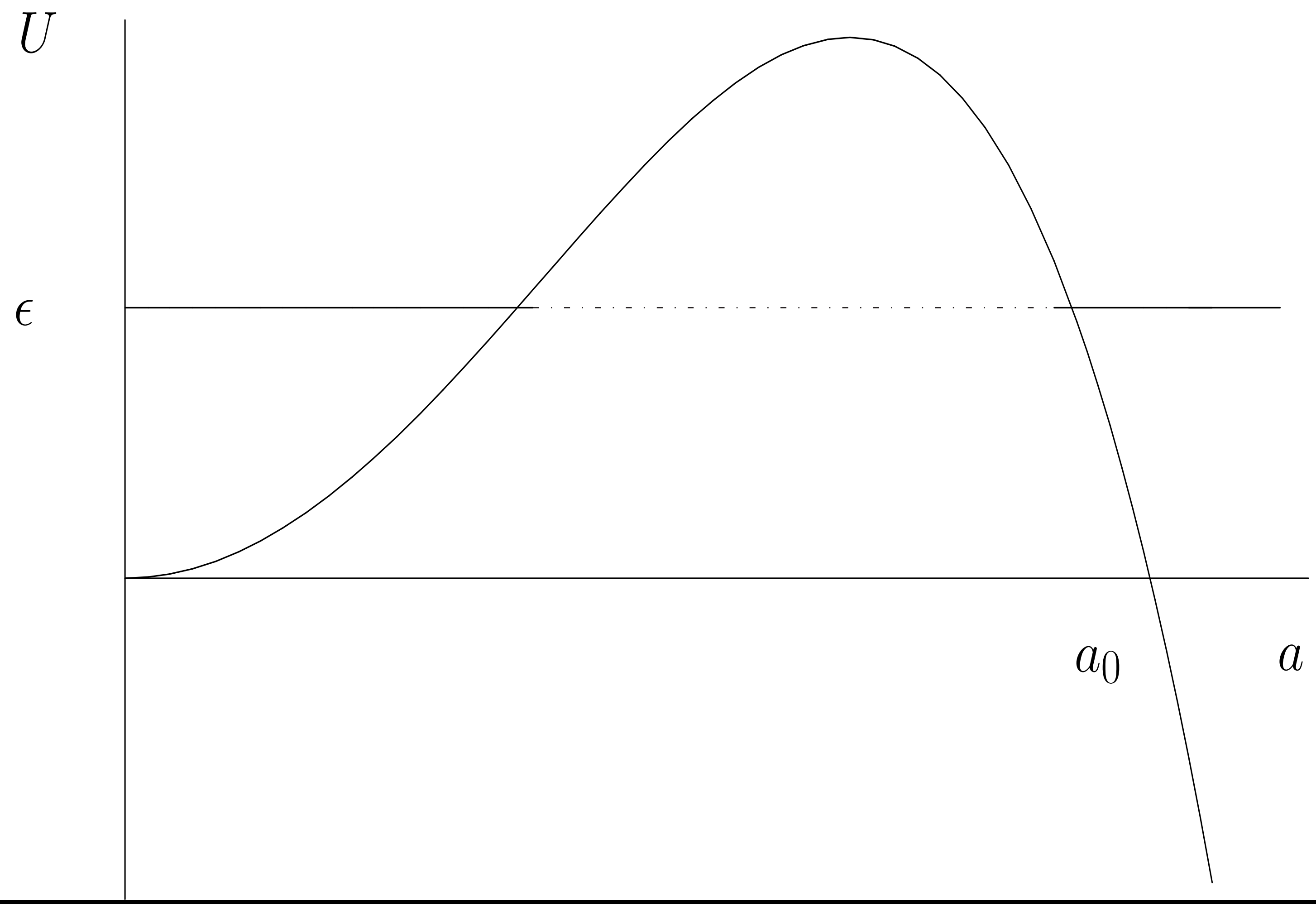
$$\rho = \rho_v + \epsilon / a^4 .$$

The evolution equation will be

$$p^2 + a^2 - a^4 / a_0^2 = \epsilon$$

with $a_0 = (3/4)\rho_v^{-1/2}$,
(units $G = 1$, gauge $N = 1$).

Effective potential $U(a) = a^2 - a^4/a_0^2$.



Tunneling probability

$$P \sim \exp\left(-2\int_{a_1}^{a_2} |p(a)|da\right).$$

One can consider limit $\epsilon \to 0$:
from 'nothing' to a_0

$$P \sim \exp\left(-2\int_{0}^{a_0} |p(a)|da\right) = \exp\left(-\frac{3}{8\rho_v}\right).$$

QC formalism

Establishing of the QC
1950s: ADM formalism (hamiltonian approach to GR)
1960s: Wheeler-De Witt equation (ψ, Einstein-Schrodinger equation)
1970-1980s: Boundary conditions (creation from 'nothing')

Expectations from QC: Initial conditions of the universe
Original density fluctuations

Spacetime $\mathcal{M}$ to slices Σ_t:

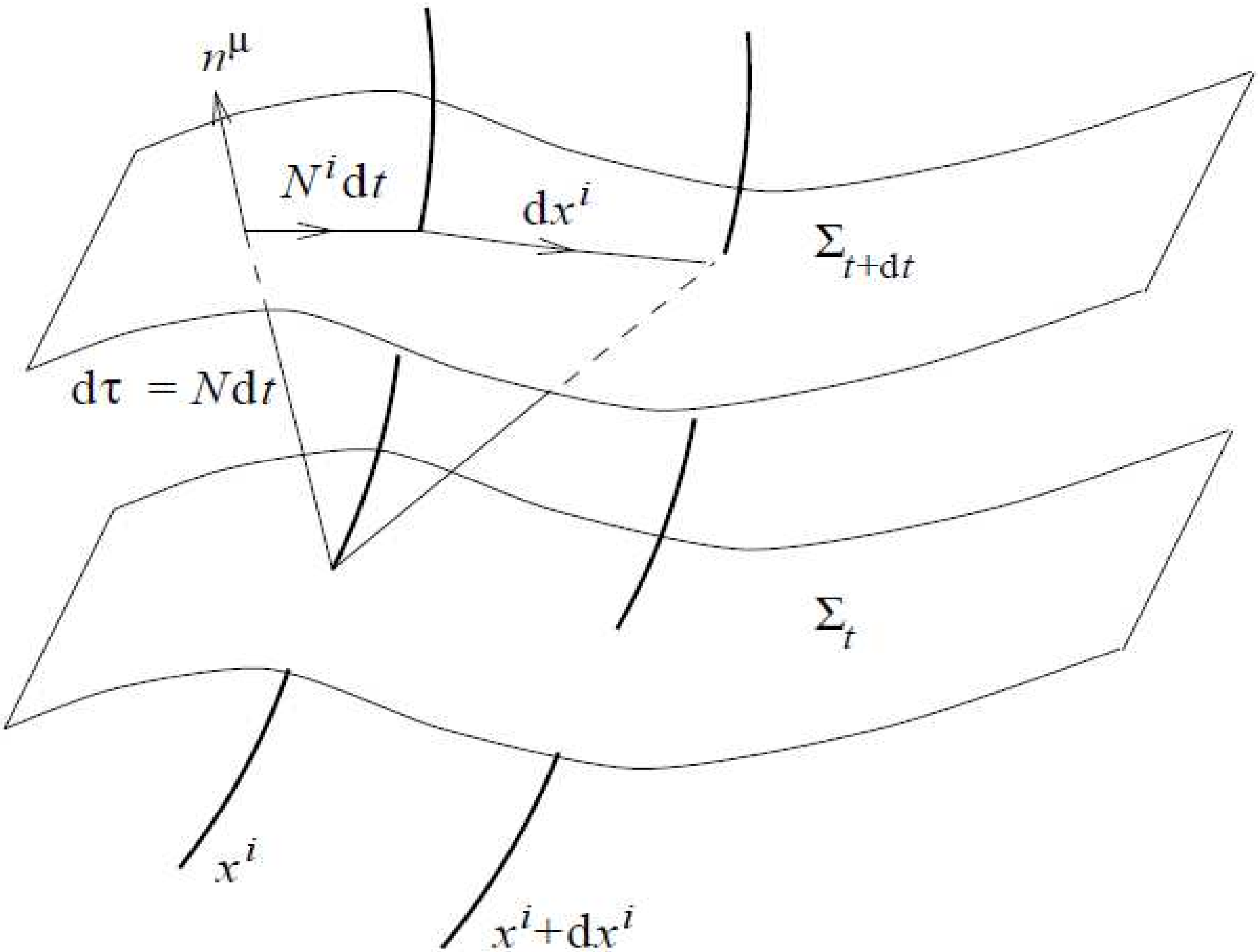
$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\omega^0 \otimes \omega^0 + h_{ij} \omega^i \otimes \omega^i,$$

where $\omega^0 = N dt$ and $N(t, x^k)$ is a lapse function;
$\omega^i = dx^i + N^i dt$ and $N^i(t, x^k)$ is shift vector;
$h_{ij}(t, x^k)$ — intrinsic metric of Σ_t.
In components:

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_k N^k & N_i \\ N_i & h_{ij} \end{pmatrix}$$

with $N_k = h_{kj} N^j$.

3+1 decomposition



Extrinsic curvature

$$K_{ij} \equiv n_{i;j} = \frac{1}{2}\left(N_{i|j} + N_{j|i} - \frac{\partial h_i j}{\partial t}\right)$$

where ';' and '|' are covariant derivatives w.r.t. g and h.
One can arrive to the action of decomposed GR with Λ:

$$\frac{1}{4\kappa^2}\int dt d^3x N\sqrt{h}(K_{ij}K^{ij} - K^2 + R^{(3)} - 2\Lambda).$$

Canonical momenta will be

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} = -\frac{\sqrt{h}}{4\kappa^2}\left(K^{ij} - h^{ij}K\right),$$

$$\pi^0 \equiv \frac{\delta L}{\delta \dot{N}} = 0, \ \pi^i \equiv \frac{\delta L}{\delta \dot{N}_i} = 0$$

One can add matter lagrangian $[-(\partial_\mu \phi)^2 - V(\phi)]\sqrt{-g}$ and decompose it.

$$\pi_\phi \equiv \frac{\delta L}{\delta \dot{\phi}} = \frac{\sqrt{h}}{N}(\dot{\phi} - N^i \phi_{,i}).$$

One can obtain the hamiltonian

$$H = \int d^3x(\pi^0\dot{N} + \pi^i\dot{N}_i + N\mathcal{H} + N_i\mathcal{H}^i),$$

$$\mathcal{H} = 4\kappa^2\mathcal{G}_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{h}}{4\kappa^2}(R^{(3)} - \Lambda) + \frac{\sqrt{h}}{2}\left(\frac{\pi_\phi^2}{h} + h^{ij}\phi_{,i}\,\phi_{,j} + 2V\right),$$

$$\mathcal{H}^i = -2\pi^{ij}_{|j} + h^{ij}\phi_{,i}\,\pi_\phi$$

Wheeler-De Witt metric:

$$\mathcal{G}_{ijkl} = \frac{1}{2\sqrt{h}}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}).$$

On the equations of motion

$$\mathcal{H} = 0,\ \mathcal{H}^i = 0$$

(dynamical constraints).

Quantization

$$\pi^{ij} \to -i\frac{\delta}{\delta h_{ij}}, \ \pi^{i} \to -i\frac{\delta}{\delta N_i},$$

$$\pi^{0} \to -i\frac{\delta}{\delta N}, \ \pi_{\phi} \to -i\frac{\delta}{\delta \phi}.$$

vanishing momenta implies

$$\hat{\pi}^0\psi = 0 = -i\frac{\partial \psi}{\partial N}, \ \hat{\pi}^i\psi = 0 = -i\frac{\partial \psi}{\partial N_i}.$$

Constraint $\hat{\mathcal{H}}^i\psi = 0$ reveals coordinate invariance of ψ on Σ.
Wheeler-De Witt equation $\hat{\mathcal{H}}\psi = 0$:

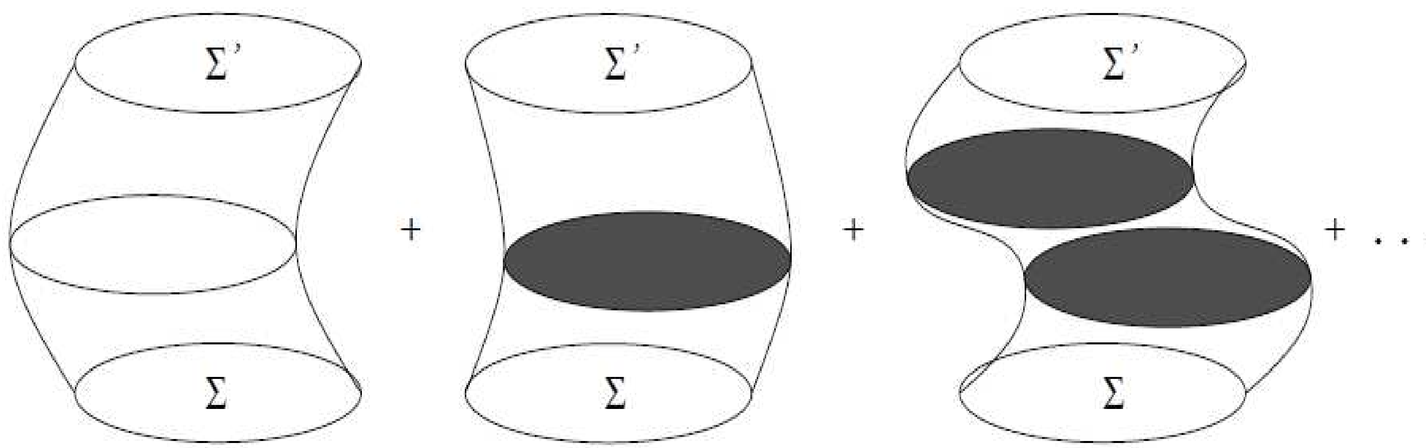
$$\left[-4\kappa^2\mathcal{G}_{ijkl}\frac{\partial^2}{\partial h_{ij}\partial h_{kl}} + \frac{\sqrt{h}}{4\kappa^2}(2\Lambda - R^{(3)} + 4\kappa^2\hat{T}_\phi)\right]\psi = 0,$$

where

$$\hat{T}_\phi = -\frac{1}{2h}\frac{\partial^2}{\partial\phi^2} + h^{ij}\phi_{,i}\,\phi_{,j} + V(\phi).$$

Path integral approach:

$$< h'_{ij}, \phi', \Sigma' | h_{ij} \phi, \Sigma >= \int [d\phi][dg] e^{iS(g,\phi)}.$$



Wave-function of the universe creation:

$$\psi = \int^{h,\phi} [d\phi][dg] e^{-S_E(g,\phi)}$$

('No-boundary' boundary condition);

$$\psi = \int_{\emptyset}^{h,\phi} [d\phi][dg] e^{iS(g,\phi)}$$

('Tunneling' boundary condition).

Expectations from QC: Initial conditions of the universe?
-The problem of eternal inflation
Original density fluctuations?
-Hard to calculate inhomogeneous universes.