

Cosmology of modified gravity

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Appearance, definition and forms of DE

$f(R)$ gravity

Viability conditions

New results on $f(R)$ DE models

Combined inflationary–DE models

Conclusions about $f(R)$ models

Appearance of DE

Why going beyond the pure Einstein gravity? – Existence of dark energy (DE).

Two cases where DE shows itself:

- 1) inflation in the early Universe – primordial DE,
- 2) present accelerated expansion of the Universe – present DE.

Present view of the history of our Universe in one line:

? \longrightarrow *DS* \implies *FRWRD* \implies *FRWMD* \implies \overline{DS} \longrightarrow ?

Definition of DE

DE – unclustered component, seen by gravitational interaction only. Quantitative definition - through gravitational field equations conventionally written in the Einstein form:

$$\frac{1}{8\pi G} \left(R^\nu_\mu - \frac{1}{2} \delta^\nu_\mu R \right) = - \left(T^\nu_{\mu(vis)} + T^\nu_{\mu(DM)} + T^\nu_{\mu(DE)} \right) ,$$

$G = G_0 = \text{const}$ - the Newton gravitational constant measured in laboratory.

In the absence of direct interaction between DM and DE:

$$T^\nu_{\mu(DE); \nu} = 0 .$$

Possible forms of DE

- ▶ Physical DE

New non-gravitational field of matter. DE proper place – in the **rhs** of gravity equations.

- ▶ Geometrical DE

Modified gravity. DE proper place – in the **lhs** of gravity equations.

- ▶ Λ - intermediate case.

Observations: $T_{\mu}^{\nu}(DE)$ is very close to $\Lambda\delta_{\mu}^{\nu}$ for the concrete solution describing our Universe (less than 10% deviation at the 2σ level for averaged quantities like \bar{w}_{DE}).

$f(R)$ gravity

$$S = \frac{1}{16\pi G} \int f(R) \sqrt{-g} d^4x + S_m$$

$$f(R) = R + F(R), \quad R \equiv R^\mu{}_\mu .$$

The effective energy-momentum tensor of DE in $f(R)$ gravity:

$$8\pi G T_{\mu}^{\nu}(DE) = F'(R) R^\nu{}_\mu - \frac{1}{2} F(R) \delta^\nu{}_\mu + (\nabla_\mu \nabla^\nu - \delta^\nu{}_\mu \nabla_\gamma \nabla^\gamma) F'(R) .$$

De Sitter solutions in the absence of matter: roots $R = R_{DS}$ of the algebraic equation

$$Rf'(R) = 2f(R) .$$

Degrees of freedom

I. In quantum language: particle content.

1. **Graviton** – spin 2, massless, transverse traceless.

2. **Scalaron** – spin 0, massive, mass - R -dependent:

$$M^2(R) = \frac{1}{3f''(R)} \text{ in the WKB-regime.}$$

II. Equivalently, in classical language: number of free functions of spatial coordinates at an initial Cauchy hypersurface.

Six, instead of four for GR – two additional functions describe massive scalar waves.

Thus, $f(R)$ gravity is a **non-perturbative** generalization of GR. It is equivalent to scalar-tensor gravity with $\omega_{BD} = 0$ (if $f''(R) \neq 0$).

Why R-dependence only?

For almost all other geometric invariants –

$R_{\mu\nu}R^{\mu\nu}$, $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, $R_{,\mu}R^{,\mu}$ etc. (where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor) – ghosts appear if the theory is taken in full, in the non-perturbative regime.

The only known exception: $f(R, G)$ with $f_{RR}f_{GG} - f_{RG}^2 = 0$, where $G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Gauss-Bonnet invariant, does not possess ghosts but has other problems.

For $f_{RR}f_{GG} - f_{RG}^2 \neq 0$, a ghost was found very recently (A. de Felice, S. Tsujikawa, arViv:1006.4399).

Possible microscopic origin of $f(R)$

1. Quantum-gravitational loop corrections from all quantum field of matter including gravity itself.

Recent development: asymptotically safe gravity (existence of a fixed UV point).

Cannot produce "R-dominance" – corrections depending on $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ are of the same order as those depending on R .

2. Reduction from higher-dimensional models.

The same problem.

Viable ways.

3. A non-minimally coupled scalar field with a large negative coupling ξ (for this choice of signs, $\xi_{conf} = \frac{1}{6}$):

$$L = \frac{R}{16\pi G} - \frac{\xi R \phi^2}{2} + \frac{1}{2} \phi_{,\mu} \phi^{,\mu} - V(\phi), \quad \xi < 0, \quad |\xi| \gg 1.$$

Recent development: Higgs inflation (F. Bezrukov and M. Shaposhnikov, 2008). In the limit $|\xi| \gg 1$, the potential $V(\phi) = \frac{\lambda(\phi^2 - \phi_0^2)^2}{4}$ just produces $f(R) = \frac{1}{16\pi G} \left(R + \frac{R^2}{6M^2} \right)$ with $M^2 = \lambda/24\pi|\xi|^2 G$ (for this model, $\xi G \phi_0^2 \ll 1$).

4. Emergent gravity.

Less (not more!) symmetry at a more fundamental level.

Appearance of $f(R)$ gravity in the simplest case:

F. R. Klinkhamer and G. E. Volovik, JETP Lett. 88, 289 (2008) (arXiv:0807.3896).

Background FRW equations in $f(R)$ gravity

$$H \equiv \frac{\dot{a}}{a}, \quad R = 6(\dot{H} + 2H^2)$$

The trace equation (4th order)

$$\frac{3}{a^3} \frac{d}{dt} \left(a^3 \frac{df'(R)}{dt} \right) - Rf'(R) + 2f(R) = 8\pi G(\rho_m - 3p_m)$$

The 0-0 equation (3d order)

$$3H \frac{df'(R)}{dt} - 3(\dot{H} + H^2)f'(R) + \frac{f(R)}{2} = 8\pi G \rho_m$$

Conditions for viable $f(R)$ models

I. Conditions of classical and quantum stability:

$$f'(R) > 0, \quad f''(R) > 0.$$

Even the saturation of these inequalities should be avoided:

1. $f'(R_0) = 0$: a generic anisotropic space-like curvature singularity forms.
2. $f''(R_0) = 0$: a weak singularity forms, loss of predictability of the Cauchy evolution.

$$a(t) = a_0 + a_1(t - t_s) + a_2(t - t_s)^2 + a_3|t - t_s|^{5/2} + \dots$$

The metric is in C^2 , but not C^3 , continuous across this singularity, and there is no unambiguous relation between the coefficients a_3 for $t < t_s$ and $t > t_s$.

II. Conditions for the existence of the Newtonian limit:

$$|F| \ll R, \quad |F'| \ll 1, \quad RF'' \ll 1$$

for $R \gg R_{now}$ and up to some very large R .

The same conditions for smallness of deviations from GR.

III. Laboratory and Solar system tests.

No deviation from the Newton law up to 50μ .

No deviation from the Einstein values of the post-Newtonian coefficients β and γ up to 10^{-4} in the Solar system.

IV. Existence of a future stable (or at least metastable) de Sitter asymptote:

$$f'(R_{DS})/f''(R_{DS}) \geq R_{DS} .$$

Required since observed properties of DE are close to that of a cosmological constant.

V. Cosmological tests:

among them the anomalous growth of matter perturbations for recent redshifts

$$\left(\frac{\delta\rho}{\rho}\right)_m \propto t^{\frac{\sqrt{33}-1}{6}}$$

at the matter-dominated stage for $k \gg M(R)a$, where

$$M^2(R) = 1/3F''(R) .$$

VI. $f(R)$ cosmology should not destroy previous successes of present and early Universe cosmology in the scope of GR, including the existence of the matter-dominated stage driven by non-relativistic matter preceded by the radiation-dominated stage with the correct BBN and, finally, inflation.

Inflationary $f(R)$ models

1. The simplest one (A.A. Starobinsky, 1980):

$$f(R) = R + \frac{R^2}{6M^2}$$

with small one-loop quantum gravitational corrections producing the scalaron decay.

During inflation ($H \gg M$): $H = \frac{M^2}{6}(t_f - t)$, $|\dot{H}| \ll H^2$.

Remains viable: $n_s = 1 - \frac{2}{N} \approx 0.96$, $r = \frac{12}{N^2} \approx 0.004$
where $N \sim 55$ is the number of e-folds between the first Hubble radius crossing during inflation of the present Hubble scale and the end of inflation.

Observations: $n_s = 0.963 \pm 0.012$; $r < 0.24$ (95% CL).

2. Analogues of chaotic inflation: $F(R) \approx R^2 A(R)$ for $R \rightarrow \infty$ with $A(R)$ being a slowly varying function of R , namely

$$|A'(R)| \ll \frac{A(R)}{R}, \quad |A''(R)| \ll \frac{A(R)}{R^2}.$$

3. Analogues of new inflation, $R \approx R_1$:

$$F'(R_1) = \frac{2F(R_1)}{R_1}, \quad F''(R_1) \approx \frac{2F(R_1)}{R_1^2}.$$

Thus, all inflationary models in $f(R)$ gravity are close to the simplest one over some range of R .

An example of a viable DE model

A. A. Starobinsky, JETP Lett. **86**, 157 (2007)

$$f(R) = R + \lambda R_0 \left(\frac{1}{\left(1 + \frac{R^2}{R_0^2}\right)^n} - 1 \right)$$

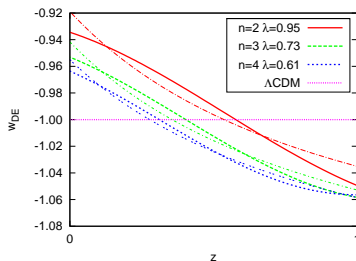
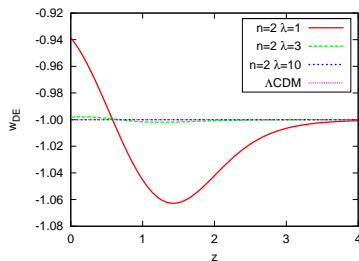
with $n \geq 2$. Similar models in:

1. W. Hu and I. Sawicki, Phys. Rev. D **76**, 064004 (2007).
2. A. Appleby and R. Battye, Phys. Lett. B **654**, 7 (2007).

Phantom boundary crossing

H. Motohashi, A. A. Starobinsky, J. Yokoyama,
Progr. Theor. Phys. **123**, 887 (2010).

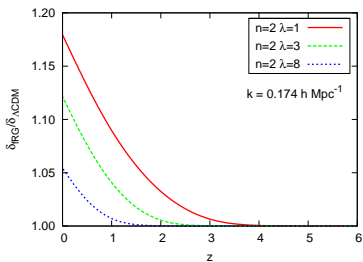
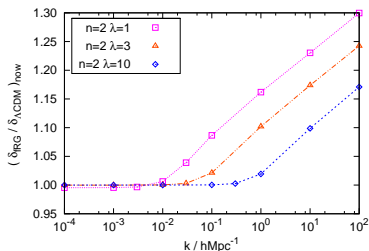
Generic feature: phantom behaviour for $z > 1$,
crossing of the phantom boundary $w_{DE} = -1$ for $z < 1$.



Anomalous growth of perturbations

Deeply in the sub-horizon regime:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\text{eff}}\rho\delta = 0, \quad G_{\text{eff}} = \frac{G}{f'} \frac{1 + 4\frac{k^2}{a^2} \frac{f''}{f'}}{1 + 3\frac{k^2}{a^2} \frac{f''}{f'}}.$$



Still not the end of the story!

Structure of corrections to GR

$$R = R^{(0)} + \delta R_{ind} + \delta R_{osc} ,$$

$$R^{(0)} = 8\pi G T_m \propto a^{-3} ,$$

$$\delta R_{ind} = (RF'(R) - 2F(R) - 3\nabla_\mu \nabla^\mu F'(R))_{R=R^{(0)}} ,$$

$$R \gg R_0, \quad \delta R_{ind} \approx \text{const} = -F(\infty) = 4\Lambda(\infty) .$$

No Dolgov-Kawasaki instability.

$$MD : \quad \delta R_{osc} \propto t^{-(3n+4)} \sin (c_1 t^{-(2n+1)} + c_2) ,$$

$$RD : \quad \delta R_{osc} \propto t^{-3(3n+4)/4} \sin (c_3 t^{-(3n+1)/2} + c_4) .$$

$\delta a/a$ is small but $\delta R_{osc}/R^{(0)}$ diverges for $t \rightarrow 0$.

δR_{osc} should be very small just from the beginning – a problem for those $f(R)$ models which do not let R become negative due to crossing of the $f''(R) = 0$ point.

The "scalon overproduction" problem.

Three new problems

In the early Universe:

- ▶ Unlimited growth of $M(R)$ for $t \rightarrow 0$: when $M(R)$ exceeds M_{Pl} , quantum-gravitational loop corrections invalidate the use of an effective quasi-classical $f(R)$ gravity.
- ▶ Unlimited growth of the amplitude of δR oscillations for $t \rightarrow 0$ (the "scalon overproduction" problem).
- ▶ "Big Boost" singularity before the Big Bang:

$$a(t) = a_0 + a_1(t-t_0) + a_2|t-t_0|^k + \dots, \quad 1 < k = \frac{2n+1}{n+1} < 2,$$

if $F(R) \propto R^{-2n}$ for $R \rightarrow \infty$.

Curing all three problems

S. A. Appleby, R. A. Battye and A. A. Starobinsky,
JCAP **1006**, 005 (2010).

Add $\frac{R^2}{6M^2}$ to $f(R)$ with M not less than the scale of inflation.
Then the first and third problems go away. The second
problem still remains, but (any) inflation can solve it.

However, in all known inflationary models R may be negative
during reheating after inflation (e.g. when $V(\phi) = 0$).

Necessity of an extension of $f(R)$ to $R < 0$ keeping $f''(R) > 0$.
As a result, a non-zero g -factor ($0 < g < 1/2$) arises:

$$g = \frac{f'(R) - f'(-R)}{2f'(R)}, \quad R_0 \ll R \ll M^2.$$

An example: g -extended R^2 -corrected AB model

$$f(R) = (1 - g)R + g\epsilon \log \left[\frac{\cosh(R/\epsilon - b)}{\cosh b} \right] + \frac{R^2}{6M^2} .$$

Combined inflationary–DE models

If $M \approx 3 \times 10^{-6} M_{Pl}$, the scalaron can play the role of an inflaton, too. Then the inflationary predictions are formally the same as for the pure $R + R^2/6M^2$ inflationary model which does not describe the present DE:

$$n_s = 1 - \frac{2}{N}, \quad r = \frac{12}{N^2}.$$

However, N is different, $N \sim 70$ for the unified model (versus $N \sim 55$ for the purely inflationary one) because the stage of reheating after inflation becomes completely different: it consists of unequal periods with $a \approx \text{const}$ and $a \propto t^{1/2}$.

Duration of the periods in terms of $\ln t$:

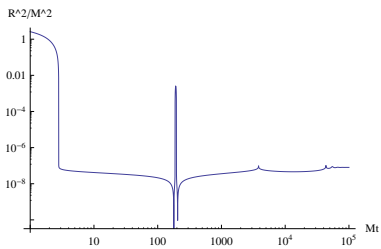
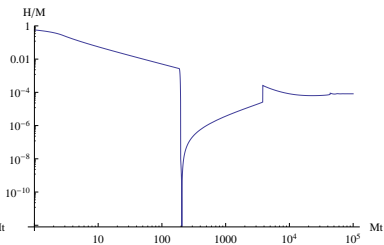
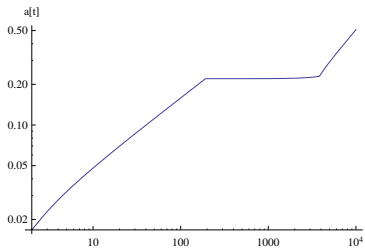
$-\ln(1 - 2g)$ and $-2 \ln(1 - 2g)$ respectively.

So, $a(t) \propto t^{1/3}$ on average for a long time after the end of inflation, in contrast to

$$a(t) \propto t^{2/3} \left(1 + \frac{2}{3Mt} \sin M(t - t_1) \right)$$

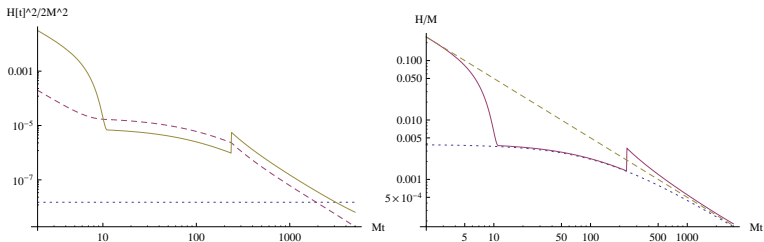
for the pure inflationary $f(R) = R + R^2/6M^2$ model.

Observable prediction which is, however, degenerate with other inflationary models in $f(R)$ gravity.



Reheating – due to gravitational particle creation which occurs mainly at the end of inflation. Less efficient than in the pure inflationary $f(R) = R + R^2/6M^2$ model,

$$t = t_{\text{reh}} \sim M^{-4} M_{\text{Pl}}^3 \sim 10^{-18} \text{ s} .$$



Conclusions about $f(R)$ models

- ▶ The simplest pioneer inflationary $f(R)$ model remains viable. Critical test: measurement of r (primordial GW).
- ▶ Much more problems with models of present DE. Still a narrow class among all $f(R)$ models of present DE remains viable: it is possible to construct models satisfying all existing cosmological, Solar system and laboratory data, and distinguishable from Λ CDM.
- ▶ To achieve this, previously constructed viable $f(R)$ DE models should be extended to large R with the $\sim R^2$ asymptotic behaviour and to negative R keeping $f'(R) > 0$, $f''(R) > 0$ at least up to the scale of inflation.

- ▶ Unified description of primordial DE producing inflation and present DE in the scope of $f(R)$ gravity is possible for the specific choice of M : $M \approx 3 \times 10^{-6} M_{Pl}$.
- ▶ Combined inflationary – DE $f(R)$ models have a significantly different reheating stage after inflation as compared to pure inflationary $f(R)$ models, with strongly non-linear oscillations of the scale factor $a(t)$.
- ▶ The most critical test for all $f(R)$ models of present dark energy: anomalous growth of density perturbations in the matter component at recent redshifts $z \sim 1 - 3$.