Introduction to supersymmetric models (in the main, on the example of SUSY mechanics)

Sergey Fedoruk

BLTP JINR, Dubna, Russia

International Advanced School on Modern Mathematical Physics
BLTP JINR, Dubna, Russia
September 5-15, 2010

Plan

- Lecture 1: Grounds for supersymmetry
- Lecture 2: 1D SUSY in usual space
- Lecture 3: 1D SUSY in superspace
- Lecture 4: Superconformal mechanics
- Lecture 5: Supersymmetric models in harmonic superspace

Lecture 1: Grounds for supersymmetry

- Unification in elementary particle physics
- Haag-Lopushanski-Sohnius theorem: Graded symmetry algebra
- Brief sketch on supermatrix and supergroups
- Super–Poincare algebra
- Conformal supersymmetry
- Wess–Zumino model

In elementary particle physics, the hope is that we will eventually achieve a unified scheme which combines all particles and all their interactions into one consistent theory.

Modern particles:

$$\text{Bosons}:\quad \underbrace{\overbrace{A_\mu \sim (\vec{E},\vec{B})}^{\text{Maxwell Theory},\ U(1)} \ \oplus\ W^\pm_\mu,\ W^0_\mu}_{\text{Electroweek Theory},\ SU(2)\times U(1)} \ \oplus\ \underbrace{G^r_\mu,\ r{=}1,\ldots,8}_{\text{Strong Interaction},\ SU(3)} \ \oplus\ \underbrace{g_{\mu\nu}}_{\text{Gravity}}$$

$$\text{Fermions}:\qquad \psi^i_\alpha\,,\qquad \bar\psi_{\dot\alpha i}$$

$\mu,\nu = 0,1,2,3$, $\alpha = 1,2$, $\dot\alpha = 1,2$ are the Lorentz indices; i is internal symmetry index

$$\left.\begin{array}{l}\left.\begin{array}{l}[L_{\mu\nu},L_{\rho\lambda}] = i\,(\eta_{\nu\rho}L_{\mu\lambda}+\eta_{\mu\lambda}L_{\nu\rho}-(\mu\leftrightarrow\nu))\quad \mathrm{SL(2,C)}\\ [P_\mu,P_\nu]=0\,,\quad [L_{\mu\nu},P_\lambda]=i\,(\eta_{\nu\lambda}P_\mu+\eta_{\mu\lambda}P_\nu)\end{array}\right\}\ \text{Poincare algebra}\\ [D,P_\mu]=i\,P_\mu\,,\quad [D,K_\mu]=-i\,K_\mu\,,\quad [D,L_{\mu\nu}]=0\\ [K_\mu,K_\nu]=0\,,\quad [P_\mu,K_\nu]=-2i\,(\eta_{\mu\nu}D+L_{\mu\nu})\,,\quad [L_{\mu\nu},K_\lambda]=i\,(\eta_{\nu\lambda}K_\mu+\eta_{\mu\lambda}K_\nu)\end{array}\right\}\ \text{conformal algebra}$$

$$[T^i_j,T^k_l]=i\,(\delta^i_l T^k_j-\delta^k_j T^i_l)\,,\quad (T^i_j)^+=-T^j_i\,,\quad T^i_i=0 \qquad \text{internal summetry algebra SU(n)}$$

$$[T^i_j,L_{\mu\nu}]=[T^i_j,P_\mu]=[T^i_j,K_\mu]=[T^i_j,D]=0$$

Coleman, Mandula, 1967: it is impossible to unify space–time symmetry with internal symmetries in frame of local relativistic field theory in four dimension with finite number of massive particles.

$$W^\mu=\tfrac12\epsilon^{\mu\nu\lambda\rho}P_\mu L_{\lambda\rho}\,,\qquad W^2=-m^2\vec{J}^2\ \ (P^2\neq0)\,,\qquad W_\mu=\Lambda P_\mu\ \ (P^2=0)$$

$[T^i_j,P^2]=[T^i_j,W^2]=[T^i_j,\Lambda]=0$ – all particles of an irreducible multiplet must have the same mass and the same spin (helicity)

Bypass of the Coleman–Mandula theorem:
Haag, Lopushanski, Sohnius, 1975 proved that in the context of relativistic field theory the only models which lead to the unification problem are supersymmetric theories.
Symmetry algebras of the supersymmetric models are graded Lie algebras or Lie superalgebras

$$[B_A, B_B] = i\, c_{AB}^{C} B_C\,, \qquad [B_A, Q_K] = i\, g_{AK}^{M} Q_M\,, \qquad \{Q_K, Q_M\} = i\, f_{KM}^{A} B_A$$

B_A are even (commuting, bosonic) elements; Q_K are odd (anticommuting, fermionic) elements
Graded Jacobi identities

$$[[G_1, G_2\}, G_3\} + \text{graded cyclic} = 0$$

(there is additional minus sign if two fermionic operators are interchanged)
Bosonic subalgebra B_A are defined by Coleman–Mandula theorem.
On the fermionic operators Q_M it is realized the representation of the bosonic subalgebra.
Q_M generate supersymmetric tansformations

$$Q\,|\text{boson} > = |\text{fermion} >\,, \qquad Q\,|\text{fermion} > = |\text{boson} >$$

Parity: $q(B) = 0, \quad q(Q) = 1, \quad q(|\text{boson} >) = 0, \quad q(|\text{fermion} >) = 1$

Simple example of SUSY algebra: BRST symmetry

$$[B_A, Q] = 0\,, \qquad \{Q, Q\} = 0$$

Q is BRST charge

$$X = \left(\begin{array}{c|c} B_1 & F_1 \\ \hline F_2 & B_2 \end{array} \right) ;$$

$B_{1,2}$ are ordinary matrices,
$F_{1,2}$ are fermionic matrices

$$\mathrm{str}\, X = \mathrm{tr}\, B_1 - \mathrm{tr}\, B_2 \,, \qquad \mathrm{str}\, XY = \mathrm{str}\, YX$$

$$\mathrm{sdet} \begin{pmatrix} B_1 & F_1 \\ 0 & 1 \end{pmatrix} = \det B_1, \quad \mathrm{sdet} \begin{pmatrix} 1 & F_1 \\ 0 & B_2 \end{pmatrix} = \det B_2^{-1}; \qquad \mathrm{sdet}\, XY = \mathrm{sdet}\, X \cdot \mathrm{sdet}\, Y$$

$$\begin{pmatrix} B_1 & F_1 \\ F_2 & B_2 \end{pmatrix} = \begin{pmatrix} 1 & F_1 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} B_1 - F_1 B_2^{-1} F_2 & 0 \\ B_2^{-1} & 1 \end{pmatrix}, \quad \mathrm{sdet}\, X = \det (B_1 - F_1 B_2^{-1} F_2) \cdot \det B_2^{-1}$$

$$\mathrm{sdet}\, X = \exp \{ \mathrm{str}\, (\ln X) \}$$

$$\mathrm{OSp}(m|n) : \qquad G = e^X = \left(\begin{array}{c|c} \mathrm{Sp}(n) & F_1 \\ \hline F_2 & \mathrm{SO}(m) \end{array} \right)$$

$$\mathrm{U}(m,n|p) : \qquad G = e^X = \left(\begin{array}{c|c} \mathrm{U}(m,n) & F_1 \\ \hline F_2 & \mathrm{U}(p) \end{array} \right)$$

$$\mathrm{SU}(m,n|p) : \qquad G = e^X \,, \quad \mathrm{str}\, X = 0$$

For $m + n = p$ the identity matrix obeys $\mathrm{tr}\, B_1 = \mathrm{tr}\, B_2$ and generates $\mathrm{U}(1)$ subgroup.
The quotient $\mathrm{PSU}(m,n|p) = \mathrm{SU}(m,n|p)/\mathrm{U}(1)$ is simple and is often denoted just $\mathrm{SU}(m,n|p)$.

Notations used:

$$(\gamma_\mu)_a{}^b = \begin{pmatrix} 0 & (\sigma_\mu)_{\alpha\dot\beta} \\ (\tilde\sigma_\mu)^{\dot\alpha\beta} & 0 \end{pmatrix}, \qquad a = 1,..,4 \quad \alpha = 1,2, \ \dot\alpha = 1,2$$

$$\Psi_a = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix}, \qquad \bar\chi_{\dot\alpha} = \overline{(\psi_\alpha)} \text{ for a Majorana spinor}$$

$$\sigma_\mu\tilde\sigma_\nu + \sigma_\nu\tilde\sigma_\mu = 2\eta_{\mu\nu}, \quad \sigma_{\mu\nu} = i\sigma_{[\mu}\tilde\sigma_{\nu]}, \quad \tilde\sigma_{\mu\nu} = i\tilde\sigma_{[\mu}\sigma_{\nu]}$$

$$\psi^\alpha=\epsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha=\epsilon_{\alpha\beta}\psi^\beta, \quad \bar\psi^{\dot\alpha}=\epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}, \quad \bar\psi_{\dot\alpha}=\epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta},$$

$$\epsilon_{\alpha\beta}=-\epsilon_{\beta\alpha}, \quad \epsilon_{\dot\alpha\dot\beta}=-\epsilon_{\dot\beta\dot\alpha}, \quad \epsilon_{12}=-\epsilon^{12}=1$$

$$(\tilde\sigma_\mu)^{\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}(\sigma_\mu)_{\beta\dot\beta}, \quad (\sigma^\mu)_{\alpha\dot\alpha} = (1,\vec\sigma)_{\alpha\dot\alpha}, \quad (\tilde\sigma^\mu)^{\dot\alpha\alpha} = (1,-\vec\sigma)^{\dot\alpha\alpha}$$

$$x_{\alpha\dot\beta} = \sigma^\mu_{\alpha\dot\beta}x_\mu, \quad x^{\dot\alpha\beta} = \tilde\sigma^{\dot\alpha\alpha}_\mu x^\mu, \quad x^\mu = x^{\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\alpha}$$

$$P_\mu, L_{\mu\nu}, T^i_j \quad \oplus \quad Q^i_\alpha, \bar{Q}_{\dot\alpha i} = (Q^i_\alpha)^+ \quad \oplus \quad Z^{ij}, \bar{Z}_{ij} = (Z^{ij})^+$$

$$\{Q^i_\alpha, \bar{Q}_{\dot\beta j}\} = 2\delta^i_j (\sigma^\mu)_{\alpha\dot\beta} P_\mu, \qquad \{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij}, \qquad \{\bar{Q}_{\dot\alpha i}, \bar{Q}_{\dot\beta j}\} = \epsilon_{\dot\alpha\dot\beta} \bar{Z}_{ij},$$

$$[P_\mu, Q^i_\alpha] = 0, \quad [P_\mu, \bar{Q}_{\dot\alpha i}] = 0, \qquad [L_{\mu\nu}, Q^i_\alpha] = -\tfrac{1}{2}(\sigma_{\mu\nu})_\alpha^\beta Q^i_\alpha, \quad [L_{\mu\nu}, \bar{Q}_{\dot\alpha i}] = \tfrac{1}{2}(\tilde\sigma_{\mu\nu})^{\dot\beta}_{\dot\alpha} Q_{\dot\alpha i},$$

$$[T^i_j, Q^k_\alpha] = \delta^k_j Q^i_\alpha - \tfrac{1}{N}\delta^i_j Q^k_\alpha, \qquad [T^i_j, \bar{Q}_{\dot\alpha k}] = -\delta^i_k \bar{Q}_{\dot\alpha j} + \tfrac{1}{N}\delta^i_j \bar{Q}_{\dot\alpha k}$$

$Z^{ij}, \bar{Z}_{ij} = (Z^{ij})^+$ are central charges, $[Z,P] = [Z,L] = [Z,Q] = [Z,Z] = 0$

In massless case, $P^2 = 0$, : $\quad Z^{ij} = 0, \ \bar{Z}_{ij} = 0$

Basic properties of Poincare supersymmetry

- $[P^2, Q] = 0$ - all particles of any supermultiplet have the same mass
- $0 \le \sum_i \sum_\alpha \{Q^i_\alpha, \bar{Q}_{\dot\alpha i}\} = 4N\, P_0$ - the energy is non-negative
- $\{|\mathrm{bosons}>\} \ \stackrel{Q}{\rightarrow} \ \{|\mathrm{fermions}>\} \ \stackrel{Q}{\rightarrow} \ \{|\mathrm{bosons}>\}$ translated by P_μ $\Rightarrow$
  There are an equal number of bosons and fermions
  when translations are an invertible operator

$$SU(2,2|N)$$

$$P_\mu, L_{\mu\nu}, \underbrace{T^i_j,\ R}_{U(N)} \quad \oplus \quad K_\mu, D \quad \oplus \quad Q^i_\alpha, \bar{Q}_{\dot\alpha i} = (Q^i_\alpha)^+ \quad \oplus \quad S_{\alpha i}, \bar{S}^i_{\dot\alpha} = (S_{\alpha i})^+$$

$$\{S_{\alpha i}, \bar{S}^j_{\dot\alpha}\} = 2\delta^j_i (\sigma^\mu)_{\alpha\dot\alpha} K_\mu \,,$$

$$\{Q^i_\alpha, S^\beta_j\} = -\delta^i_j (\sigma^{\mu\nu})^\beta_\alpha L_{\mu\nu} - 4i\delta^\beta_\alpha T^i_j - 2i\delta^\beta_\alpha \delta^i_j D + \tfrac{2(4-N)}{N}\, \delta^\beta_\alpha \delta^i_j R \,,$$

$$[K_\mu, Q^i_\alpha] = (\sigma_\mu)_{\alpha\dot\alpha}\, \bar{S}^{\dot\alpha i} \,, \quad [K_\mu, \bar{Q}_{\dot\alpha i}] = -\,(\sigma_\mu)_{\alpha\dot\alpha}\, S^\alpha_i \,,$$

$$[P_\mu, S_{\alpha i}] = (\sigma_\mu)_{\alpha\dot\alpha}\, \bar{Q}^{\dot\alpha}_i \,, \quad [P_\mu, \bar{S}^i_{\dot\alpha}] = -\,(\sigma_\mu)_{\alpha\dot\alpha}\, Q^{\alpha i} \,,$$

$$[D,Q] = \tfrac{i}{2}\, Q \,, \qquad [D,\bar{Q}] = \tfrac{i}{2}\, \bar{Q} \,, \qquad [D,S] = -\tfrac{i}{2}\, S \,, \qquad [D,\bar{S}] = -\tfrac{i}{2}\, \bar{S} \,,$$

$$[R,Q] = -\tfrac{1}{2}\, Q \,, \qquad [R,\bar{Q}] = \tfrac{1}{2}\, \bar{Q} \,, \qquad [R,S] = \tfrac{1}{2}\, S \,, \qquad [R,\bar{S}] = -\tfrac{1}{2}\, \bar{S} \,,$$

Simple 4D supersymmetric field theory: Wess–Zumino model

$$\underbrace{\phi(x),\ \bar\phi(x),\ \ F(x),\ \bar F(x),}_{\text{bosons (c}-\text{number)}}\qquad \underbrace{\psi_\alpha(x),\ \bar\psi_{\dot\alpha}(x)}_{\text{fermions (Grassmann)}}$$

$\varepsilon_\alpha, \bar\varepsilon_{\dot\alpha}$ – Grassmann parameters of SUSY translations , $\{\varepsilon_\alpha, \varepsilon_\beta\} = \{\varepsilon_\alpha, \bar\varepsilon_{\dot\beta}\} = 0$

$$\delta\phi = (\varepsilon^\alpha Q_\alpha + \bar\varepsilon_{\dot\alpha}\bar Q^{\dot\alpha})\phi\,, \qquad \ldots, \qquad \delta\bar\psi_{\dot\alpha} = (\varepsilon^\alpha Q_\alpha + \bar\varepsilon_{\dot\alpha}\bar Q^{\dot\alpha})\bar\psi_{\dot\alpha}$$

$$(\delta_1\delta_2 - \delta_2\delta_1)\phi = 2(\varepsilon_1\sigma^\mu\bar\varepsilon_2 - \varepsilon_2\sigma^\mu\bar\varepsilon_1)P_\mu\phi, \quad \ldots, \quad (\delta_1\delta_2 - \delta_2\delta_1)\bar\psi_{\dot\alpha} = 2(\varepsilon_1\sigma^\mu\bar\varepsilon_2 - \varepsilon_2\sigma^\mu\bar\varepsilon_1)P_\mu\bar\psi_{\dot\alpha}$$

$$\delta\phi = -\varepsilon^\alpha\psi_\alpha\,, \quad \delta\psi_\alpha = -2i\sigma^\mu_{\alpha\dot\alpha}\bar\varepsilon^{\dot\alpha}\partial_\mu\phi - 2\varepsilon_\alpha F\,, \quad \delta F = -i\sigma^\mu_{\alpha\dot\alpha}\bar\varepsilon^{\dot\alpha}\partial_\mu\psi^\alpha$$

$$S = \int d^4x\,\mathcal{L}\,, \qquad \mathcal{L} = -\tfrac{i}{2}\,\psi\sigma^\mu\partial_\mu\bar\psi - \bar\phi\Box\phi \underbrace{+\bar F F + m\,(\phi F + \bar\phi\bar F)}_{-m\phi\bar\phi} - \tfrac{1}{4}\,m\,(\psi\psi + \bar\psi\bar\psi)$$

$F(x), \bar F(x)$ – auxiliary fields; its equations of motion is purely algebraic: $F + m\bar\phi = 0$

$\delta\phi = -\varepsilon^\alpha\psi_\alpha\,, \quad \delta\psi_\alpha = -2i\sigma^\mu_{\alpha\dot\alpha}\bar\varepsilon^{\dot\alpha}\partial_\mu\phi + 2m\,\varepsilon_\alpha\phi$ - on-shell SUSY transformations (nonlinear in case of the interaction, closed only on field equations of motion)

Lecture 2: 1D SUSY in usual space

- 1D super-Poincare and superconformal symmerties
- Example of 1D field theory: relativistic particle
- 1D field theory with global SUSY
- Hamiltonian analysis and supercharges
- $N = 1$ supergravity in 1D
- Quantization: spin $1/2$ particle in pseudoclassical approach

1*D* *N*-extended super-Poincare algebra

$$\{Q_a, Q_b\} = 2\delta_{ab}H\,, \qquad (Q_a)^+ = Q\,, \qquad a = 1, ..., N$$

1*D* *N*-extended superconformal algebra

1*D* superconformal algebra $\supset$ 1*D* conformal symmetry $\mathrm{SO}(1,2) \sim \mathrm{Sp}(2) \sim \mathrm{SU}(1,1)$

$$\sim \left(\begin{array}{c|c} \mathrm{Sp}(2) & Q+S \\ \hline Q-S & \mathrm{SO}(N) \end{array} \right), \qquad \sim \left(\begin{array}{c|c} \mathrm{SU}(1,1) & Q+S \\ \hline Q-S & \mathrm{SU}(M) \end{array} \right)$$

$$\{Q,Q\} \sim H, \quad \{S,S\} \sim K, \quad \{Q,S\} \sim D+J, \qquad (H,K,D)\subset su(1,1), \quad J\subset o(N)\,\mathrm{or}\, su(M)$$

$N{=}1$: $\mathrm{OSp}(1|2)$

$N{=}2$: $\mathrm{OSp}(2|2) \sim \mathrm{SU}(1,1|1)$

$N{=}4$: $D(2,1;\alpha)$

$\alpha = -1/2,\ \alpha = 1$: $D(2,1;\alpha) \sim \mathrm{OSp}(4|2)$

$\alpha = 0,\ \alpha = -1$: $D(2,1;\alpha) \sim \mathrm{SU}(1,1|2) \oplus_S \mathrm{SU}(2)$

$$D(2,1;\alpha): \quad \{Q^{ai'i}, Q^{bk'k}\} = 2\left(\epsilon^{ik}\epsilon^{i'k'}T^{ab} + \alpha\epsilon^{ab}\epsilon^{i'k'}J^{ik} - (1+\alpha)\epsilon^{ab}\epsilon^{ik}I^{i'k'}\right),$$

$$[T^{ab}, T^{cd}] = i(\epsilon^{ac}T^{bd} + \epsilon^{bd}T^{ac}), \quad ..., \quad [T^{ab}, Q^{ci'i}] = i\epsilon^{c(a}Q^{b)i'i}, ...$$

$$Q^{21'i}{=}-Q^i, \quad Q^{22'i} = -\bar{Q}^i, \quad Q^{11'i}{=}S^i, \quad Q^{12'i} = \bar{S}^i, \qquad T^{22}{=}H, \quad T^{11}{=}K, \quad T^{12}{=}-D.$$

Bosonic generators T^{ab}, J^{ik} and $I^{i'k'}$ form $su(1,1)$, $su(2)$ and $su'(2)$ algebras.

t - time coordinate; 1D field theory - mechanics

$$\tilde{S} = \int dt\, \tilde{L}, \qquad \tilde{L} = \tfrac{1}{2}\, \dot{\phi}^A \dot{\phi}^B \eta_{AB}$$ - 1D massless Klein-Gordon action

Global invariance: $t' = t + a, \;\; \phi'(t') = \phi(t)$

Local invariance: 1D gravity

4D: $g_{\mu\nu} = \eta_{mn} e^m_\mu e^n_\nu$, e^m_μ are vielbein fields

$\Gamma^\lambda_{\mu\nu} = g^{\lambda\rho}\, \tfrac{1}{2}\,(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}), \quad R^\rho_{\mu\nu\lambda} = \partial_\mu \Gamma^\rho_{\nu\lambda} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - (\mu \leftrightarrow \nu)$

1D: $g_{\mu\nu} = e^2, \;\; g^{\mu\nu} = e^{-2}, \;\; \sqrt{g} = e, \;\; \Gamma = (\ln e)^{\cdot}, \quad R \equiv 0$

$$S = \int dt\, L, \qquad L = \tfrac{1}{2}\, \sqrt{g} g^{\mu\nu} \partial_\mu \phi \cdot \partial_\mu \phi + \tfrac{1}{2}\, \sqrt{g} m^2 = \tfrac{1}{2}\,(e^{-1}\dot{\phi} \cdot \dot{\phi} + e m^2)$$

$$H = \tfrac{1}{2}\, e\,(p \cdot p - m^2), \qquad T_1 = p \cdot p - m^2 \approx 0, \qquad T_2 = p_e \approx 0$$

Quantization:

$$[\hat{A}, \hat{B}] = i\{A, B\}_P : \quad \hat{\phi} = x, \quad \hat{p} = -i\partial_x, \quad \hat{e} = e, \quad \hat{p}_e = -i\partial_e$$

$$\Phi = \Phi(x, e, t), \qquad i\partial_t \Phi = \hat{H}\Phi, \quad \hat{T}_{1,2}\Phi = 0$$

$$\Rightarrow \qquad (\Box + m^2)\Phi(x) = 0, \qquad \text{Klein} - \text{Gordon equation}$$

Quantization 1D matter fields in 1D gravity background $\Rightarrow$ spin 0 target space field

Note: String action $\quad S_{string} = \int d^2\sigma \mathcal{L}_{string}, \qquad \mathcal{L}_{string} = T\, \sqrt{g}\, g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu$

$$S = \int dt\, L, \qquad L = \tfrac{1}{2}\,\dot\phi^2 + \tfrac{i}{2}\,\psi\dot\psi$$

$$[\phi(t_1),\phi(t_2)] = \phi(t_1)\phi(t_2) - \phi(t_2)\phi(t_1) = 0, \qquad \{\psi(t_1),\psi(t_2)\} = \psi(t_1)\psi(t_2) + \psi(t_2)\psi(t_1) = 0$$

$$\phi^+ = \phi, \qquad \psi^+ = \psi; \qquad (AB)^+ = B^+A^+$$

$Q: \quad \phi \to \psi, \quad \psi \to \phi \qquad \Rightarrow \qquad$ the parameter $\varepsilon = \varepsilon^+$ must be anticommuting

$$[S/\hbar] = 0, \quad \hbar = 1 \qquad \Rightarrow \qquad [L] = +1, \quad [t] = -1 \qquad \Rightarrow \qquad [\phi] = -1/2, \quad [\psi] = 0$$

$$\delta\phi = i\varepsilon\psi \qquad \Rightarrow \qquad [\varepsilon] = -1/2 \qquad \Rightarrow \qquad \delta\psi \sim \varepsilon\dot\phi$$

$$\delta\phi = i\varepsilon\psi, \qquad \delta\psi = -\varepsilon\dot\phi$$

$$\delta L = \tfrac{i}{2}\,(\varepsilon\psi\dot\phi)^{\cdot} + i\dot\varepsilon\psi\dot\phi = 0, \qquad \varepsilon = const, \qquad \phi|_{t=\pm\infty} = \phi|_{t=\pm\infty} = 0$$

$$[\delta_1,\delta_2]\,\phi = 2i\varepsilon_1\varepsilon_2\dot\phi, \qquad [\delta_1,\delta_2]\,\psi = 2i\varepsilon_1\varepsilon_2\dot\psi$$

Note: In $N > 1$ 1D and $D > 1$ $\qquad [\delta_1,\delta_2]\,\psi = 2i\varepsilon_1\varepsilon_2\dot\psi + (\text{eq.of motion})$

$$p = \frac{\partial L}{\partial \dot\phi}, \quad \pi = \frac{\partial^r L}{\partial \dot\psi} \qquad \Rightarrow \qquad p = \dot\phi, \quad \pi = \tfrac{i}{2}\,\psi$$

$$H_0 = p\dot\phi + \pi\dot\psi - L = \tfrac{1}{2}\,p^2$$

$$G \equiv \pi - \tfrac{i}{2}\,\psi \approx 0 \qquad - \quad \text{the constraint}$$

$$H = H_0 + \lambda G$$

$$\{\phi, p\}_P = 1, \qquad \{\psi, \pi\}_P = 1$$

$$\{G, G\}_P = -i \neq 0 \qquad - \quad \text{second class constraint}$$

$$\dot G = \{H, G\}_P = 0 \qquad \Rightarrow \qquad \lambda = 0 \qquad \Rightarrow \qquad H = \tfrac{1}{2}\,p^2$$

$$\{A, B\}_D = \{A, B\}_P - \{A, G\}_P \{G, G\}_P^{-1} \{G, B\}_P$$

$$\{\phi, p\}_D = 1, \quad \{\psi, \psi\}_D = -i \qquad \Rightarrow \qquad [\phi, p] = i, \quad \{\psi, \psi\} = 1$$

$$\delta S = \int dt\,\dot\Lambda = \int dt \left(\frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot q}\,\delta \dot q\right) \quad \Rightarrow \quad \int dt\,\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\,\delta q - \Lambda\right) = \int dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\delta q = 0$$

$$p\,\delta q - \Lambda = const \qquad \text{on} - \text{shell}$$

$$p\,\delta\phi + \pi\,\delta\psi - \Lambda = i\varepsilon p\psi = i\,\varepsilon Q,$$

$$Q = p\psi, \qquad \{Q, Q\}_D = -2i\,H \qquad \Rightarrow \qquad \{Q, Q\} = 2H$$

$$S = \int dt \, (\tfrac{1}{2}\,\dot\phi^2 + \tfrac{i}{2}\,\psi\dot\psi), \qquad \delta S = i \int dt \, \dot\varepsilon\psi\dot\phi \qquad !!! \;\; \varepsilon = \varepsilon(t)$$

Introduce the gauge fermionic field (the gravitino) $\chi = \chi^+$: $\delta\chi = \dot\varepsilon + ...$

$$\text{New term}: \qquad S' = -i \int dt \, \chi\psi\dot\phi, \qquad \delta S' = -i \int dt \, \varepsilon\chi(i\psi\dot\psi + \dot\phi\dot\phi)$$

The first term can be canceled by adding a new term in $\delta\psi$

The last term can only be canceled by introducing a new field h (the graviton) and coupling $\dot\phi^2$

Local SUSY is a theory of gravity!!!

$$\text{New term}: \qquad S'' = -\int dt \, h\dot\phi\dot\phi$$

$$L_{1D\,SUGRA} = \tfrac{1}{2}\,\dot\phi^2 + \tfrac{i}{2}\,\psi\dot\psi - i\,\chi\psi\dot\phi - h\dot\phi\dot\phi$$

$$\text{Local SUSY}: \qquad \delta\phi = i\varepsilon\psi, \qquad \delta\psi = -\varepsilon(1-2h)\dot\phi + i\varepsilon\chi\psi,$$
$$\delta h = -i\varepsilon(1-2h)\chi, \qquad \delta\chi = (1-2h)\dot\varepsilon$$
$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\phi = 2i\varepsilon_1\varepsilon_2(1-2h)\dot\phi - \varepsilon_1\varepsilon_2\chi\psi = \hat\xi\dot\phi + i\hat\varepsilon\psi$$

$$\text{Gravity transformation}: \qquad \delta\phi = \xi\dot\phi, \qquad \delta\psi = \xi\dot\psi, \qquad \delta h = \tfrac{1}{2}\,\dot\xi + \xi\dot h - \dot\xi h, \qquad \delta\chi = \xi\dot\chi$$

Note: $\delta g_{\mu\nu} = \xi^\lambda\partial_\lambda g_{\mu\nu} + \partial_\mu\xi^\lambda g_{\lambda\nu} + \partial_\nu\xi^\lambda g_{\nu\lambda}$, $\delta e^m_\mu = \xi^\lambda\partial_\lambda e^m_\mu + \partial_\mu\xi^\lambda e^m_\lambda$,

$\delta e^\mu_m = \xi^\lambda\partial_\lambda e^\mu_m - e^\lambda_m\partial_\lambda\xi^\mu$, $\quad e^\mu_m = 1 - 2h$

$$\phi = (\phi_\mu), \qquad \psi = (\psi_\mu), \qquad \mu = 0, 1, ..., D-1, \qquad e^{-1} = 1 - 2h$$

$$S = \int dt\, L, \qquad L = \tfrac{1}{2}\, e^{-1} \dot\phi^\mu \dot\phi_\mu + \tfrac{i}{2}\, \psi^\mu \dot\psi_\mu - i\, \chi \dot\phi^\mu \psi_\mu$$

$$p_\mu = e^{-1}\dot\phi_\mu - i\chi\psi_\mu, \quad \pi_\mu = \tfrac{i}{2}\, \psi_\mu, \quad p_e = 0, \quad \pi_\chi = 0$$

$$H_0 = p_\mu \dot\phi^\mu + \pi_\mu \dot\psi^\mu + p_e \dot e - L = \tfrac{1}{2}\, e\, p^\mu p_\mu + \tfrac{i}{2}\, \chi\, p^\mu \psi_\mu$$

$p_e \approx 0, \quad \pi_\chi \approx 0, \quad G_\mu \equiv \pi_\mu - \tfrac{i}{2}\, \psi_\mu \approx 0$ — primary constraints

$$H = H_0 + \lambda^\mu G_\mu + \lambda_e p_e + \lambda_\chi \pi_\chi$$

$$\{\phi_\mu, p_\nu\}_P = \eta_{\mu\nu}, \qquad \{\psi_\mu, \pi_\nu\}_P = \eta_{\mu\nu} \qquad \{e, p_e\}_P = 1 \qquad \{\chi, p_\chi\}_P = 1$$

$\{G_\mu, G_\nu\}_P = -i\eta_{\mu\nu}$ — second class constraints

$$\dot G_\mu = \{H, G_\mu\}_P = 0 \qquad \Rightarrow \qquad \lambda_\mu = 0$$

$$\dot p_e = \{H, p_e\}_P = 0 \qquad \Rightarrow \qquad T \equiv p^\mu p_\mu \approx 0$$

$$\dot\pi_\chi = \{H, \pi_\chi\}_P = 0 \qquad \Rightarrow \qquad D \equiv p^\mu \psi_\mu \approx 0$$

$$\{A,B\}_D = \{A,B\}_P - \{A,G^\mu\}_P \{G_\mu, G_\nu\}_P^{-1} \{G^\nu, B\}_P$$
$$\{\phi_\mu, p_\nu\}_D = \eta_{\mu\nu}, \qquad \{\psi_\mu, \psi_\nu\}_D = -i\eta_{\mu\nu}$$

Quantization:

$$[\hat{A},\hat{B}] = i\{A,B\}_D : \qquad [\hat{\phi}_\mu, \hat{p}_\nu] = i\,\eta_{\mu\nu}, \quad \{\hat{\psi}_\mu, \hat{\psi}_\nu\} = \eta_{\mu\nu}$$
$$\hat{\phi}_\mu = x_\mu, \quad \hat{p}_\mu = -i\partial_\mu, \quad \hat{\psi}_\mu = \tfrac{1}{\sqrt{2}}\,\gamma_\mu$$

$\Psi = \Psi_a(x), \qquad \gamma^\mu \partial_\mu \Psi = 0, \qquad \partial^\mu \partial_\mu \Psi = 0$ Dirac field

Quantization 1D matter fields in 1D supergravity background $\Rightarrow$ spin $\frac{1}{2}$ target space field

Note: Fermionic string action: $S_{f\text{–}string} = \int d^2\sigma \mathcal{L}_{f\text{–}string}$,

$$\mathcal{L}_{f\text{–}string} = T\sqrt{g}\left\{ g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - \bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi_\mu - 2\bar{\chi}_\alpha \gamma^\beta \gamma^\alpha \psi^\mu \left(\partial_\beta X_\mu + \tfrac{1}{2}\,\bar{\chi}_\beta \psi_\mu\right)\right\}$$

$$\gamma^\alpha = e^\alpha_a \gamma^a, \qquad \alpha = 1,2, \qquad a = 1,2, \qquad \mu = 0,1,...,D-1$$

Lecture 3: $1D$ SUSY in superspace

- Superfields in superspace
- $1D$ supergravity in superspace
- Extended SUSY in superspace
- $N=2$ supersymmetric models
  - Real superfield
  - Chiral superfield

Superspace: Supersymmetry is realized by coordinate transformations

Q describes fermionic transformations $\longrightarrow$ translations in odd direction of extended space

Usual 1D space: (t) $\Rightarrow$

$N{=}1$, 1D superspace: (t,θ), where $\theta=\bar\theta$ is Grassmann coordinate, $\theta\theta\equiv 0$

$$Q=Q^+=\partial_\theta+i\,\theta\,\partial_t\,,\qquad H=H^+=i\,\partial_t\,;\qquad \{Q,Q\}=2\,H\,,\qquad [H,Q]=0$$

$$\delta t=\varepsilon Q\cdot t,\quad \delta\theta=\varepsilon Q\cdot\theta\ :\qquad \delta t=i\,\varepsilon\theta,\qquad \delta\theta=\varepsilon$$

$N{=}1,1D$ superfield : $\Phi(t,\theta)=\phi(t)+i\,\theta\psi(t)$

$$\Phi'(t',\theta')=\Phi(t,\theta),\quad \delta\Phi=\Phi(t',\theta')-\Phi(t,\theta)=\varepsilon Q\cdot\Phi=\delta\phi+i\,\theta\delta\psi\quad\Rightarrow$$

$$\delta\phi=i\,\varepsilon\psi,\quad \delta\psi=-\varepsilon\dot\phi$$

Integration over odd variable : $\int d\theta\, f(\theta)=\int d\theta\, f(\theta{+}\alpha)\quad\Rightarrow\quad \int d\theta\,\theta=1,\quad \int d\theta\,\alpha=0$

Covariant derivatives : $D_\theta=\partial_\theta-i\,\theta\,\partial_t\equiv D\,,\qquad D_t=\partial_t\,,\qquad \{Q,D\}=0\,,\quad [Q,\partial_t]=0$

$$S=\int dt\,d\theta\,\mathcal{L}(\Phi,\partial_t\Phi,D\Phi),\qquad \delta S=\int dt\,d\theta\,Q\,[\ldots]=\int \underbrace{dt\,d\theta\,\partial_\theta\,[\ldots]}_{=0,\,\partial_\theta[\ldots]\text{ contains no }\theta}+\int dt\,d\theta\,\underbrace{i\theta\partial_t\,[\ldots]}_{\text{the total derivative}}$$

Any action, built from superfields and covariant derivatives ∂_t and D, is always supersymmetric

Examples of the $N=1$ supermultiplets

$$\Phi(t,\theta) = \phi(t) + i\,\theta\psi(t) \qquad - \qquad \text{even superfield}$$

$$S = \tfrac{i}{2}\int dt\,d\theta\,\partial_t\Phi\,D\Phi = \tfrac{1}{2}\int dt\left(\dot{\phi}^2 + i\,\psi\dot{\psi}\right)$$

$(1,1,0)$ supermultiplet

$$\Psi(t,\theta) = \psi(t) + \theta F(t) \qquad - \qquad \text{odd superfield}$$

$$S = \tfrac{1}{2}\int dt\,d\theta\,\Psi\,D\Psi = \tfrac{1}{2}\int dt\left(i\,\psi\dot{\psi} + F^2\right) \quad \xrightarrow{F=0} \quad \tfrac{i}{2}\int dt\,\psi\dot{\psi}$$

$(0,1,1)$ supermultiplet

The supermultiplet $(m,n,n-m)$ contains $\begin{cases} m & \text{physical bosons} \\ n & \text{fermions} \\ n-m & \text{auxiliary bosons} \end{cases}$

Supergravity: local translations and local supertranslations $\Rightarrow$
general coordinate transformations in superspace

4D scalar matter in the curved space : $\sim \int d^4x \, \det(e^m_\mu)\, \eta^{mn} \underbrace{e^\mu_m \partial_\mu \phi}_{\mathcal{D}_m \phi} \cdot \underbrace{e^\nu_n \partial_\nu \phi}_{\mathcal{D}_n \phi}$

4D space $\rightarrow$ 1D superspace $\Downarrow$

$$S = \frac{i}{2} \int dt\, d\theta \, \mathrm{sdet}({E_M}^A)\, \underbrace{{E_{\underline{t}}}^M \partial_M \Phi}_{\mathcal{D}_{\underline{t}} \Phi} \, \underbrace{{E_{\underline{\theta}}}^N \partial_N \Phi}_{\mathcal{D}_{\underline{\theta}} \Phi}$$

${E_M}^A(t,\theta)$ — the super vierbein (supermatrix)

$M, N = (1,2) = (t,\theta)$ are curved indices; $A, B = (1,2) = (\underline{t},\underline{\theta})$ are flat indices

${E_A}^M(t,\theta)$ — the inverse super vierbein, ${E_A}^M {E_M}^B = \delta^B_A$

$\partial_M = (\partial_t, \partial_\theta)$ — curved derivatives

$$\left.\begin{array}{l} \mathcal{D}_{\underline{t}} = E_{\underline{t}}^M \partial_M = E_{\underline{t}}^t \partial_t + E_{\underline{t}}^\theta \partial_\theta \\ \mathcal{D}_{\underline{\theta}} = E_{\underline{\theta}}^N \partial_N = E_{\underline{\theta}}^t \partial_t + E_{\underline{\theta}}^\theta \partial_\theta \end{array}\right\} \quad \text{— covariant derivatives}$$

General coordinate transformations in superspace:

$$\delta E_M{}^A = \xi^N \partial_N E_M{}^A + \partial_M \xi^N E_N{}^A, \qquad \delta\Phi = \xi^M \partial_M \Phi, \qquad \xi^M(t,\theta) \ - \ \text{2 local parameters}$$
$$\delta E_A{}^M = \xi^N \partial_N E_A{}^M - E_A{}^N \partial_N \xi^M,$$

The extra symmetry which acts on tangent vectors:

$$\delta E_M{}^A = E_M{}^B \delta_B^t \alpha^A, \qquad \delta\,\mathrm{sdet}(E_M{}^A) = \alpha^t\,\mathrm{sdet}(E_M{}^A), \qquad \alpha^A(t,\theta) \ - \ \text{2 local parameters}$$
$$\delta E_A{}^M = -\delta_A^t \alpha^B E_B{}^M, \qquad \delta\Phi = 0,$$

Gauge fixing for 3 local transformations eliminates 3 from 4 superfields in $E_M{}^A$. Possible choice:

$$E_M{}^t = E \overset{o}{E}_M{}^t, \qquad E_M{}^\theta = E^{1/2} \overset{o}{E}_M{}^\theta, \qquad E(t,\theta) \ - \ \text{residual gauge superfield}$$

$$\overset{o}{E}_M{}^A = \begin{pmatrix} 1 & 0 \\ i\theta & 1 \end{pmatrix}, \qquad \overset{o}{E}_A{}^M = \begin{pmatrix} 1 & 0 \\ -i\theta & 1 \end{pmatrix}, \qquad - \ \text{flat vielbein}$$

$$\overset{o}{E}_A{}^M \partial_M = (\partial_t, D), \qquad E_A{}^M = \begin{pmatrix} E^{-1} & 0 \\ -i\theta E^{-1/2} & E^{-1/2} \end{pmatrix}, \qquad \mathrm{sdet}(E_M{}^A) = E^{1/2}$$

$$S = \frac{i}{2} \int dt\, d\theta\, E^{-1}\, \dot\Phi\, D\Phi$$

$$E(t,\theta) = e(t) - i\theta\chi(t), \qquad \Phi(t,\theta) = \phi(t) + i\theta\psi(t), \qquad S = \frac{1}{2}\int dt\, e^{-1}(\dot\phi^2 + i\,\psi\dot\psi - i\,e^{-1}\chi\dot\phi\psi)$$

Replacement $\psi \to e^{1/2}\psi$, $\chi \to e^{3/2}\chi$ yields the component action considered.
Residual gauge transformations coincide with local SUSY of the component action considered.

N-extended 1*D* superspace:

$$(t, \theta_i), \qquad \theta_k = (\overline{\theta_k}), \qquad \{\theta_i, \theta_k\} = 0, \qquad i, j, k = 1, ..., N$$

Realization of super-Poincare algebra in superspace:

$$Q_k = Q_k^+ = \frac{\partial}{\partial\theta_k} + i\,\theta_k \frac{\partial}{\partial t}\,, \qquad H = H^+ = i\,\partial_t\,; \qquad \{Q_k, Q_j\} = 2\,\delta_{kj}\,H\,, \qquad [H, Q_k] = 0$$

$$\delta t = \varepsilon_k Q_k \cdot t, \quad \delta\theta_k = \varepsilon_j Q_j \cdot \theta_k \;\; : \qquad \delta t = i\,\varepsilon_k\theta_k, \qquad \delta\theta_k = \varepsilon_k$$

General supersfield:

$$\Phi(t, \theta_k) = \phi(t) + \theta_k\psi_k(t) + \theta_{k_1}\theta_{k_2}\phi_{k_1k_2}(t) + \theta_{k_1}\theta_{k_2}\theta_{k_3}\psi_{k_1k_2k_3}(t) + \cdots + \theta_{k_1}\ldots\theta_{k_N}\phi_{k_1\ldots k_N}(t)$$

Off-shell contents:

$$\left.\begin{array}{l} 2^{N-1}\ \text{bosonic (fermionic) component fields}\ \phi, \phi_{k_1k_2}, \ldots \\ 2^{N-1}\ \text{fermionic (bosonic) component fields}\ \psi_{k_1}, \psi_{k_1k_2k_3}, \ldots \end{array}\right\} \quad \text{if}\ \Phi(t, \theta_k)\ \text{is bosonic (fermionic)}$$

Covariant derivatives:

$$D_k = \frac{\partial}{\partial\theta_k} - i\,\theta_k \frac{\partial}{\partial t}\,, \qquad \{Q_j, D_k\} = 0$$

$$F(D_k)\,\Phi = 0 \qquad - \qquad \text{covariant constraint}$$

On-shell (physical) contents of a model is defined by the action.

$N{=}2$ $1D$ supersymmetric models are similar to the models with $N{=}1$ $4D$ SUSY

Real $N{=}2$, $1D$ superspace: (t, θ_1, θ_2), $\theta_1 = \theta_1^+$, $\theta_2 = \theta_2^+$

$$Q_1 = \frac{\partial}{\partial \theta_1} + i\,\theta_1\,\partial_t\,, \quad Q_2 = \frac{\partial}{\partial \theta_2} + i\,\theta_2\,\partial_t\,, \qquad H = i\,\partial_t\,;$$

$$\{Q_1, Q_1\} = 2\,H, \qquad \{Q_2, Q_2\} = 2\,H, \qquad \{Q_1, Q_2\} = 0, \qquad [H, Q_1] = [H, Q_2] = 0$$

$$\delta t = i\,(\varepsilon_1\theta_1 + \varepsilon_2\theta_2), \qquad \delta\theta_1 = \varepsilon_1, \qquad \delta\theta_2 = \varepsilon_2$$

Complex $N{=}2$, $1D$ superspace:

$$(t, \theta, \bar\theta), \qquad \theta = \tfrac{1}{\sqrt{2}}\,(\theta_1 + i\,\theta_2),\ \bar\theta = \theta^+ = \tfrac{1}{\sqrt{2}}\,(\theta_1 - i\,\theta_2)$$

$$Q = \frac{\partial}{\partial \theta} + i\,\bar\theta\,\partial_t\,, \quad \bar Q = \frac{\partial}{\partial \bar\theta} + i\,\theta\,\partial_t\,, \qquad H = i\,\partial_t$$

$$\{Q, \bar Q\} = 2\,H, \qquad \{Q, Q\} = \{\bar Q, \bar Q\} = 0, \qquad [H, Q] = [H, \bar Q] = 0$$

$$\delta t = i\,(\varepsilon\bar\theta + \bar\varepsilon\theta), \qquad \delta\theta = \varepsilon, \qquad \delta\bar\theta = \bar\varepsilon, \qquad \bar\varepsilon = \varepsilon^+$$

General $N{=}2$, $1D$ superfield:

$$\Phi(t, \theta) = \phi(t) + \theta\psi(t) + \bar\theta\chi(t) + \theta\bar\theta F(t)$$

$$\delta\phi = \varepsilon\psi + \bar\varepsilon\chi, \qquad \delta\psi = -i\,\bar\varepsilon\dot\phi + \bar\varepsilon F, \qquad \delta\chi = -i\,\varepsilon\dot\phi - \varepsilon F, \qquad \delta F = -i\,\varepsilon\dot\psi + i\,\bar\varepsilon\dot\chi$$

Covariant derivatives : $D = \dfrac{\partial}{\partial \theta} - i\,\bar\theta\,\partial_t\,,\quad \bar D = \dfrac{\partial}{\partial \bar\theta} - i\,\theta\,\partial_t\,,\qquad \{D, Q\} = \{D, \bar Q\} = 0$

$\Phi^+ = \Phi$ – the real superfield; $\bar D\,\Phi = 0$ – the chiral superfield

Real superfield:

$$\Phi(t,\theta) = \Phi^{+} = \phi(t) + \theta\psi(t) - \bar\theta\bar\psi(t) + \theta\bar\theta F(t), \qquad \phi^{+} = \phi, \quad F^{+} = F, \quad \psi^{+} = \bar\psi$$

Off-shell SUSY transformations:

$$\delta\phi = \varepsilon\psi - \bar\varepsilon\bar\psi, \qquad \delta\psi = -i\,\bar\varepsilon\dot\phi + \bar\varepsilon F, \qquad \delta\bar\psi = i\,\varepsilon\dot\phi + \varepsilon F, \qquad \delta F = -i\,(\varepsilon\dot\psi + \bar\varepsilon\dot{\bar\psi})$$

$$S = \tfrac{i}{2}\int dt\, d\theta\, d\bar\theta\, \bar D\Phi\, D\Phi = \tfrac{1}{2}\int dt\,\left\{\dot\phi^2 + i\,(\psi\dot{\bar\psi} - \dot\psi\bar\psi) + F^2\right\}$$

$$\mathrm{On-shell:} \qquad \ddot\phi = 0, \qquad \dot\psi = 0, \qquad \dot{\bar\psi} = 0, \qquad F = 0 \qquad (1,2,1)\ \mathrm{multiplet}$$

On-shell action:

$$S = \tfrac{1}{2}\int dt\,\left\{\dot\phi^2 + i\,(\psi\dot{\bar\psi} - \dot\psi\bar\psi)\right\}$$

On-shell SUSY transformations:

$$\delta\phi = \varepsilon\psi - \bar\varepsilon\bar\psi, \qquad \delta\psi = -i\,\bar\varepsilon\dot\phi, \qquad \delta\bar\psi = i\,\varepsilon\dot\phi$$

$$[\delta_1,\delta_2]\psi = i\,(\varepsilon_1\bar\varepsilon_2 - \varepsilon_2\bar\varepsilon_1)\,\dot\psi - \underbrace{2i\,\bar\varepsilon_1\bar\varepsilon_2\dot{\bar\psi}}_{=0\ \mathrm{on-shell}}$$

On-shell SUSY transformations are closed only on equations of motion.

Chiral superfield:

$$\bar{D}\,\Phi = 0 \quad \to \quad \Phi(t,\theta) = \phi(t) + \theta\psi(t) - i\,\theta\bar{\theta}\dot{\phi}(t), \qquad \phi,\ \psi \quad - \quad \text{complex fields}$$

$(2, 2, 0)$ multiplet

$$\Phi(t,\theta) = \phi(t) + \theta\psi(t) - i\,\theta\bar{\theta}\dot{\phi}(t) = \phi(t_L) + \theta\psi(t_L) = \Phi(t_L,\theta)$$

Chiral $N{=}2$, $1D$ subspace:

$$(t_L,\theta), \qquad t_L \equiv t - i\,\theta\bar{\theta}$$

$$\delta t = i\,(\varepsilon\bar{\theta} + \bar{\varepsilon}\theta), \quad \delta\theta = \varepsilon, \quad \delta\bar{\theta} = \bar{\varepsilon} \quad \Rightarrow \quad \delta t_L = 2i\,\bar{\varepsilon}\theta, \qquad \delta\theta = \varepsilon$$

Supercharges in superspace $(t_L,\theta,\bar{\theta})$:

$$Q = \frac{\partial}{\partial\theta}\,, \qquad \bar{Q} = \frac{\partial}{\partial\bar{\theta}} + 2i\,\theta\,\partial_{t_L}$$

SUSY transformations of component fields:

$$\delta\phi = \varepsilon\psi\,, \qquad \delta\psi = -2i\,\bar{\varepsilon}\dot{\phi}$$

SUSY invariant action:

$$S = -\tfrac{1}{2}\int dt\,d\theta\,d\bar{\theta}\,\bar{D}\Phi\,\bar{D}\bar{\Phi} = \tfrac{1}{2}\int dt\,\left\{4\dot{\phi}\dot{\bar{\phi}} - i\,(\psi\dot{\bar{\psi}} - \dot{\psi}\bar{\psi})\right\}$$

Lecture 4: Superconformal mechanics

- Conformal mechanics: peculiarities
- Matrix models as multiparticle conformal mechanics
- $\mathcal{N}=2$ superconformal mechanics
- Brief sketch on $\mathcal{N}=4$ superconformal mechanics

Conformal mechanics action:

$$S = \frac{1}{2}\int dt\,\left(\dot{x}^2 - \frac{g}{x^2}\right)$$

Conformal invariance:

$$\delta t = a + b\,t + c\,t^2 \equiv f(t), \qquad \delta x = \tfrac{1}{2}\,\dot{f}\,x\,, \qquad \delta S = \int dt\,\dot{\Lambda}\,, \quad \Lambda = \tfrac{1}{2}\,\ddot{f}\,x^2$$

Conserved charges $\left(p = \dot{x}; \qquad \frac{d}{dt}\,(H\,\delta t - p\,\delta x + \Lambda) = 0\right)$:

$$\begin{aligned} H &= \tfrac{1}{2}\,\left(p^2 + \tfrac{g}{x^2}\right) \\ D &= tH - \tfrac{1}{2}\,xp \\ K &= t^2H - txp + \tfrac{1}{2}\,x^2 \end{aligned}$$

$$\frac{d}{dt}K = \frac{\partial}{\partial t}K + \{K,H\}_P = 0, \quad \frac{d}{dt}D = \frac{\partial}{\partial t}D + \{D,H\}_P = 0, \quad H \ - \ \text{the Hamiltonian}$$

$$\{H,D\}_P = H, \quad \{K,D\}_P = -K, \quad \{H,K\}_P = 2D \qquad - \qquad \text{dynamical symmetry}$$

$$[\mathbf{A},\mathbf{B}] = i\{A,B\}_P : \quad [\mathbf{H},\mathbf{D}] = i\,\mathbf{H}, \quad [\mathbf{K},\mathbf{D}] = -i\,\mathbf{K}, \quad [\mathbf{H},\mathbf{K}] = 2i\,\mathbf{D} \quad - \quad sl(2,\mathbb{R}) \text{ algebra}$$

Properties of the conformal mechanics:

- If $\mathbf{H}\,|E>=E\,|E>$, then $\mathbf{H}\,e^{i\alpha D}\,|E>=e^{2\alpha}\,E\,|E>$ $\Rightarrow$ the spectrum of **H** is continuous;
- The eigenspectrum of **H** includes all $E>0$ values, for each of which there exists a plane wave narmalizable state;
- The spectrum of **H** does not have an endpoint (ground state), the state with $E=0$ is not even plane wave normalizible.

It is awkward to describe the conformal theory in terms of **H** eigenstates.

The $sl(2,\mathbb{R})$ algebra in the Virasoro form:

$$\mathbf{R} = \tfrac{1}{2}\,(a\,\mathbf{H} + \frac{1}{a}\,\mathbf{K}), \qquad \mathbf{L}_{\pm} = -\tfrac{1}{2}\,(a\,\mathbf{H} - \frac{1}{a}\,\mathbf{K} \mp i\,\mathbf{D}); \qquad a \text{ is a parameter}$$

$$[\mathbf{R},\mathbf{L}_{\pm}] = \pm\,\mathbf{L}_{\pm}, \qquad [\mathbf{L}_{+},\mathbf{L}_{-}] = -2\,\mathbf{R}$$

R is the $u(1)$ generator in $sl(2,\mathbb{R}) \sim o(1,2)$ algebra.

The eigenvalues of

$$\mathbf{R}|_{t=0,\,a=1} = \tfrac{1}{2}\,\left(p^2 + \frac{g}{x^2} + x^2\right)$$

are given by a discrete series

$$r_n = r_0 + n, \quad n = 0,1,2,\ldots; \qquad r_0 = \tfrac{1}{2}\left(1 + \sqrt{g + \tfrac{1}{4}}\right)$$

- the hermitian $n\times n$-matrix field $X_a^b(t)$, $(\overline{X_a^b}) = X_b^a$,
- complex commuting $\mathrm{U}(n)$-spinor field $Z_a(t)$, $\bar{Z}^a = (\overline{Z_a})$,
- n^2 non–propagating "gauge fields" $A_a^b(t)$, $(\overline{A_a^b}) = A_b^a$.

$$S_0 = \int dt \left[\tfrac{1}{2}\,\mathrm{Tr}\,(\nabla X \nabla X) + \tfrac{i}{2}\,(\bar{Z}\nabla Z - \nabla\bar{Z} Z) + c\,\mathrm{Tr}A \right],$$

$$\nabla X = \dot{X} + i[A,X], \qquad \nabla Z = \dot{Z} + iAZ\ .$$

The 1D conformal $\mathrm{SO}(1,2)$ symmetry:

$$\delta t = f, \qquad \delta X_a^b = \tfrac{1}{2}\,\dot{f} X_a^b, \qquad \delta Z_a = 0, \qquad \delta A_a^b = -\dot{f} A_a^b, \qquad \partial_t^3 f = 0$$

The local $\mathrm{U}(n)$ symmetry, $g(\tau) \in \mathrm{U}(n)$:

$$X \to gXg^{\dagger}, \quad Z \to gZ, \quad A \to gAg^{\dagger} + i\dot{g}g^{\dagger}.$$

The $\mathrm{U}(n)$ gauge fixing : $\qquad X_a^b = x_a\delta_a^b, \qquad \bar{Z}^a = Z_a.$

The algebraic equations of motion

$$(Z_a)^2 = c \quad (\text{which implies } c > 0); \qquad A_a^b = \frac{Z_a Z_b}{2(x_a - x_b)^2}, \ {}_{a\neq b}$$

As result, we arrive at the standard Calogero action

$$S = \tfrac{1}{2} \int dt \left[\sum_a \dot{x}_a\dot{x}_a - \sum_{a\neq b} \frac{c^2}{(x_a - x_b)^2} \right], \qquad H = \tfrac{1}{2} \left[\sum_a p_a p_a + \sum_{a\neq b} \frac{c^2}{(x_a - x_b)^2} \right],$$

The $\mathcal{N}=2$ superconformal group OSp(2|2)∼SU(1,1|1)

$$\{Q,\bar Q\}=2H,\quad \{S,\bar S\}=2K,\quad \{Q,\bar S\}=2(D-U),\quad \{S,\bar Q\}=2(D+U),$$

$$i\left[P,\begin{pmatrix}S\\ \bar S\end{pmatrix}\right]=-\begin{pmatrix}Q\\ \bar Q\end{pmatrix},\qquad i\left[K,\begin{pmatrix}Q\\ \bar Q\end{pmatrix}\right]=\begin{pmatrix}S\\ \bar S\end{pmatrix},$$

$$i\left[D,\begin{pmatrix}Q\\ \bar Q\end{pmatrix}\right]=\frac12\begin{pmatrix}Q\\ \bar Q\end{pmatrix},\qquad i\left[D,\begin{pmatrix}S\\ \bar S\end{pmatrix}\right]=-\frac12\begin{pmatrix}S\\ \bar S\end{pmatrix},$$

$$i\left[U,\begin{pmatrix}Q\\ \bar Q\end{pmatrix}\right]=\frac12\begin{pmatrix}Q\\ -\bar Q\end{pmatrix},\qquad i\left[U,\begin{pmatrix}S\\ \bar S\end{pmatrix}\right]=-\frac12\begin{pmatrix}S\\ -\bar S\end{pmatrix}$$

The closure of S, $\bar S$ with Q, $\bar Q$ $\Rightarrow$ the full OSp(2|2).

We obtain the superconformal transformations by nonlinear realization method.

Coset realization of $N=2$ superspace:

$$\mathcal{G}=\{H,Q,\bar Q,U\},\qquad \mathcal{H}=\{U\},\qquad \mathcal{K}=\{H,Q,\bar Q\}$$

$$\mathcal{K}(t,\theta,\bar\theta)=e^{itH+\theta Q+\bar\theta\bar Q},\qquad t,\ \theta,\ \bar\theta \text{ are the coordinates on the coset}$$

$$e^{\varepsilon Q+\bar\varepsilon\bar Q}\,e^{itH+\theta Q+\bar\theta\bar Q}=e^{it'H+\theta' Q+\bar\theta'\bar Q}:\qquad \delta t=i(\varepsilon\bar\theta+\bar\varepsilon\theta),\quad \delta\theta=\varepsilon,\quad \delta\bar\theta=\bar\varepsilon$$

Note : $$e^A\,e^B=\exp\left\{A+B+\tfrac12[A,B]+\tfrac1{12}([A,[A,B]]+[[A,B],B])+\cdots\right\}$$

Coset realization of SU(1,1|1):

$$\mathcal{G}=\{H,D,K,Q,\bar{Q},S,\bar{S},U\},\qquad \mathcal{H}=\{U\},\qquad \mathcal{K}=\{H,D,K,Q,\bar{Q},S,\bar{S}\}$$

$$\mathcal{K}=e^{itH}\,e^{\theta Q+\bar\theta\bar Q}\,e^{iuD}\,e^{izK}\,e^{\zeta S+\bar\zeta\bar S}$$

$$e^{\varepsilon Q+\bar\varepsilon\bar Q}\,\mathcal{K}=\mathcal{K}'\mathcal{H},\qquad e^{\eta S+\bar\eta\bar S}\,\mathcal{K}=\mathcal{K}'\mathcal{H}$$

Note : $\quad e^{A}\,B\,e^{-A}=e^{A}\wedge B,\qquad 1\wedge B\equiv B,\quad A\wedge B\equiv[A,B],\quad A^2\wedge B\equiv[A,[A,B]],\quad\cdots$

$$\delta t=i(\varepsilon\bar\theta+\bar\varepsilon\theta),\qquad \delta\theta=\varepsilon,\qquad \delta\bar\theta=\bar\varepsilon;$$

$$\delta' t=i(\eta\bar\theta+\bar\eta\theta)t,\qquad \delta'\theta=\eta(t-i\theta\bar\theta),\qquad \delta'\bar\theta=\bar\eta(t+i\theta\bar\theta)$$

$$\delta'(dtd^2\theta)=0,\qquad \delta' D=-2i\,\eta\bar\theta\,D,\quad \delta'\bar D=-2i\,\bar\eta\theta\,\bar D$$

$$\mathcal{X}=x(t)+\theta\psi-\bar\theta\bar\psi(t)+\theta\bar\theta F(t),\qquad \delta'\mathcal{X}=i(\eta\bar\theta+\bar\eta\theta)\,\mathcal{X}$$

$$S=\int dtd^2\theta\left(\tfrac12 D\mathcal{X}\bar D\mathcal{X}+\gamma\ln\mathcal{X}\right)=\tfrac12\int dt\left\{\dot x^2+i\,(\psi\dot{\bar\psi}-\dot\psi\bar\psi)-\frac{\gamma^2+\gamma\psi\bar\psi}{x^2}\right\}$$

Multi-particle generalization (N=2 superconformal Calogero):

$$S=\int dt\,d^2\theta\left(\tfrac12\sum_a D\mathcal{X}_a\bar D\mathcal{X}_a+\gamma\sum_{a\neq b}\ln|\mathcal{X}_a-\mathcal{X}_b|\right)$$

The standard $\mathcal{N}=4$, 1D superspace:

$$\left\{ t, \theta_k, \bar\theta^k = (\theta_k)^+ \right\}, \qquad k = 1,2$$

Supersymmetry transformations from the $\mathcal{N}=4$, 1D superconformal group $D(2,1;\alpha)$:

$$\delta t = i(\theta_k \bar\varepsilon^k - \varepsilon_k \bar\theta^k), \qquad \delta\theta_k = \varepsilon_k, \qquad \delta\bar\theta^k = \bar\varepsilon^k;$$

$$\delta' t = -i(\eta_k \bar\theta^k - \bar\eta^k \theta_k)t + (1+2\alpha)\theta_j\bar\theta^j(\eta_k\bar\theta^k + \bar\eta^k\theta_k),$$

$$\delta'\theta_k = \eta_k t - 2i\alpha\theta_k\theta_j\bar\eta^j + 2i(1+\alpha)\theta_k\bar\theta^j\eta_j - i(1+2\alpha)\eta_k\theta_j\bar\theta^j$$

$$\text{Covariant derivatives}: \qquad D^k = \frac{\partial}{\partial\theta_k} + i\,\bar\theta^k\partial_t \qquad \bar D_k = \frac{\partial}{\partial\bar\theta^k} + i\,\theta^k\partial_t$$

Some types of the $\mathcal{N}=4$, 1D superfields:

- $D^k D_k \mathcal{X} = m$, $\bar D^k \bar D_k \mathcal{X} = m$, $[D^k, \bar D_k]\mathcal{X} = 0$ - scalar superfield, (1,4,3) multiplet
- $D^{(i}V^{jk)} = 0$, $\bar D^{(i}V^{jk)} = 0$ - vector superfield, (3,4,1)

Superconformal models ($\mathcal{X} = (V^{ik}V_{ik})^{1/2}$ for vector superfield):

$$S \sim \int dt\, d^4\theta\; \mathcal{X}^{-1/2} \qquad \text{for } \alpha \neq -1; \qquad S \sim \int dt\, d^4\theta\; \mathcal{X}\ln\mathcal{X} \qquad \text{for } \alpha = -1$$

$$\text{In components}: \qquad S \sim \int dt \left[\dot x^2 + i\,(\psi_k\dot{\bar\psi}^k - \dot\psi_k\bar\psi^k) - \frac{g + F(\psi,\bar\psi)}{x^2}\right]$$

More general formulations of $\mathcal{N}=4$, 1D models is achieved in harmonic superspace

Lecture 5: Supersymmetric models in harmonic superspace

- Harmonic superspace for $\mathcal{N}=4$, $1D$ SUSY models
- Harmonics and harmonic analysis
- Harmonic superfield models
- Multiparticle $\mathcal{N}=4$ superconformal models

$\mathcal{N}=4$, $1D$ SUSY algebra : $\quad \left\{ H,\ Q^k,\ \bar{Q}_k = (Q^k)^+,\ \overbrace{\underbrace{J^{(ik)}}_{su_L(2)},\ \underbrace{I^{(i'k')}}_{su_R(2)}}^{R-\text{symmetry}} \right\}, \quad i,k = 1,2$

Standard $\mathcal{N}=4$, $1D$ superspace : $\quad \left\{ H,\ Q^k,\ \bar{Q}_k = (Q^k)^+,\ J^{ik},\ I^{i'k'} \right\} / \left\{ J^{ik},\ I^{i'k'} \right\}$

Standard superspace coordinates : $\quad \left\{ t, \theta_k, \bar{\theta}^k = (\theta_k)^+ \right\}$

$su_L(2)$ algebra : $\quad J^{(ik)} = \left\{ J^{\pm}, J^0 \right\}, \qquad J^0 \ -\ u(1)$ generator

$\mathcal{N}=4$, $1D$ harmonic superspace : $\quad \left\{ H,\ Q^k,\ \bar{Q}_k = (Q^k)^+,\ J^{ik},\ I^{i'k'} \right\} / \left\{ J^0,\ I^{i'k'} \right\}$

Harmonic superspace coordinates : $\quad \left\{ t, \theta_k, \bar{\theta}^k, u_i^{\pm} \right\}$

Harmonic coordinates $u_i^{\pm}$ parametrize the sphere $S^2 \sim SU(2)/U(1)$

Parametrize $S^2 \sim SU(2)/U(1)$ by two $SU(2)$ spinors

$$u_i^{\pm}, \qquad u_i^- = (\overline{u^{+i}})$$

which subject to the constraint

$$u^{+i}u_i^- = 1 \quad \rightarrow \quad u_i^+ u_k^- - u_k^+ u_i^- = \epsilon_{ik}$$

and are defined up to a $U(1)$ phase transformations

$$u_i^+ \rightarrow e^{i\alpha} u_i^+, \qquad u_i^- \rightarrow e^{-i\alpha} u_i^-$$

$$||u|| = \begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{pmatrix} \in SU(2), \qquad ||u|| \rightarrow g\,||u||\,h\,, \quad g \in SU(2), \quad h \in U(1)$$

i, k are $SU(2)$ indices; $\pm$ are $U(1)$ charges

Any function on $S^2 \sim SU(2)/U(1)$ must have a definite $U(1)$ charge q

$$\Phi^{(q)}(u) = \sum_{n=0}^{\infty} \phi^{i_1 \ldots i_{n+q} j_1 \ldots j_n} u_{i_1}^+ \ldots u_{i_{n+q}}^+ u_{j_1}^- \ldots u_{j_n}^- \qquad \text{for} \quad n \geq 0$$

Harmonic functions are defined up to the transformations $\Phi^{(q)} \rightarrow e^{i\alpha q}\Phi^{(q)}$.

The use of such parametrization of S^2 has the advantage of manifest $SU(2)$ covariance

Covariant derivatives on the harmonic sphere S^2:

$$D^{\pm\pm} = u_i^{\pm} \frac{\partial}{\partial u_i^{\mp}} \equiv \partial^{\pm\pm}\,, \qquad D^0 = u_i^{+} \frac{\partial}{\partial u_i^{+}} - u_i^{-} \frac{\partial}{\partial u_i^{-}} \equiv \partial^0$$

$$[D^{++}, D^{--}] = D^0\,, \qquad [D^0, D^{\pm\pm}] = \pm 2\, D^{\pm\pm}$$

Harmonic fields satisfy

$$D^0\, \Phi^{(q)} = q\, \Phi^{(q)}$$

Harmonic integrals:

$$\int du\, u^+_{(i_1} \ldots u^+_{i_m} u^-_{j_1} \ldots u^-_{j_n)} = 0\,,$$

$$\int du\, 1 = 1\,,$$

$$\int du\, F^{(q)} = 0 \qquad \text{if} \quad q \neq 0$$

Central basis in harmonic superspace:

$$\left\{ t, \theta_k, \bar{\theta}^k, u_i^\pm \right\} \equiv \{ z, u \}$$

The $\mathcal{N}=4$, 1D Poincare supersymmetry:

$$\delta t = i(\theta_k \bar{\varepsilon}^k - \varepsilon_k \bar{\theta}^k), \qquad \delta\theta_k = \varepsilon_k, \qquad \delta\bar{\theta}^k = \bar{\varepsilon}^k, \qquad \delta u_i^\pm = 0$$

Analytic basis in harmonic superspace:

$$\{ t_A, \theta^\pm, \bar{\theta}^\pm, u_i^\pm \} \equiv \{ z_A, u \}, \qquad \theta^\pm = \theta^i u_i^\pm, \quad \bar{\theta}^\pm = \bar{\theta}^i u_i^\pm, \qquad t_A = t - i\left(\theta^+\bar{\theta}^- + \theta^-\bar{\theta}^+\right)$$

Analytic superspace

$$\{ t_A, \theta^+, \bar{\theta}^+, u_i^\pm \} \equiv \{ \zeta, u \}$$

is closed under $\mathcal{N}=4$ Poincare SUSY (and under $\mathcal{N}=4$ superconformal symmetry)

$$\delta t_A = -2i(\varepsilon^-\bar{\theta}^+ + \theta^+\bar{\varepsilon}^-), \qquad \delta\theta^+ = \varepsilon^+ = \varepsilon^i u_i^+, \qquad \delta\bar{\theta}^+ = \bar{\varepsilon}^+ = \bar{\varepsilon}^i u_i^+, \qquad \delta u_i^\pm = 0$$

Covariant derivatives $D^\pm = D^i u_i^\pm$, $\bar{D}^\pm = \bar{D}^i u_i^\pm$ in analytic basis:

$$D^+ = \frac{\partial}{\partial\theta^-}, \qquad \bar{D}^+ = -\frac{\partial}{\partial\bar{\theta}^-}, \qquad D^- = -\frac{\partial}{\partial\theta^+} + 2i\bar{\theta}^-\partial_A, \qquad \bar{D}^- = \frac{\partial}{\partial\bar{\theta}^+} + 2i\theta^-\partial_A$$

$$D^+\Psi(z,u) = \bar{D}^+\Psi(z,u) = 0 \qquad \Rightarrow \qquad \Psi = \Psi(\zeta, u)$$

Vector superfield (3,4,1) multiplet

$$D^+V^{++}=\bar D^+V^{++}=0,\qquad D^{++}V^{++}=0$$

Central basis:

$$D^{++}V^{++}=0\quad\Rightarrow\quad V^{++}=V^{ik}(z)u_i^+u_k^+$$
$$D^+V^{++}=\bar D^+V^{++}=0\quad\Rightarrow\quad D^{(i}V^{kl)}=\bar D^{(i}V^{kl)}=0$$

Analytic basis:

$$D^+V^{++}=\bar D^+V^{++}=0\quad\Rightarrow\quad V^{++}=V^{++}(\zeta,u)$$
$$D^{++}V^{++}=0\quad\Rightarrow\quad V^{++}=v^{ik}u_i^+u_k^++\theta^+\psi^iu_i^++\bar\theta^+\bar\psi^iu_i^++i\theta^+\bar\theta^+(F+2\dot v^{ik}u_i^+u_k^+)$$

$$S=\gamma\int dt\,d^4\theta\,du\,\mathcal{L}\left(V^{++},D^{--}V^{++},(D^{--})^2V^{++},u\right)$$
$$+\gamma'\int dt\,d\theta^+d\bar\theta^+\,du\,\mathcal{L}^{++}\left(V^{++},u\right)$$

first term $\Rightarrow\quad \gamma\int dt\,\mathcal{H}(v)\left(\dot v^{ik}\dot v_{ik}+F^2\right)$

second term $\Rightarrow\quad \gamma'\int dt\left\{F\mathcal{V}(v)+\dot v^{ik}\mathcal{A}_{ik}(v)\right\}$

$\partial_{ik}\mathcal{A}_{lt}-\partial_{lt}\mathcal{A}_{ik}=(\epsilon_{il}\partial_{kt}-\epsilon_{kt}\partial_{il})\mathcal{V}\quad -\quad$ monopole – like potential

Hypermultiplet (4,4,0) multiplet

$$D^+ q^+_a = \bar{D}^+ q^+_a = 0, \qquad D^{++} q^+_a = 0, \qquad (\widetilde{q^+_a}) = \epsilon^{ab} q^+_b\,, \qquad a,b = 1,2$$

Central basis:

$$D^{++} q^+_a = 0 \quad \Rightarrow \quad q^+_a = q^i_a(z) u^+_i$$
$$D^+ q^+_a = \bar{D}^+ q^+_a = 0 \quad \Rightarrow \quad D^{(i} q^{k)}_a = \bar{D}^{(i} q^{k)}_a = 0$$

Analytic basis:

$$D^+ q^+_a = \bar{D}^+ q^+_a = 0 \quad \Rightarrow \quad q^+_a = q^+_a(\zeta, u)$$
$$D^{++} q^+_a = 0 \quad \Rightarrow \quad q^+_a = f^i_a u^+_i + \theta^+ \chi_a + \bar\theta^+ \bar\chi_a + 2i\theta^+ \bar\theta^+ \dot f^i_a u^-_i$$

$$S = \gamma \int dt\, d^4\theta\, du\, \mathcal{L}\left(q^+_a, D^{--} q^+_a, u\right)$$
$$+\gamma' \int dt\, d\theta^+ d\bar\theta^+ du\, \mathcal{L}^{++}\left(q^+_a, u\right)$$

$$\text{first term} \quad \Rightarrow \quad \gamma \int dt\, G^{ab}(f)\, \dot f^i_a \dot f_{ib}$$
$$\text{second term} \quad \Rightarrow \quad \gamma' \int dt\, \dot f^{ia} \mathcal{A}_{ia}(f)$$
$$\mathcal{A}_{ia} \quad - \quad \text{self-dual gauge potential}$$

The $\mathcal{N}=4$ superconformal matrix model ($\mu_H = dudtd^4\theta,\ \mu_A^{(-2)} = dud\zeta^{(-2)}$):

$$S = -\frac{1}{2}\int \mu_H \mathrm{Tr}(\mathcal{X}^2) + \frac{1}{2}\int \mu_A^{(-2)} \mathcal{V}_0\, \widetilde{\mathcal{Z}}^+ \mathcal{Z}^+ + \frac{i}{2}\, c \int \mu_A^{(-2)}\, \mathrm{Tr}\, V^{++}\,,$$

Superfield contents:

- hermitian matrix superfields $\mathcal{X} = (\mathcal{X}_a^b)$:

$$\mathcal{D}^{++}\mathcal{X} = 0, \qquad \mathcal{D}^{+}\mathcal{D}^{-}\mathcal{X} = 0, \quad (\mathcal{D}^{+}\bar{\mathcal{D}}^{-} + \bar{\mathcal{D}}^{+}\mathcal{D}^{-})\,\mathcal{X} = 0;$$

- analytic superfields $\mathcal{Z}_a^+(\zeta, u)$: $\qquad \mathcal{D}^{++}\mathcal{Z}^+ = 0;$
- the gauge matrix connection $V^{++}(\zeta, u)$.

$$\mathcal{D}^{++} = D^{++} + i\,V^{++}, \quad \mathcal{D}^{++}\mathcal{X} = D^{++}\mathcal{X} + i\,[V^{++}, \mathcal{X}], \quad \text{etc.}$$

The superfield $\mathcal{V}_0(\zeta, u)$ is defined by the integral transform ($\mathcal{X}_0 \equiv \mathrm{Tr}\,(\mathcal{X})$)

$$\mathcal{X}_0(t, \theta_i, \bar{\theta}^i) = \int du\, \mathcal{V}_0\left(t_A, \theta^+, \bar{\theta}^+, u^\pm\right)\Big|_{\theta^\pm = \theta^i u_i^\pm,\ \bar{\theta}^\pm = \bar{\theta}^i u_i^\pm}.$$

Symmetries

- The $\mathcal{N}=4$ superconformal symmetry $D(2,1;\alpha)$ with $\alpha=-\frac{1}{2}\simeq \mathrm{OSp}(4|2)$:

$$\delta'\mathcal{X}=-\Lambda_0\,\mathcal{X},\ \ \delta'\mathcal{Z}^+=\Lambda\,\mathcal{Z}^+,\ \ \delta'V^{++}=0,\quad \Lambda=2i\alpha(\bar\eta^-\theta^+-\eta^-\bar\theta^+),\ \Lambda_0=2\Lambda-D^{--}D^{++}\Lambda$$

It is important that just the field multiplier $\mathcal{V}_0$ in the action provides this invariance.

- The local $\mathrm{U}(n)$ invariance:

$$\mathcal{X}'=e^{i\lambda}\mathcal{X}e^{-i\lambda},\qquad \mathcal{Z}^{+\prime}=e^{i\lambda}\mathcal{Z}^+,\qquad V^{++\prime}=e^{i\lambda}V^{++}e^{-i\lambda}-i\,e^{i\lambda}(D^{++}e^{-i\lambda}),$$

where $\lambda_a^b(\zeta,u^\pm)\in u(n)$ is the 'hermitian' analytic matrix parameter, $\widetilde{\lambda}=\lambda$.

Using gauge freedom we choose the WZ gauge: $\quad V^{++}=-2i\,\theta^+\bar\theta^+A(t_A)$.
In the WZ gauge: $\quad S_4=S_b+S_f$,

$$S_b=\int dt\left[\mathrm{Tr}\left(\nabla X\nabla X+c\,A\right)+\tfrac{i}{2}X_0\left(\bar Z_k\nabla Z^k-\nabla\bar Z_k Z^k\right)+\tfrac{n}{8}(\bar Z^{(i}Z^{k)})(\bar Z_i Z_k)\right],$$

$$S_f=-i\mathrm{Tr}\int dt\left(\bar\Psi_k\nabla\Psi^k-\nabla\bar\Psi_k\Psi^k\right)-\int dt\,\frac{\Psi_0^{(i}\bar\Psi_0^{k)}(\bar Z_i Z_k)}{X_0},$$

where $\quad \mathcal{X}=X(t_A)+\theta^-\Psi^i(t_A)u_i^++\bar\theta^-\bar\Psi^i(t_A)u_i^++\ldots,\qquad \mathcal{Z}^+=Z^i(t_A)u_i^++\ldots$
$X_0\equiv\mathrm{Tr}(X),\quad \Psi_0^i\equiv\mathrm{Tr}(\Psi^i),\quad \bar\Psi_0^i\equiv\mathrm{Tr}(\bar\Psi^i)$.

- imposing the gauge $X_a^b = 0$, $a \neq b$,
- eliminating A_a^b, $a \neq b$, by the equations of motion,
- introducing the new fields $Z'^i_a = (X_0)^{1/2} Z^i_a$ (omit the primes):

$$S_b = \int dt \Big\{ \sum_a \dot{x}_a \dot{x}_a + \tfrac{i}{2} \sum_a (\bar{Z}^a_k \dot{Z}^k_a - \dot{\bar{Z}}^a_k Z^k_a) + \sum_{a \neq b} \frac{\mathrm{Tr}(S_a S_b)}{4(x_a - x_b)^2} - \frac{n\,\mathrm{Tr}(\hat{S}\hat{S})}{2(X_0)^2} \Big\},$$

where $\quad (S_a)_i{}^j \equiv \bar{Z}^a_i Z^j_a, \quad (\hat{S})_i{}^j \equiv \sum_a \Big[(S_a)_i{}^j - \tfrac{1}{2}\delta_i^j (S_a)_k{}^k \Big]$.

The fields Z^k_a are subject to the constraints

$$\bar{Z}^a_i Z^i_a = c \qquad \forall\, a\,.$$

$$\tfrac{i}{2} \int dt \sum_a (\bar{Z}^a_k \dot{Z}^k_a - \dot{\bar{Z}}^a_k Z^k_a) \qquad \Rightarrow \qquad [\bar{Z}^a_i, Z^j_b]_D = i\delta^a_b \delta^j_i\,.$$

Thus the quantities S_a for each a form $u(2)$ algebras

$$[(S_a)_i{}^j, (S_b)_k{}^l]_D = i\delta_{ab} \left\{ \delta_i^l (S_a)_k{}^j - \delta_k^j (S_a)_i{}^l \right\}.$$

Modulo center-of-mass conformal potential, the bosonic limit

$$S'_b = \int dt \Big\{ \sum_a \dot{x}_a \dot{x}_a + \sum_{a \neq b} \frac{\mathrm{Tr}(S_a S_b)}{4(x_a - x_b)^2} \Big\}$$

is none other than the integrable U(2)-spin Calogero model

Bibliography

At the introductory level:

- J.Wess, J.Bagger, “Supersymmetry and Supergravity”
- P.West, “Introduction to Supersymmetry and Supergravity”
- M.Kaku, “Introduction to Superstrings”

At the comprehensive level:

- S.J.Gates, Jr., M.T.Grisaru, M.Rocek, W.Siegel, “Superspace or One Thousand and One Lessons in Supersymmetry”
- M.B.Green, J.H.Schwarz, E.Witten, “Superstring theory”
- A.S.Galperin, E.A.Ivanov, V.I Ogievetsky, E.S.Sokatchev “Harmonic superspace”