Heavy Quarks on the lattice

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Lectures at HISS school, September 2011

Plan

Introduction
- Naive HQ on the lattice

Continuum HQET
- Action
- Propagator
- Symmetries
- Renormalizability

Lattice HQET
- Action
- Propagator

Renormalization and matching
- Perturbative matching

Structure of the $1/M$ expansion
- Toy model
- HQET at order 1/m

Non-perturbative HQET
- Tests
- Non-perturbative matching
- Large volume

Some results

[arXiv:1008.0710]
and Les Houches school 2009
Oxford University Press
look there for more references

Introduction: Particle Physics

- **Observations** ($e, \mu, \dots Z, \dots t$, Lorenz invariance . . .)

  + Principles (Unitarity, Causality, Renormalizability)

  + theory calculations including lattice QCD (spectrum, F_π)

- **Standard Model of Particle Physics**

  local Quantum Field Theory (gauge theory)
  QED + Salam-Weinberg + QCD + GR

Introduction: the successfull Standard Model

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- very constrained: 3 coupling constants
- enormous predictivity

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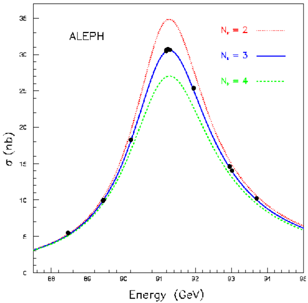

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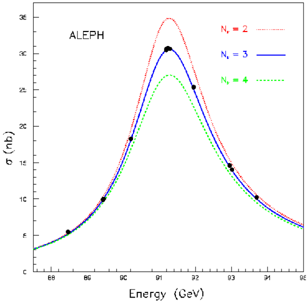

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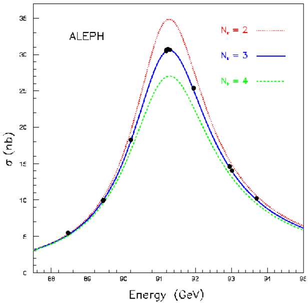
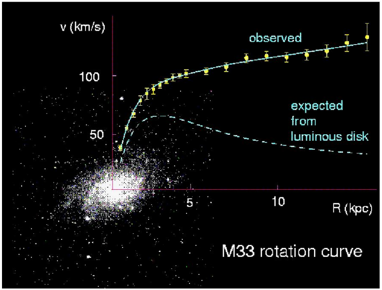


- too successfull (all particle physics experiments match)

Introduction: the incomplete Standard Model

But from other sources we know that there are missing pieces

- dark matter
- too little CP-violation for the observed matter / antimatter asymmetry

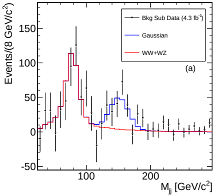


- There is an intense search for deviations from the Standard Model in particle physics experiments

Two Frontiers

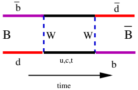
to search for missing pieces

- High Energy
  - Tevatron
  - LHC

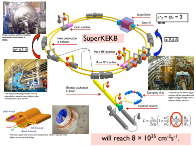


[CDF: arXiv:1104.0699]

- High Intensity
  - virtual (quantum) effects



  - less tested interactions



[Yutaka Ushiroda, May 2008]

High Intensity Frontier

Less tested interactions: quark-flavour changing interactions

$$\mathscr{L}_{\rm int} = \ldots g_{\rm weak} W^+_\mu \bar{U} \gamma_\mu (1-\gamma_5) D' \ldots$$

- B-decays

$$D' = \underbrace{\begin{pmatrix} \text{d'} \\ \text{s'} \\ \text{b'} \end{pmatrix}}_{\text{weak int.}} = V_{\rm CKM} \underbrace{\begin{pmatrix} \text{d} \\ \text{s} \\ \text{b} \end{pmatrix}}_{\text{strong int.}} = V_{\rm CKM}\, D$$

$$V_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & \color{red}{V_{ub}} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

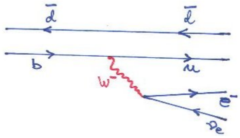
Confinement: V_{ij} are *not* directly measurable.
QCD matrix elements (or assumptions/approximations) are needed.

b to u transitions

- "clean" transitions: $B = b\bar{u} \to W \to l\nu$

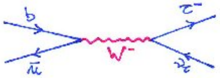
1. inclusive: $B \to X_u l\nu$
   optical theorem + heavy quark expansion
   $\to$ perturbatively calculable: (accuracy?)
   double expansion in $\alpha_s(m_\mathrm{b}) \approx 0.2$, $\Lambda_{QCD}/m_\mathrm{b} \approx 0.1$
2. semileptonic: $B \to \pi l\nu$

   (three-body, form factor)



3. leptonic: $B \to l\nu$

   (decay constant)



b to u transitions

- V_{ub} “puzzle”

b to u transitions

- $V_{\rm ub}$ "puzzle"

G. Isidori – Quark flavour mixing with right-handed currents *Euroflavour2010, Munich*

Motivation

Exp. side: RH currents provide a natural solution to the "V_{ub} puzzle"

$$B(B \to \pi\, l\nu) \propto |V_{ub}|^2$$
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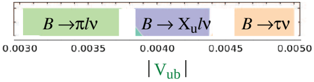
Within SM →



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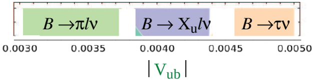

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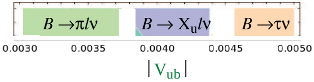

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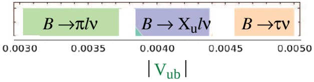
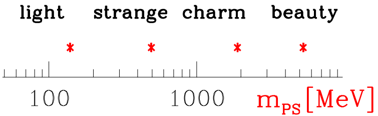


- Precise & reliable lattice calculations are needed to check whether such puzzles are for real or others are there.
- V_{ub} is one example. Others such as $B\bar{B}$ oscillations. . . .

The challenge of B-physics on the lattice

multiple scale problem
always difficult
for a numerical treatment



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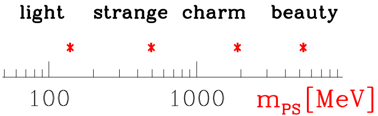

lattice cutoffs:

$$\Lambda_{\rm UV} = a^{-1}$$
$$\Lambda_{\rm IR} = L^{-1}$$

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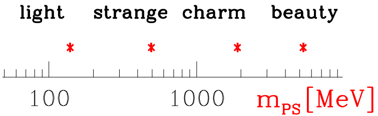

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$$L^{-1} \ll m_\pi, \ldots, m_{\mathrm{D}}, m_{\mathrm{B}} \ll a^{-1}$$

$$\mathrm{O}(\mathrm{e}^{-Lm_\pi}) \qquad\qquad m_{\mathrm{D}} a \lesssim 1/2$$

$$\downarrow \qquad\qquad \downarrow$$

$$L \gtrsim 4/m_\pi \sim 6\,\mathrm{fm} \qquad\qquad a \approx 0.05\,\mathrm{fm}$$

$$L/a \gtrsim 120$$

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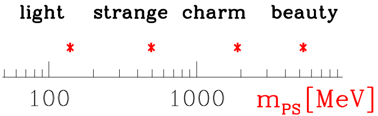

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beauty not yet accomodated: we'll discuss what to do

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$$E^2 = m^2 + \mathbf{p}^2$$

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$$E^2 = m^2 + \mathbf{p}^2 + \mathrm{O}(a^2)$$

$$\Uparrow$$

enhanced by $m \gg |\mathbf{p}|$

Dispersion relation

- Wilson fermion action $\quad [\,\partial_\mu^* f(x) = \frac{1}{a}(f(x) - f(x - a\hat{\mu}))\,]$

$$
\begin{aligned}
S_{\rm lat} &= a^4 \sum_x \overline{\psi}(x) \left\{ m_0 + \frac{\partial_\mu + \partial_\mu^*}{2}\gamma_\mu - a\frac{\partial_\mu^*\partial_\mu}{2} \right\} \psi(x) \\
&= \int_{-\pi/a}^{\pi/a} \mathrm{d}^4 p\, \tilde{\overline{\psi}}(p) \left\{ m_0 + i\tilde{p}_\mu\gamma_\mu + \tfrac{a}{2}\hat{p}^2 \right\} \tilde{\psi}(p)
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- propagator

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Dispersion relation

Transfer matrix representation for euclidean correlation functions, $\langle\, .\, \rangle$:

– generally

$$\langle O(x_0)O(0)\rangle \overset{x_0\to\infty}{\sim} B\mathrm{e}^{-Ex_0}$$

– here (free theory) expect

$$\begin{aligned} G(x_0,\mathbf{p}) &= \langle\tilde\psi_\alpha(x_0,\mathbf{p})\tilde{\overline{\psi}}_\beta(0,-\mathbf{p})\rangle \\ &= B_{\alpha\beta}\,\mathrm{e}^{-Ex_0}\,, \qquad \tilde\psi_\alpha(x_0,\mathbf{p}) = a^3\sum_{\mathbf{x}}\psi_\alpha(x)\mathrm{e}^{i\mathbf{px}} \end{aligned}$$

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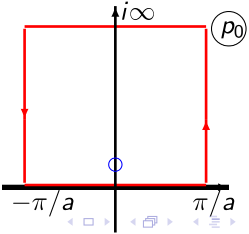

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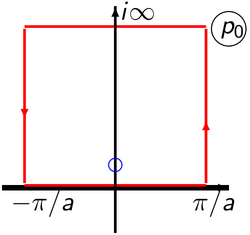

$$G_W(x_0,\mathbf{p}) = \int_{-\pi/a}^{\pi/a} \mathrm{d}p_0\, \mathrm{e}^{ip_0x_0}\; G_W(p)$$

$E(\mathbf{p})$ from $G_W(p)^{-1} = 0$: $\Downarrow$ **Exercise**

$$2\cosh(Ea) = \frac{1 + a^2\tilde{\mathbf{p}}^2}{A} + A\,, \quad A = 1 + am_0 + \tfrac{a^2}{2}\hat{\mathbf{p}}^2$$

Interpretation:

1. Renormalization (in the free theory!)

$\mathbf{p} = 0$: $\mathrm{e}^{aE} + \mathrm{e}^{-aE} = A + \frac{1}{A} \rightarrow E = \frac{1}{a}\log(1 + am_0) \equiv m_\mathrm{R}$

Now expand in a: $2\cosh(Ea) = 2 + a^2E^2 + \frac{1}{12}a^4E^4 + \ldots$

Dispersion relation

$\log(1 + am_0) \equiv m_{\mathrm{R}}$ and expand in a:

$2\cosh(Ea) = 2 + a^2 E^2 + \frac{1}{12} a^4 E^4 + \ldots$

find:

$$E^2 = \mathbf{p}^2 + m_{\mathrm{R}}^2 - \left(\tfrac{1}{3}\mathbf{p}^4 + \tfrac{2}{3} m_{\mathrm{R}}^2 \mathbf{p}^2 + \tfrac{1}{3}\sum_k p_k^4\right) a^2 + \mathrm{O}(a^4)$$

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cutoff effects are

- enhanced by large m_R
- $\mathrm{O}(a^2)$ (in the free Wilson theory automatically)
- break O(3) symmetry (not H(3))

Dispersion relation

Numerical example, relevant for B-physics:

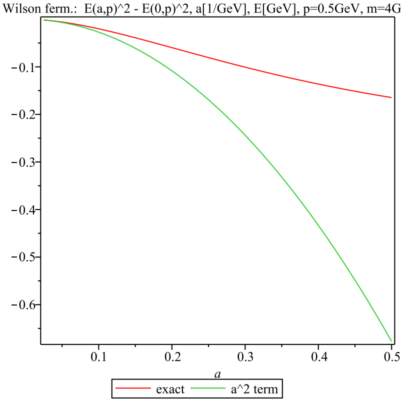
- Wilson discretization

$$|\mathbf{p}| = 0.5\mathrm{GeV}$$

$$a = \frac{1}{2\mathrm{GeV}} \cdots \frac{1}{4\mathrm{GeV}}$$

$$m_\mathrm{R} = 4\mathrm{GeV}$$

units: GeV



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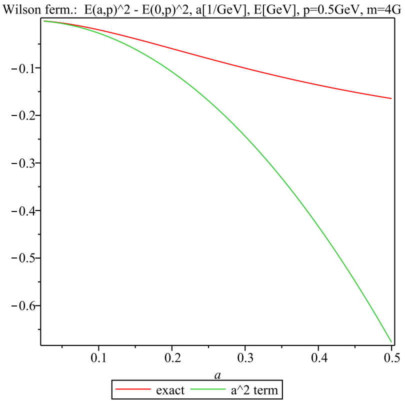

- asymptotics (a^2-behavior needs $am < 1/2$)
- $am = 1/2 \dots 1/4$ needed; therefore: **charm: yes**, **beauty: no**

Relativistic heavy quark action

[El Khadra, Kronfeld, Mackenzie; Aoki, Kuramashi; Christ, Li, Lin]

$$S_{\rm lat} = a^4 \sum_x \overline{\psi}(x) \left\{ m_0 + \frac{\partial_0 + \partial_0^*}{2}\gamma_0 + \xi \frac{\partial_k + \partial_k^*}{2}\gamma_k - a\frac{\partial_\mu^*\partial_\mu}{2} \right\} \psi(x)$$

$$\xi = \xi(am_0)$$

- $|\mathbf{p}| = 0.5\mathrm{GeV}$
  $a = \frac{1}{2\mathrm{GeV}} \ldots \frac{1}{4\mathrm{GeV}}$
  $m_{\rm R} = 4\mathrm{GeV}$
  $E^2 = \mathbf{p}^2 + m_{\rm R}^2$
  $-(\frac{1}{3}\mathbf{p}^4 + \frac{1}{3}\sum_k p_k^4)\, a^2$
  $+\mathrm{O}(a^4)$

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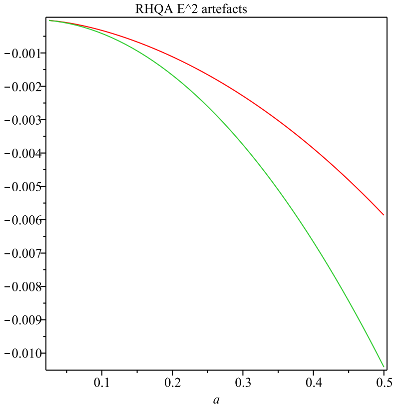

Options to do B-physics on the lattice

- "relativistic heavy quark actions"
- extrapolations in the quark mass
- effective theories: expansions in $\Lambda/m_{\rm b}$

  Heavy Quark Effective Theory

  Nonrelativistic QCD

Options to do B-physics on the lattice

- "relativistic heavy quark actions"
  consistent beyond tree-level?
  at the non-perturbative level?
- extrapolations in the quark mass
- effective theories: expansions in $\Lambda/m_{\rm b}$
  Heavy Quark Effective Theory
  Nonrelativistic QCD

Options to do B-physics on the lattice

- "relativistic heavy quark actions"
  consistent beyond tree-level?
  at the non-perturbative level?
- extrapolations in the quark mass
  need continuum limit before the extrapolation
- effective theories: expansions in $\Lambda/m_{\rm b}$

  Heavy Quark Effective Theory

  Nonrelativistic QCD

Plan

Introduction

- Naive HQ on the lattice

Continuum HQET

- Action
- Propagator
- Symmetries
- Renormalizability

Lattice HQET

- Action
- Propagator

Renormalization and matching

- Perturbative matching

Structure of the $1/M$ expansion

- Toy model
- HQET at order 1/m

Non-perturbative HQET

- Tests
- Non-perturbative matching
- Large volume

Some results

[arXiv:1008.0710]
and Les Houches school 2009
Oxford University Press
look there for more references

On continuum HQET

Want to describe hadrons with a single very heavy quark
e.g. a B-meson

like a hydrogen atom

hydrogen atom	:	heavy proton	+	light electron
B-meson	:	heavy b-quark	+	light anti-quark
b-baryons	:	heavy b-quark	+	two light quarks

...

$m_{\rm b} \to \infty$

- Rest-frame of $B \leftrightarrow$ rest-frame of b (quark)
- antiquarks can't be created

$$D_k \psi = 0$$

On continuum HQET

Dirac Lagrangian:

$$\overline{\psi}\{D_\mu\gamma_\mu + m\}\psi \overset{D_k\psi=0}{\longrightarrow} \overline{\psi}\{D_0\gamma_0 + m\}\psi = \mathscr{L}_{\rm h}^{\rm stat} + \underbrace{\mathscr{L}_{\bar{\rm h}}^{\rm stat}}_{\text{anti-quark}}$$

$$\mathscr{L}_{\rm h}^{\rm stat} = \overline{\psi}_{\rm h}(m + D_0)\psi_{\rm h}\,, \quad P_+\psi_{\rm h} = \psi_{\rm h}\,, \quad \overline{\psi}_{\rm h}P_+ = \overline{\psi}_{\rm h}\,, \quad P_\pm = \frac{1\pm\gamma_0}{2}$$

Corrections

by treating $D_k\gamma_k$ perturbatively: $D_k\psi \ll m\psi$

Couple quark and anti-quark fields.

"Deriving" the form of the continuum HQET Lagrangian

Decouple by Fouldy Wouthuysen-Tani (FTW) transformations

$$\begin{aligned} \psi &\rightarrow \psi' = \mathrm{e}^{S'}\mathrm{e}^{S}\psi\,, \quad S = \tfrac{1}{2m} D_k\gamma_k = -S^\dagger\,, S' = \frac{1}{4m^2}\gamma_0\gamma_k F_{k0} \\ \overline{\psi} &\rightarrow \overline{\psi}' = \overline{\psi}\mathrm{e}^{S}\mathrm{e}^{S'} \end{aligned}$$

... rename $\psi' \rightarrow \psi$

$$\begin{aligned} \mathscr{L} &= \mathscr{L}_{\mathrm{h}}^{\mathrm{stat}} + \tfrac{1}{2m}\mathscr{L}_{\mathrm{h}}^{(1)} + \mathscr{L}_{\bar{\mathrm{h}}}^{\mathrm{stat}} + \tfrac{1}{2m}\mathscr{L}_{\bar{\mathrm{h}}}^{(1)} + \mathrm{O}(\tfrac{1}{m^2}) \\ \mathscr{L}_{\mathrm{h}}^{(1)} &= -(\mathcal{O}_{\mathrm{kin}} + \mathcal{O}_{\mathrm{spin}})\,, \quad \mathscr{L}_{\bar{\mathrm{h}}}^{(1)} = -(\bar{\mathcal{O}}_{\mathrm{kin}} + \bar{\mathcal{O}}_{\mathrm{spin}})\,, \\ & \mathcal{O}_{\mathrm{kin}}(x) = \overline{\psi}_{\mathrm{h}}(x)\,\mathbf{D}^2\,\psi_{\mathrm{h}}(x)\,,\ \mathcal{O}_{\mathrm{spin}}(x) = \overline{\psi}_{\mathrm{h}}(x)\,\boldsymbol{\sigma}\cdot\mathbf{B}(x)\,\psi_{\mathrm{h}}(x)\,, \\ & \sigma_k = \tfrac{1}{2}\epsilon_{ijk}\sigma_{ij}\,, \quad B_k = i\tfrac{1}{2}\epsilon_{ijk}F_{ij}\,, \end{aligned}$$

Comments on the “Derivation”

- It is classical: in a path integral $D_k\psi \ll m\psi$ is not satisfied. All momentum components are integrated over.
- Have not “integrated out any components”
- The derivation is order by order in $1/m$
- We now have the classical effective Lagrangian.
  Its renormalization could need more terms.
  $\rightarrow$ to be discussed.

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- We now have the classical effective Lagrangian.
  Its renormalization could need more terms.
  $\rightarrow$ to be discussed.
- Lagrangian for a b-hadron at rest.
  $B \rightarrow l\nu, B \rightarrow \pi l\nu, B \leftrightarrow \bar{B}, \ldots$: ok

The static quark propagator

The propagator $G_{\rm h}(x,y)$ satisfies

$$(\partial_{x_0} + A_0(x) + m) G_{\rm h}(x,y) = \delta(x-y)\, P_+$$

Solution

$$\begin{aligned} G_{\rm h}(x,y) \quad &= \quad \theta(x_0 - y_0)\, \exp(-m\,(x_0 - y_0))\, \delta(\mathbf{x} - \mathbf{y})\, P_+ \\ &\cdot \mathcal{P} \exp\left\{ -\int_{y_0}^{x_0} \mathrm{d}z_0 A_0(z_0, \mathbf{x}) \right\} \end{aligned}$$

$\mathcal{P}$: path ordering

- explicit solution (check it as an exercise)
- $\delta(\mathbf{x} - \mathbf{y})$: static

Mass dependence

$$G_{\rm h}(x,y) \;=\; \theta(x_0-y_0)\,\exp(-m\,(x_0-y_0))\,\delta(\mathbf{x}-\mathbf{y})\,P_+ \cdot \mathcal{P}\exp\left\{-\int_{y_0}^{x_0}{\rm d}z_0\,A_0(z_0,\mathbf{x})\right\}$$

explicit factor $\exp(-m\,|x_0-y_0|)$ for any gauge field

also after path integration over the gauge fields

$$C_{\rm h}(x,y;m) = C_{\rm h}(x,y;0)\,\exp(-m\,(x_0-y_0))\,.$$

example:

$$C_{\rm h}^{\rm PP}(x,y;m) = \langle\overline{\psi}_{\rm l}(x)\gamma_5\psi_{\rm h}(x)\,\overline{\psi}_{\rm h}(y)\gamma_5\psi_{\rm l}(y)\rangle\,,$$

$\psi_{\rm l}(x)$: a light-quark fermion field

- remove m from Lagrangian $\qquad \mathscr{L}_{\rm h}^{\rm stat} = \overline{\psi}_{\rm h}(D_0+\epsilon)\psi_{\rm h}$

$$\mathscr{L}_{\bar{\rm h}}^{\rm stat} = \overline{\psi}_{\bar{\rm h}}(-D_0+\epsilon)\psi_{\bar{\rm h}}$$

- all energies are shifted by m $\qquad E_{{\rm h}/\bar{\rm h}}^{\rm QCD} = E_{{\rm h}/\bar{\rm h}}^{\rm stat} + m$

Heavy Quark Symmetries

1. Flavor

   F heavy quarks

$$\begin{aligned} \psi_\mathrm{h} &\rightarrow \psi_\mathrm{h} = (\psi_{\mathrm{h}1}, \ldots, \psi_{\mathrm{h}F})^T \\ \mathscr{L}_\mathrm{h}^\mathrm{stat} &= \overline{\psi}_\mathrm{h}(D_0 + \epsilon)\psi_\mathrm{h} \,. \end{aligned}$$

   symmetry

$$\psi_\mathrm{h}(x) \rightarrow V\,\psi_\mathrm{h}(x)\,, \quad \overline{\psi}_\mathrm{h}(x) \rightarrow \overline{\psi}_\mathrm{h}(x)V^\dagger\,, \qquad V \in \mathrm{SU}(F)$$

   emerges as $(F = 2)$

$$m_\mathrm{b} - m_\mathrm{c} = c \times \Lambda_\mathrm{QCD}\,, \quad \text{or} \quad m_\mathrm{b}/m_\mathrm{c} = c'\,, \quad m_\mathrm{b} \rightarrow \infty$$

   c or c' fixed when taking $m_\mathrm{b} \rightarrow \infty$

Heavy Quark Symmetries

2. Spin

$$\psi_{\rm h}(x) \to {\rm e}^{i\alpha_k\sigma_k}\,\psi_{\rm h}(x)\,, \qquad \overline{\psi}_{\rm h}(x) \to \overline{\psi}_{\rm h}(x){\rm e}^{-i\alpha_k\sigma_k}\,,$$

$$\sigma_k = \frac{1}{2}\epsilon_{ijk}\sigma_{ij} \equiv \begin{pmatrix}\sigma_k & 0 \\ 0 & \sigma_k\end{pmatrix}\,,$$

in Dirac representation where

$$\gamma_0 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\,,\ P_+ = \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}\,,\ P_- = \begin{pmatrix}0 & 0 \\ 0 & 1\end{pmatrix}\,.$$

Heavy Quark Symmetries

3. Local Flavor-number

$$\psi_{\rm h}(x) \to {\rm e}^{i\eta(\mathbf{x})}\,\psi_{\rm h}(x)\,, \qquad \overline{\psi}_{\rm h}(x) \to \overline{\psi}_{\rm h}(x){\rm e}^{-i\eta(\mathbf{x})}\,,$$

is a symmetry for any local phase $\eta(\mathbf{x})$.

For every point $\mathbf{x}$ there is a corresponding Noether charge

$$Q_{\rm h}(x) = \overline{\psi}_{\rm h}(x)\psi_{\rm h}(x)\,[= \overline{\psi}_{\rm h}(x)\gamma_0\psi_{\rm h}(x)\,]$$
$$\partial_0 Q_{\rm h}(x) = 0\ \forall x$$

$Q_{\rm h}(x)$: local (heavy) Flavor number

- All heavy quark symmetries are broken at order $1/m$.
  But it is essential to have them at the lowest order.

Summary first lecture

- B-physics is interesting for searching for deviations from the standard model
- $m_{\rm b} \ll a^{-1}$ impossible to realize for a while to come
- Expansion in $\Lambda/m_{\rm b} \sim 1/10$
- Lowest order Term

$$\mathscr{L}_{\rm h}^{\rm stat} = \overline{\psi}_{\rm h}(D_0 + \epsilon)\psi_{\rm h}\,, \quad P_+\psi_{\rm h} = \psi_{\rm h}$$

  $m_{\rm b}$ scale is removed

- Symmetries
  $SU(2)$ Flavor (for $m_{\rm b} \to \infty$ and $m_{\rm c} \to \infty$)
  spin symmetry
  local flavor number

Renormalizability of the static theory

$$\mathscr{L}_{\rm h}^{\rm stat} = \overline{\psi}_{\rm h}(D_0 + \epsilon)\psi_{\rm h}$$

- local Lagrangian with field

$$\mathcal{O}_1(x) = \overline{\psi}_{\rm h}(x) D_0 \psi_{\rm h}(x)\,, \quad [\mathcal{O}_1] = 4$$

- standard wisdom: renormalized by adding all local fields with $[\mathcal{O}_{\rm j}] \leq 4$

$$\mathcal{O}_2(x) = \overline{\psi}_{\rm h}(x)\psi_{\rm h}(x)\,, \quad [\mathcal{O}_2] = 3$$

  no other fields compatible with the symmetries
- complete renormalized Lagrangian

$$\begin{aligned} \mathscr{L}_{\rm h}^{\rm stat} &= \overline{\psi}_{\rm h}(D_0 + \delta m + \epsilon)\psi_{\rm h} \\ \delta m &= (e_1 g_0^2 + e_2 g_0^4 + \ldots)\,\Lambda_{\rm cut} \end{aligned}$$

  - δm given for massless light quarks
  - $1 \times \mathcal{O}_1(x)$ possible by choosing wave function renormalization
  - Energies of *any state* are

$$E_{\rm h/\bar{h}}^{\rm QCD} = E_{\rm h/\bar{h}}^{\rm stat}\Big|_{\delta m=0} + \delta m + m = E_{\rm h/\bar{h}}^{\rm stat}\Big|_{\delta m=0} + m_{\rm bare}$$

Renormalizability of the static theory

Note: none of this is proven (e.g. to all orders of PT), but

- worked out in PT so far
- NP tests (later)

Predictions (if charm is heavy enough)

$$E_{\mathrm{h}/\bar{\mathrm{h}}}^{\mathrm{QCD}} = E_{\mathrm{h}/\bar{\mathrm{h}}}^{\mathrm{stat}}\Big|_{\delta m=0} + \delta m + m_f = E_{\mathrm{h}/\bar{\mathrm{h}}}^{\mathrm{stat}}\Big|_{\delta m=0} + m_{\mathrm{bare}}^f$$

Considering different levels (e.g. radial excitations or different angular momentum:

$$E_{\mathrm{b}}' - E_{\mathrm{b}} = E_{\mathrm{c}}' - E_{\mathrm{c}} + \mathrm{O}(\Lambda_{\mathrm{QCD}}/m_{\mathrm{c}})$$

Also predictions for decays:

$B \to \tau\bar{\nu}_\tau$

$$\text{amplitude} \propto \langle 0|\bar{u}(x)\gamma_\mu\gamma_5 b(x)|B(p)\rangle = p_\mu \mathrm{e}^{ipx} F_{\mathrm{B}}$$

$$\mathbf{p} = 0 : \quad \langle 0|\bar{u}(0)\gamma_0\gamma_5 b(0)|B(p)\rangle = m_{\mathrm{B}} F_{\mathrm{B}}$$

Normalization of states, scaling of decay constants

relativistic normalization of states

$$\langle \mathbf{p}|\mathbf{p}'\rangle_{\rm rel} \quad = \quad (2\pi)^3\, 2E(\mathbf{p})\, \delta(\mathbf{p}-\mathbf{p}')\,.$$

The factor $E(\mathbf{p})$ introduces a spurious mass-dependence.

non-relativistic normalization is

$$\begin{aligned} \langle \mathbf{p}|\mathbf{p}'\rangle_{\rm NR} &\equiv \langle \mathbf{p}|\mathbf{p}'\rangle = 2\,(2\pi)^3\,\delta(\mathbf{p}-\mathbf{p}') \\ |\mathbf{p}\rangle_{\rm rel} &= \sqrt{E(\mathbf{p})}\,|\mathbf{p}\rangle\,. \end{aligned}$$

To lowest order in $1/m_{\rm b}$ the FTW transformation is trivial:

$$A_0^{\rm HQET}(x) = A_0^{\rm stat}(x) + {\rm O}(1/m_{\rm b})\,,\; A_0^{\rm stat}(x) = \bar{u}(x)\gamma_0\gamma_5\psi_{\rm h}(x)\,.$$

$$\rightarrow \qquad \langle 0|A_0^{\rm stat}(0)|B^-(\mathbf{p}=0)\rangle = \underbrace{\Phi^{\rm stat}}_{\rm mass-independent}$$

$$\Phi^{\rm stat} = m_{\rm B}^{-1/2}\, p_0\, F_{\rm B} = m_{\rm B}^{1/2} F_{\rm B} = m_{\rm D}^{1/2} F_{\rm D}$$

in the limit $m_{\rm b}\rightarrow\infty, m_{\rm c}\rightarrow\infty$; rather doubtful for charm.
also up to logarithmic corrections (see later)

Static action on the lattice

- no chiral symmetry for a static quark
- discretize à la Wilson (with $r = 1$)

$$D_0\gamma_0 \quad \rightarrow \quad \tfrac{1}{2}\{(\nabla_0 + \nabla_0^*)\gamma_0 - a\nabla_0^*\nabla_0\}\,,$$

$\left(\nabla_\mu^*\psi(x) = \frac{1}{a}[\psi(x) - \psi(x - a\hat{\mu})]\,,\ \ \nabla_\mu\psi(x) = \frac{1}{a}[\psi(x + a\hat{\mu}) - \psi(x)]\right)$

with $P_+\psi_{\rm h} = \psi_{\rm h}\,,\ P_-\psi_{\bar{\rm h}} = \psi_{\bar{\rm h}}$, get lattice identities

$$D_0\,\psi_{\rm h}(x) = \nabla_0^*\psi_{\rm h}(x)\,, \quad D_0\,\psi_{\bar{\rm h}}(x) = \nabla_0\psi_{\bar{\rm h}}(x)\,.$$

- convenient normalization factor $\rightarrow$

$$\begin{aligned}\mathscr{L}_{\rm h} &= \frac{1}{1 + a\delta m}\overline{\psi}_{\rm h}(x)[\nabla_0^* + \delta m]\psi_{\rm h}(x)\,,\\ \mathscr{L}_{\bar{\rm h}} &= \frac{1}{1 + a\delta m}\overline{\psi}_{\bar{\rm h}}(x)[-\nabla_0 + \delta m]\psi_{\bar{\rm h}}(x)\,.\end{aligned}$$

Static action on the lattice

The following points are worth noting.

- Formally, this is just a one-dimensional Wilson fermion replicated for all space points $\mathbf{x}$
- No doubler modes
- Positive hermitian transfer matrix for Wilson fermions can be taken over
- The choice of the backward derivative for the quark and the forward derivative for the anti-quark is selected by the Wilson term. Selects forward/backward propagation; an ϵ-prescription is not needed
- First written down by Eichten and Hill.
- **Preserves all the continuum heavy quark symmetries**

Propagator

$$\frac{1}{1+a\,\delta m}(\nabla_0^* + \delta m) G_{\rm h}(x,y) = \delta(x-y)P_+ \equiv a^{-4}\prod_\mu \delta_{\frac{x_\mu}{a}\frac{y_\mu}{a}} P_+ \,.$$

Writing

$$G_{\rm h}(x,y) = g(n_0,k_0;\mathbf{x})\delta(\mathbf{x}-\mathbf{y})P_+\,, \quad x_0 = an_0\,,\; y_0 = ak_0$$

simple recursion for $g(n_0+1,k_0;\mathbf{x})$ in terms of $g(n_0,k_0;\mathbf{x})$
solution

$$\begin{aligned} g(n_0,k_0;\mathbf{x}) &= \theta(n_0-k_0)(1+a\delta m)^{-(n_0-k_0)}\mathcal{P}(y,x;0)^\dagger\,, \\ \mathcal{P}(x,x;0) &= 1\,, \quad \mathcal{P}(x,y+a\hat{0};0) = \mathcal{P}(x,y;0)U(y,0)\,, \end{aligned}$$

where

$$\theta(n_0-k_0) = \begin{cases} 0 & n_0 < k_0 \\ 1 & n_0 \geq k_0\,. \end{cases}$$

$$\begin{aligned} G_{\rm h}(x,y) &= \theta(x_0-y_0)\,\delta(\mathbf{x}-\mathbf{y})\,\exp\left(-\widehat{\delta m}\,(x_0-y_0)\right)\,\mathcal{P}(y,x;0)^\dagger\,P_+\,, \\ &\widehat{\delta m} = \tfrac{1}{a}\ln(1+a\delta m)\,. \end{aligned}$$

Propagator

$$G_{\rm h}(x,y) \quad = \quad \theta(x_0-y_0)\,\delta(\mathbf{x}-\mathbf{y})\;\exp\left(-\,\widehat{\delta m}\,(x_0-y_0)\right)\,\mathcal{P}(y,x;0)^{\dagger}\;P_+\;,$$

$$\widehat{\delta m} = \tfrac{1}{a}\ln(1+a\delta m)\,.$$

- $\theta(0)=1$ for the lattice θ-function
- mass counter term δm just yields an energy shift

  on the lattice:

$$E^{\rm QCD}_{\rm h/\bar h} \quad = \quad E^{\rm stat}_{\rm h/\bar h}\Big|_{\delta m=0} + m_{\rm bare}\,, \qquad m_{\rm bare} = \widehat{\delta m} + m\,.$$

the split between δm and the finite m is convention dependent

Symanzik analysis of cutoff effects

I do not derive it here ... result:

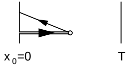
- automatic $\mathrm{O}(a)$ improvement of the action, energy levels $\mathrm{O}(a^2)$ cutoff effects
- $\mathrm{O}(a)$ improvement term for currents, eg.

$$\begin{aligned} A_0^{\mathrm{stat}} &= \overline{\psi}_{\mathrm{l}}\gamma_0\gamma_5\psi_{\mathrm{h}} \\ (A_{\mathrm{R}}^{\mathrm{stat}})_0 &= Z_{\mathrm{A}}^{\mathrm{stat}}\,(A_0^{\mathrm{stat}} + a c_{\mathrm{A}}^{\mathrm{stat}}(g_0)\,\delta A_0^{\mathrm{stat}}) \end{aligned}$$

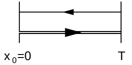
  irrespective of the light-quark action
- other (components) of currents, similarly

Numerical test of the renormalizability

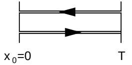
$$f_{\rm A}^{\rm stat}(x_0,\theta) = -\frac{a^6}{2}\sum_{\mathbf{y},\mathbf{z}} \left\langle (A_{\rm I}^{\rm stat})_0(x)\,\overline{\zeta}_{\rm h}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z})\right\rangle \quad :$$



$$f_1^{\rm stat}(\theta) = -\frac{a^{12}}{2L^6}\sum_{\mathbf{u},\mathbf{v},\mathbf{y},\mathbf{z}} \left\langle \overline{\zeta}_{\rm l}{}'(\mathbf{u})\gamma_5\zeta_{\rm h}{}'(\mathbf{v})\,\overline{\zeta}_{\rm h}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z})\right\rangle \quad :$$



$$f_1^{\rm hh}(x_3,\theta) = -\frac{a^8}{2L^2}\sum_{x_1,x_2,\mathbf{y},\mathbf{z}} \langle \overline{\zeta}_{\bar{\rm h}}{}'(\mathbf{x})\gamma_5\zeta_{\rm h}{}'(\mathbf{0})\,\overline{\zeta}_{\rm h}(\mathbf{y})\gamma_5\zeta_{\bar{\rm h}}(\mathbf{z})\rangle \quad :$$



In a Schrödinger functional .
Double lines are static quark propagators.
θ angle in spatial BC's.

Renormalization according to standard wisdom

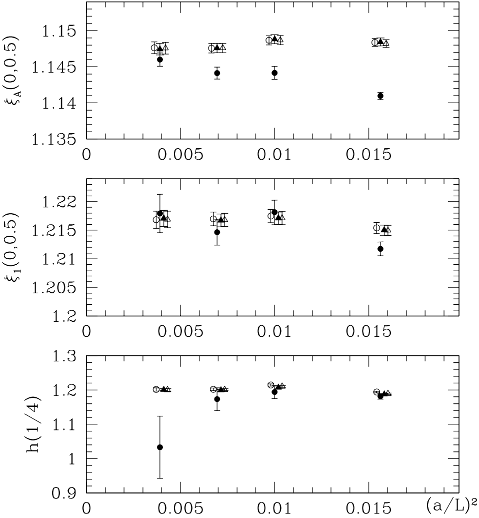
$$\left[f_{\rm A}^{\rm stat}\right]_{\rm R} = Z_{\rm A}^{\rm stat} Z_{\zeta_{\rm h}} Z_{\zeta}\; f_{\rm A}^{\rm stat}\,, \quad \left[f_1^{\rm stat}\right]_{\rm R} = Z_{\zeta_{\rm h}}^2 Z_{\zeta}^2\; f_1^{\rm stat}\,, \quad \left[f_1^{\rm hh}\right]_{\rm R} = Z_{\zeta_{\rm h}}^4\; f_1^{\rm hh}\,.$$

Numerical test of the renormalizability [Della Morte, Shindler, S., 2005]

$$\xi_{\rm A}(\theta,\theta') = \frac{f_{\rm A}^{\rm stat}(T/2,\theta)}{f_{\rm A}^{\rm stat}(T/2,\theta')}\,,$$

$$\xi_1(\theta,\theta') = \frac{f_1^{\rm stat}(\theta)}{f_1^{\rm stat}(\theta')}\,,$$

$$h(d/L,\theta) = \frac{f_1^{\rm hh}(d,\theta)}{f_1^{\rm hh}(L/2,\theta)}\,.$$



Beyond the classical theory: Renormalization and Matching

a matrix element of A_0:

QCD	HQET in static approx.
$Z_{\rm A}\,\langle f\vert A_0(x)\vert i\rangle_{\rm QCD}$	$Z_{\rm A}^{\rm stat}(\mu)\langle f\vert A_0^{\rm stat}(x)\vert i\rangle_{\rm stat}$
$\Phi^{\rm QCD}(m)$	$\Phi(\mu)$

- m: mass of heavy quark (b) in some definition (all other masses zero for simplicity)
- μ: arbitrary renormalization scale
- matching (equivalence):

$$\Phi^{\rm QCD}(m) = \widetilde{C}_{\rm match}(m,\mu) \times \Phi(\mu) + {\rm O}(1/m)$$

$$\widetilde{C}_{\rm match}(m,\mu) = 1 + \underbrace{c_1(m/\mu)}_{\gamma_0 \log(\mu/m)+{\rm const.}} \bar{g}^2(\mu) + \ldots$$

$$=$$

$M, \Lambda, \Phi_{\rm RGI}$: Renormalization Group Invariants

Beyond the classical theory: Renormalization and Matching

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- matching (equivalence):

$$\begin{aligned} \Phi^{\rm QCD}(m) &= \widetilde{C}_{\rm match}(m,\mu) \times \Phi(\mu) + {\rm O}(1/m) \\ \widetilde{C}_{\rm match}(m,\mu) &= 1 + \underbrace{c_1(m/\mu)}_{\gamma_0 \log(\mu/m) + {\rm const.}} \bar{g}^2(\mu) + \ldots \\ &= C_{\rm PS}(M/\Lambda)\,\Phi_{\rm RGI} \end{aligned}$$

$M, \Lambda, \Phi_{\rm RGI}$: Renormalization Group Invariants

The nature of the expansion

- QFT $\longrightarrow$ divergencies, $\log(M)$
- Strongly interacting:
  – non-perturbative in α – perturbative in $1/M$
- A **toy model** with the essential features

$$\Phi^{\rm QCD}(\beta) = \underbrace{C(M/\Lambda)} \quad \Phi_0(\beta) + \frac{c_1}{M}\Phi_1(\beta) + \frac{c_2}{M^2}\Phi_2(\beta) + \ldots$$

$\Phi^{\rm QCD}$: a (renormalized) observable
energy level
decay constant ($F_{\rm B}\sqrt{m_{\rm B}}$)
β: Quantum number (e.g. pseudo-scalar vector)
M: the mass of the quark (RGI)

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- QFT $\longrightarrow$ divergencies, $\log(M)$
- Strongly interacting:
  – non-perturbative in α – perturbative in $1/M$
- A **toy model** with the essential features

$$\Phi^{\rm QCD}(\beta) = \underbrace{C(M/\Lambda)}_{\frac{(\log(M/\Lambda))^r}{1+k[\log(M/\Lambda)]^{-1}}} \Phi_0(\beta) + \frac{c_1}{M}\Phi_1(\beta) + \frac{c_2}{M^2}\Phi_2(\beta) + \ldots$$

$\Phi^{\rm QCD}$: a (renormalized) observable
energy level
decay constant ($F_{\rm B}\sqrt{m_{\rm B}}$)
β: Quantum number (e.g. pseudo-scalar vector)
M: the mass of the quark (RGI)

- $C(M/\Lambda)$ not constant; not a naive expansion!

as just discussed:
– from renormalization of HQET and matching to QCD [Eichten, Hill]
– from QCD mass dependence at large masses [Shifmann, Voloshin; Politzer, Wise]

The nature of the expansion

$$\Phi^{\rm QCD}(\beta) = \Phi_0(\beta)\underbrace{\frac{(\log(M/\Lambda))^r}{1+k[\log(M/\Lambda)]^{-1}}}_{C(M/\Lambda)} + \Phi_1(\beta)\frac{c_1}{M} + \Phi_2(\beta)\frac{c_2}{M^2} + \ldots$$

$$[\log(M/\Lambda)]^{-1} \sim \alpha(M)$$

$$C(M/\Lambda) = (\log(M/\Lambda))^r\left[1+\sum_{l\geq 1}(-k)^l(\log(M/\Lambda))^{-l}\right]$$

$$= \alpha(M)^{-r}\left[1+\sum_{l\geq 1}(-k)^l\alpha(M)^l\right]$$

- simplified:
  – summable, finite radius of convergence (not in real QCD)
  – neglected log-corrections and mixing in M^{-n} terms
  – r given by anomalous dimension in HQET (γ_0)

The standard approach

- $[\log(M/\Lambda)]^{-1} \sim \alpha(M) \ll 1$
  "matching (computation of the coefficients) can be done in perturbation theory"
  " Wilson coefficients can be computed in perturbation theory"
- in our model this means

$$
\begin{aligned}
C(M/\Lambda) &= \alpha(M)^{-r}\left[1+\sum_{l=1}^{L}(-k)^l\alpha(M)^l\right] \pm \Delta[C(M/\Lambda)] \\
&= (\log(M/\Lambda))^r\left[1+\sum_{l=1}^{L}(-k)^l(\log(M/\Lambda))^{-l}\right] \pm \Delta[C(M/\Lambda)] \\
\text{error} &: \frac{\Delta[C(M/\Lambda)]}{C(M/\Lambda)} = \mathrm{O}\left(\alpha(M)^{L+1}\right)
\end{aligned}
$$

- **fine for leading term in $1/M$ if pertubation theory is well behaved**
- **but including M^{-1}-term is theoretically ill defined**

$$\log(M/\Lambda)^{-L-1} \gg \Lambda/M \quad \text{for } M \to \infty$$

The standard approach

- $[\log(M/\Lambda)]^{-1} \sim \alpha(M) \ll 1$
  "matching (computation of the coefficients) can be done in perturbation theory"
  " Wilson coefficients can be computed in perturbation theory"
- in our model this means

$$\begin{aligned}
C(M/\Lambda) &= \alpha(M)^{-r}\left[1+\sum_{l=1}^{L}(-k)^l\alpha(M)^l\right] \pm \Delta[C(M/\Lambda)] \\
&= (\log(M/\Lambda))^r\left[1+\sum_{l=1}^{L}(-k)^l(\log(M/\Lambda))^{-l}\right] \pm \Delta[C(M/\Lambda)] \\
\text{error} &: \frac{\Delta[C(M/\Lambda)]}{C(M/\Lambda)} = \mathrm{O}\left(\alpha(M)^{L+1}\right)
\end{aligned}$$

- **fine for leading term in $1/M$ if pertubation theory is well behaved**
- **but including M^{-1}-term is theoretically ill defined**

$$\log(M/\Lambda)^{-L-1} \gg \Lambda/M \quad \text{for } M \to \infty$$

- Need $C(M/\Lambda),\ c_1,\ c_2$ non-perturbatively

The standard approach

- $[\log(M/\Lambda)]^{-1} \sim \alpha(M) \ll 1$
  "matching (computation of the coefficients) can be done in perturbation theory"
  " Wilson coefficients can be computed in perturbation theory"
- in our model this means

$$\begin{aligned} C(M/\Lambda) &= \alpha(M)^{-r}\left[1+\sum_{l=1}^{L}(-k)^l\alpha(M)^l\right] \pm \Delta[C(M/\Lambda)] \\ &= (\log(M/\Lambda))^r\left[1+\sum_{l=1}^{L}(-k)^l(\log(M/\Lambda))^{-l}\right] \pm \Delta[C(M/\Lambda)] \\ \text{error} &: \frac{\Delta[C(M/\Lambda)]}{C(M/\Lambda)} = \mathrm{O}\left(\alpha(M)^{L+1}\right) \end{aligned}$$

- **fine for leading term in $1/M$ if pertubation theory is well behaved**
- **but including M^{-1}-term is theoretically ill defined**

$$\log(M/\Lambda)^{-L-1} \gg \Lambda/M \quad \text{for } M \to \infty$$

- Need $C(M/\Lambda)$, c_1, c_2 non-perturbatively
- Also matching QCD $\rightarrow$ HQET on the lattice

Including $1/m_\mathrm{b}$ corrections

($\mathrm{O}(1/m_\mathrm{b}^2)$ dropped without notice)

$$\mathscr{L}_\mathrm{h}^{(1)}(x) = -(\omega_\mathrm{kin}\,\mathcal{O}_\mathrm{kin}(x) + \omega_\mathrm{spin}\,\mathcal{O}_\mathrm{spin}(x))\,.$$

NRQCD path integral weight:

$$W_\mathrm{NRQCD} \quad\propto\quad \exp(-a^4\sum_x[\mathscr{L}_\mathrm{light}(x) + \mathscr{L}_\mathrm{h}^\mathrm{stat}(x) + \mathscr{L}_\mathrm{h}^{(1)}(x)])$$

non-renormalizable ($[\mathcal{O}_\mathrm{kin}] = [\mathcal{O}_\mathrm{spin}] = 5$), no continuum limit
the real trouble is not the effective theory, but that at the same time we want non-perturbative results: not a finite number of loops

HQET path integral weight:

$$W_\mathrm{HQET} \quad\equiv\quad \exp(-a^4\sum_x[\mathscr{L}_\mathrm{light}(x) + \mathscr{L}_\mathrm{h}^\mathrm{stat}(x)])\left\{1 - a^4\sum_x \mathscr{L}_\mathrm{h}^{(1)}(x)\right\}$$

part of the definition of HQET

Expanding correlation functions.

$$
\begin{aligned}
W_{\mathrm{HQET}} &\equiv \exp(-a^4 \sum_x [\mathscr{L}_{\mathrm{light}}(x) + \mathscr{L}_{\mathrm{h}}^{\mathrm{stat}}(x)]) \\
&\quad \times \left\{ 1 + a^4 \sum_x (\omega_{\mathrm{kin}}\, \mathcal{O}_{\mathrm{kin}}(x) + \omega_{\mathrm{spin}}\, \mathcal{O}_{\mathrm{spin}}(x)) \right\}
\end{aligned}
$$

This yields

$$
\begin{aligned}
\langle \mathcal{O} \rangle &= \langle \mathcal{O} \rangle_{\mathrm{stat}} + \omega_{\mathrm{kin}} a^4 \sum_x \langle \mathcal{O}\mathcal{O}_{\mathrm{kin}}(x) \rangle_{\mathrm{stat}} + \omega_{\mathrm{spin}} a^4 \sum_x \langle \mathcal{O}\mathcal{O}_{\mathrm{spin}}(x) \rangle_{\mathrm{stat}} \\
&\equiv \langle \mathcal{O} \rangle_{\mathrm{stat}} + \omega_{\mathrm{kin}} \langle \mathcal{O} \rangle_{\mathrm{kin}} + \omega_{\mathrm{spin}} \langle \mathcal{O} \rangle_{\mathrm{spin}} \,,
\end{aligned}
$$

with

$$
\langle \mathcal{O} \rangle_{\mathrm{stat}} = \frac{1}{\mathcal{Z}} \int_{\mathrm{fields}} \mathcal{O} \exp(-a^4 \sum_x [\mathscr{L}_{\mathrm{light}}(x) + \mathscr{L}_{\mathrm{h}}^{\mathrm{stat}}(x)])
$$

Expanding correlation functions.

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_{\rm stat} + \omega_{\rm kin} \langle \mathcal{O} \rangle_{\rm kin} + \omega_{\rm spin} \langle \mathcal{O} \rangle_{\rm spin} \,,$$

one more point: also fields in correlation functions need to be expanded:

$$\mathcal{O}_{\rm QCD} = A_0(x) A_0^\dagger(0)$$

$$A_0(x) \rightarrow A_0^{\rm HQET}(x) = Z_{\rm A}^{\rm HQET}\,[A_0^{\rm stat}(x) + \sum_{i=1}^{2} c_{\rm A}^{(i)} A_0^{(i)}(x)]\,,$$

$$A_0^{(1)}(x) = \overline{\psi}_{\rm l}(x) \frac{1}{2} \gamma_5 \gamma_i (\nabla_i^{\rm S} - \overleftarrow{\nabla}_i^{\rm S}) \psi_{\rm h}(x)\,, \quad A_0^{(2)}(x) = -\widetilde{\partial}_i\, A_i^{\rm stat}\,,$$

$$c_{\rm A}^{(i)} = {\rm O}(1/m) \quad [A_0^{(i)}(x)] = 4$$

symmetric derivatives:

$$\widetilde{\partial}_i = \tfrac{1}{2}(\partial_i + \partial_i^*)\,, \quad \overleftarrow{\nabla}_i^{\rm S} = \tfrac{1}{2}(\overleftarrow{\nabla}_i + \overleftarrow{\nabla}_i^*)\,, \quad \nabla_i^{\rm S} = \tfrac{1}{2}(\nabla_i + \nabla_i^*)\,.$$

Expanding correlation functions.

Example

$$C_{\rm AA,R}^{\rm QCD}(x_0) = Z_{\rm A}^2\, a^3 \sum_{\mathbf{x}} \left\langle A_0(x) A_0^\dagger(0) \right\rangle_{\rm QCD}$$

its HQET expansion

$$\begin{aligned} C_{\rm AA}^{\rm QCD}(x_0) &= {\rm e}^{-m\,x_0} (Z_{\rm A}^{\rm HQET})^2 \Big[C_{\rm AA}^{\rm stat}(x_0) + c_{\rm A}^{(1)}\, C_{\delta \rm AA}^{\rm stat}(x_0) \\ &\quad + \omega_{\rm kin}\, C_{\rm AA}^{\rm kin}(x_0) + \omega_{\rm spin}\, C_{\rm AA}^{\rm spin}(x_0) \Big] \\ &\equiv {\rm e}^{-m x_0} (Z_{\rm A}^{\rm HQET})^2\, C_{\rm AA}^{\rm stat}(x_0) \Big[1 + c_{\rm A}^{(1)}\, R_{\delta A}^{\rm stat}(x_0) \\ &\quad + \omega_{\rm kin}\, R_{\rm AA}^{\rm kin}(x_0) + \omega_{\rm spin}\, R_{\rm AA}^{\rm spin}(x_0) \Big] \end{aligned}$$

with

$$\begin{aligned} C_{\delta \rm AA}^{\rm stat}(x_0) &= a^3 \sum_{\mathbf{x}} \langle A_0^{\rm stat}(x) (A_0^{(1)}(0))^\dagger \rangle_{\rm stat} + a^3 \sum_{\mathbf{x}} \langle A_0^{(1)}(x) (A_0^{\rm stat}(0))^\dagger \rangle_{\rm stat}\,, \\ C_{\rm AA}^{\rm kin}(x_0) &= a^3 \sum_{\mathbf{x}} \langle A_0^{\rm stat}(x) (A_0^{\rm stat}(0))^\dagger \rangle_{\rm kin} \\ C_{\rm AA}^{\rm spin}(x_0) &= a^3 \sum_{\mathbf{x}} \langle A_0^{\rm stat}(x) (A_0^{\rm stat}(0))^\dagger \rangle_{\rm spin}\,. \end{aligned}$$

Expanding correlation functions.

$$C_{\rm AA}^{\rm QCD}(x_0) = {\rm e}^{-m\,x_0}(Z_{\rm A}^{\rm HQET})^2 \left[C_{\rm AA}^{\rm stat}(x_0) + c_{\rm A}^{(1)}\, C_{\delta{\rm AA}}^{\rm stat}(x_0) + \omega_{\rm kin}\, C_{\rm AA}^{\rm kin}(x_0) + \omega_{\rm spin}\, C_{\rm AA}^{\rm spin}(x_0)\right]$$

- parameters in the effective theory

$$(\omega_1,\ldots\omega_5) = (m_{\rm bare} = m+\delta m\,,\ \ln(Z_{\rm A}^{\rm HQET})\,,\ c_{\rm A}^{(1)}\,,\ \omega_{\rm kin}\,,\ \omega_{\rm spin})$$

$\omega_i = \omega_i(g_0, aM_{\rm b})$ bare parameters

- renormalization
  all terms needed for the renormalization of the correlation functions with insertions of $\mathcal{O}_{\rm kin}, \mathcal{O}_{\rm spin}$ are present in the expression
  ω_i are the necessary free coefficients
  keep $M_{\rm b}$ fixed change $g_0 \to 0$, $a \to 0$: all divergences (logarithmic and power) absorbed in ω_i
  a more detailed explanation in [R.S., arXiv:1008.0710]
- matching
  finite parts of the ω_i by matching to QCD (more precisiely later)

$$\Phi_i^{\rm HQET}(\{\omega_i\}) = \Phi_i^{\rm QCD}(M_{\rm b})$$

Expansion of energies...

$$
\begin{aligned}
m_{\rm B} &= -\lim_{x_0\to\infty} \widetilde{\partial}_0 \ln C_{\rm AA}^{\rm QCD}(x_0) \\
&= m_{\rm bare} - \lim_{x_0\to\infty} \widetilde{\partial}_0 \Big[\ln C_{\rm AA}^{\rm stat}(x_0) + c_{\rm A}^{(1)} R_{\delta A}^{\rm stat}(x_0) + \\
&\qquad + \omega_{\rm kin} R_{\rm AA}^{\rm kin}(x_0) + \omega_{\rm spin} R_{\rm AA}^{\rm spin}(x_0) \Big]_{\delta m=0} \\
&= m_{\rm bare} + E^{\rm stat} + \omega_{\rm kin} E^{\rm kin} + \omega_{\rm spin} E^{\rm spin} ,
\end{aligned}
$$

Expansion of energies...

$$
\begin{aligned}
m_{\rm B} &= -\lim_{x_0\to\infty} \widetilde{\partial}_0 \ln C_{\rm AA}^{\rm QCD}(x_0) \\
&= m_{\rm bare} - \lim_{x_0\to\infty} \widetilde{\partial}_0 \Big[\ln C_{\rm AA}^{\rm stat}(x_0) + c_{\rm A}^{(1)}\, R_{\delta A}^{\rm stat}(x_0) + \\
&\qquad + \omega_{\rm kin}\, R_{\rm AA}^{\rm kin}(x_0) + \omega_{\rm spin}\, R_{\rm AA}^{\rm spin}(x_0) \Big]_{\delta m=0} \\
&= m_{\rm bare} + E^{\rm stat} + \omega_{\rm kin} E^{\rm kin} + \omega_{\rm spin} E^{\rm spin} \,,
\end{aligned}
$$

$$
\begin{aligned}
E^{\rm stat} &= -\lim_{x_0\to\infty} \widetilde{\partial}_0 \ln C_{\rm AA}^{\rm stat}(x_0)\Big|_{\delta m=0} \,, \\
E^{\rm kin} &= -\lim_{x_0\to\infty} \widetilde{\partial}_0 R_{\rm AA}^{\rm kin}(x_0) \\
&= -\frac{1}{2L^3} \langle B|a^3 \sum_{\mathbf{z}} \mathcal{O}_{\rm kin}(0,\mathbf{z})|B\rangle_{\rm stat} = -\frac{1}{2} \langle B|\mathcal{O}_{\rm kin}(0)|B\rangle_{\rm stat} \\
E^{\rm spin} &= -\lim_{x_0\to\infty} \widetilde{\partial}_0 R_{\rm AA}^{\rm spin}(x_0) = -\frac{1}{2} \langle B|\mathcal{O}_{\rm spin}(0)|B\rangle_{\rm stat} \,, \\
0 &= \lim_{x_0\to\infty} \widetilde{\partial}_0 R_{\delta A}^{\rm stat}(x_0) \,.
\end{aligned}
$$

... and matrix elements

$$
\begin{aligned}
F_{\rm B}\sqrt{m_{\rm B}} &= \lim_{x_0\to\infty} \left\{2\exp(m_{\rm B}x_0)\, C_{\rm AA}^{\rm QCD}(x_0)\right\}^{1/2} \\
&= Z_{\rm A}^{\rm HQET}\,\Phi^{\rm stat} \lim_{x_0\to\infty} \left\{1 + \tfrac{1}{2}x_0\left[\omega_{\rm kin}E^{\rm kin} + \omega_{\rm spin}E^{\rm spin}\right]\right. \\
&\quad \left. + \tfrac{1}{2}c_{\rm A}^{(1)} R_{\delta A}^{\rm stat}(x_0) + \tfrac{1}{2}\omega_{\rm kin} R_{\rm AA}^{\rm kin}(x_0) + \tfrac{1}{2}\omega_{\rm spin} R_{\rm AA}^{\rm spin}(x_0)\right\}, \\
\Phi^{\rm stat} &= \lim_{x_0\to\infty} \left\{2\exp(E^{\rm stat}x_0)\, C_{\rm AA}^{\rm stat}(x_0)\right\}^{1/2}.
\end{aligned}
$$

... and matrix elements

$$\begin{aligned}
F_{\rm B}\sqrt{m_{\rm B}} &= \lim_{x_0\to\infty} \left\{2\exp(m_{\rm B}x_0)\, C_{\rm AA}^{\rm QCD}(x_0)\right\}^{1/2} \\
&= Z_{\rm A}^{\rm HQET}\,\Phi^{\rm stat} \lim_{x_0\to\infty} \Big\{1 + \tfrac{1}{2}x_0\left[\omega_{\rm kin}E^{\rm kin} + \omega_{\rm spin}E^{\rm spin}\right] \\
&\quad + \tfrac{1}{2}c_{\rm A}^{(1)} R_{\delta A}^{\rm stat}(x_0) + \tfrac{1}{2}\omega_{\rm kin} R_{\rm AA}^{\rm kin}(x_0) + \tfrac{1}{2}\omega_{\rm spin} R_{\rm AA}^{\rm spin}(x_0)\Big\}\,, \\
\Phi^{\rm stat} &= \lim_{x_0\to\infty} \left\{2\exp(E^{\rm stat}x_0)\, C_{\rm AA}^{\rm stat}(x_0)\right\}^{1/2}.
\end{aligned}$$

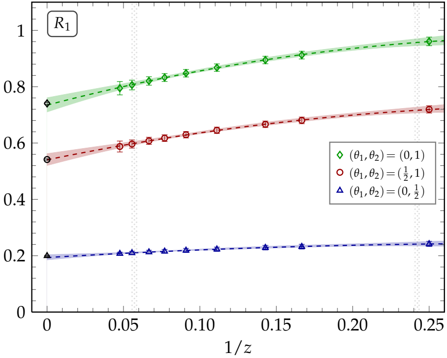
always a strict expansion

$$\begin{aligned}
(c_{\rm A}^{(1)}, \omega_{\rm kin}, \omega_{\rm spin}) &= {\rm O}(1/m_{\rm b}) \\
Z_{\rm A}^{\rm HQET}\omega_{\rm kin} &\equiv Z_{\rm A}^{\rm stat}\omega_{\rm kin} \\
&\neq (Z_{\rm A}^{\rm stat} + Z_{\rm A}^{(1/m)})\,\omega_{\rm kin}
\end{aligned}$$

Tests of HQET [Fritzsch, Jüttner, Heitger, S., Wennekers]

Example: SF boundary-to-boundary correlators

$$R_1 \quad = \quad \tfrac{1}{4}\left(\ln\left(\frac{f_1(\theta_1)k_1(\theta_1)^3}{f_1(\theta_2)k_1(\theta_2)^3}\right)\right) \qquad z = LM$$

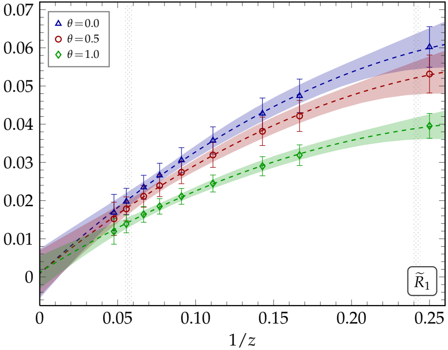


spin averaged

Tests of HQET [Fritzsch, Jüttner, Heitger, S., Wennekers]

Example: SF boundary-to-boundary correlators

$$\widetilde{R}_1 \quad = \quad \frac{3}{4}\ln\left(\frac{f_1}{k_1}\right) \propto \omega_{\rm spin} \qquad z = LM$$



spin symmetry violating

Non-perturbative determination of parameters [ALPHA Collaboration ,2001 - 2010]

static parameters

$$\omega^{\rm stat} \quad = \quad (\, m_{\rm bare}^{\rm stat} \,,\, [\ln(Z_{\rm A})]^{\rm stat} \,)^t \,, \quad N_{\rm HQET} = 2$$

parameters at first order

$$\begin{aligned} \omega^{\rm HQET} &= (\, m_{\rm bare} \,,\, \ln(Z_{\rm A}^{\rm HQET}) \,,\, c_{\rm A}^{(1)} \,,\, \omega_{\rm kin} \,,\, \omega_{\rm spin} \,)^t \quad N_{\rm HQET} = 5 \\ \omega^{(1/m)} &= \omega^{\rm HQET} - \omega^{\rm stat} \end{aligned}$$

matching: $L_1 \approx 0.5\,\mathrm{fm}$

$$\Phi_i(L_1, M, a) = \Phi_i^{\rm QCD}(L_1, M, 0)\,,\ i = 1 \ldots N_{\rm HQET}\,.$$

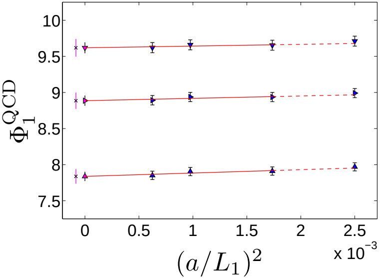
Why $L_1 = 0.5\mathrm{fm}$?

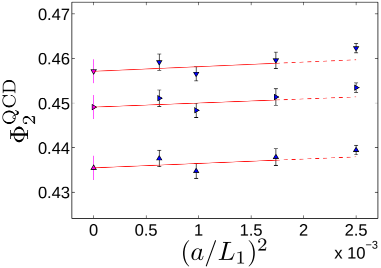
- $a = 0.012 \ldots 0.025\mathrm{fm}$ is accessible
  b-quark can be simulated, continuum limit can be taken

Non-perturbative determination of parameters [ALPHA Collaboration ,2001 - 2010]

$L_1 = 0.5\text{fm}$: $a = 0.012 \ldots 0.025\text{fm}$:

b-quark can be simulated, continuum limit can be taken





$\Phi_1 = L\, m_{\mathrm{B}}(L_1) = \mathrm{O}(z)$

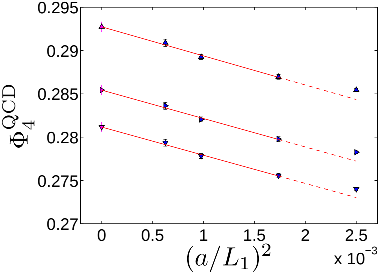
$\Phi_2 = \ln(L_1^{3/2}\,[F_{\mathrm{B}}\sqrt{m_{\mathrm{B}}}](L_1) = \mathrm{O}(z^0)$

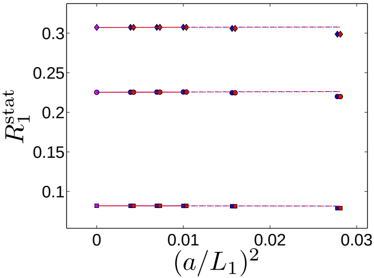
three different z

Non-perturbative determination of parameters [ALPHA Collaboration ,2001 - 2010]

$L_1 = 0.5\text{fm}$: $a = 0.012 \ldots 0.025\text{fm}$:

b-quark can be simulated, continuum limit can be taken





$$\Phi_4 = \mathrm{O}(1) = R_1^{\rm stat} + \omega_{\rm kin}\Phi_4^{(1/m)}$$

three different z three different θ

Non-perturbative determination of parameters [ALPHA Collaboration ,2001 - 2010]

$$\Phi_i(L_1, M, a) = \Phi_i^{\rm QCD}(L_1, M, 0)\,,\; i = 1 \ldots N_{\rm HQET}\,.$$

natural:

$$\begin{aligned}\Phi_1 &= L\Gamma^{\rm P} \equiv -L\widetilde{\partial}_0 \ln(-f_{\rm A}(x_0))_{x_0=L/2} \overset{L\to\infty}{\sim} Lm_{\rm B} \\ \Phi_2 &= \ln(Z_{\rm A}\frac{-f_{\rm A}}{\sqrt{f_1}}) \overset{L\to\infty}{\sim} \ln(L^{3/2}F_{\rm B}\sqrt{m_{\rm B}/2})\,,\end{aligned}$$

HQET expansion

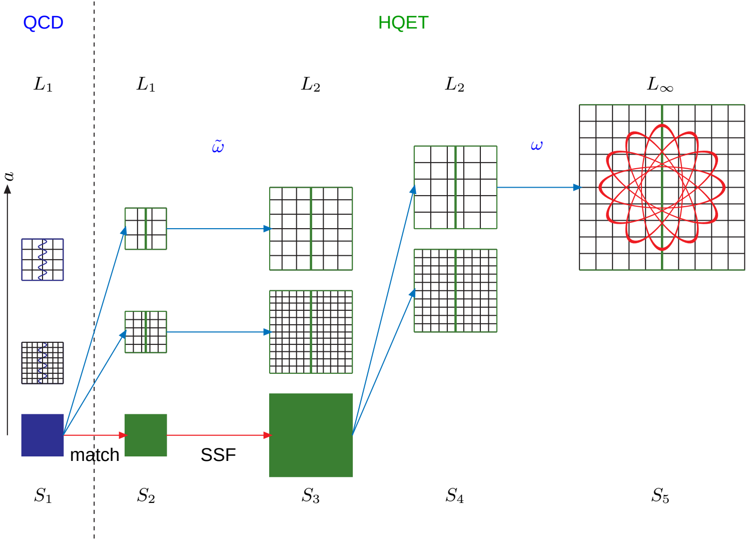
$$\begin{aligned}\Phi_1 &= L\,[m_{\rm bare} + \Gamma^{\rm stat}] + {\rm O}(1/m_{\rm b}) \\ \Phi_2 &= \ln(Z_{\rm A}^{\rm stat}) + \zeta_{\rm A} + {\rm O}(1/m_{\rm b})\end{aligned}$$

in general

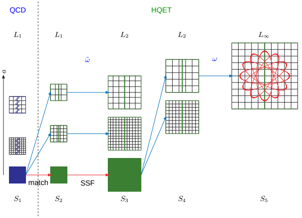
$$\begin{aligned}\Phi(L, M, a) &= \eta(L, a) + \phi(L, a)\,\omega(M, a) \\ \eta &= \begin{pmatrix}\Gamma^{\rm stat} \\ \zeta_{\rm A} \\ \ldots\end{pmatrix}, \quad \phi = \begin{pmatrix} L & 0 & \ldots \\ 0 & 1 & \ldots \\ \ldots & & \end{pmatrix}\end{aligned}$$

Full strategy to determine $\omega(M_{\rm b}, a)$, $a = 0.05\mathrm{fm} \ldots 0.1\mathrm{fm}$

(in



Full strategy (1-2a)



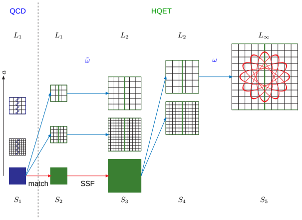
(1) continuum limit in QCD $a = 0.025\,\mathrm{fm} \ldots 0.012\,\mathrm{fm}$

$$\Phi_i^{\mathrm{QCD}}(L_1, M, 0) = \lim_{a/L_1 \to 0} \Phi_i^{\mathrm{QCD}}(L_1, M, a)\,,$$

(2a) HQET parameters $a = 0.05\,\mathrm{fm} \ldots 0.025\,\mathrm{fm}$

$$\begin{aligned} \tilde{\omega}(M, a) &\equiv \phi^{-1}(L_1, a)\,[\Phi(L_1, M, 0) - \eta(L_1, a)] \\ &= \begin{pmatrix} L_1^{-1}\Phi_1(L_1, M, 0) - \Gamma^{\mathrm{stat}}(L_1, a) \\ \Phi_2(L_1, M, 0) - \zeta_{\mathrm{A}}(L_1, a) \\ \ldots \end{pmatrix} \\ L_1/a &\gg 1\,, \quad aM_{\mathrm{b}} \text{ irrelevant} \end{aligned}$$

Full strategy



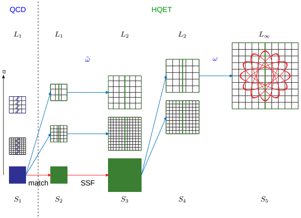
(2b) step scaling to $L_2 = 2L_1$ $a = 0.05\,\mathrm{fm}\ldots 0.025\,\mathrm{fm}$
Insert $\tilde\omega$ into $\Phi(L_2, M, a)$

$$\Phi(L_2, M, 0) = \lim_{a/L_2\to 0} \{\eta(L_2,a) + \phi(L_2,a)\,\tilde\omega(M,a)\}$$

$$= \lim_{a/L_2\to 0} \begin{pmatrix} L_2\Gamma^{\rm stat}(L_2,a) + \frac{L_2}{L_1}\Phi_1(L_1,M,0) - L_2\Gamma^{\rm stat}(L_1,a) \\ \zeta_{\rm A}(L_2,a) + \Phi_2(L_1,M,0) - \zeta_{\rm A}(L_1,a) \\ \ldots \end{pmatrix}$$

$$= \underbrace{\lim_{a/L_2\to 0} \begin{pmatrix} L_2[\Gamma^{\rm stat}(L_2,a) - \Gamma^{\rm stat}(L_1,a)] \\ \zeta_{\rm A}(L_2,a) - \zeta_{\rm A}(L_1,a) \\ \ldots \end{pmatrix}}_{\text{finite HQET SSF's}} + \underbrace{\begin{pmatrix} \frac{L_2}{L_1}\Phi_1(L_1,M,0) \\ \Phi_2(L_1,M,0) \\ \ldots \end{pmatrix}}_{\text{QCD, mass dependence}} .$$

Full strategy



(3.) Repeat (2a.) for $L_1 \longrightarrow L_2$:

$$\omega(M,a) \equiv \phi^{-1}(L_2,a)\,[\Phi(L_2,M,0) - \eta(L_2,a)]\,.$$

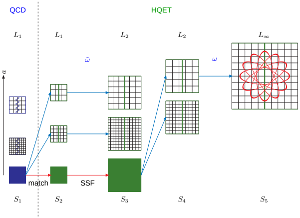
With same resolutions $L_2/a = 10\ldots 20$ now $a = 0.1\,\mathrm{fm}\ldots 0.05\,\mathrm{fm}$

(4.) insert ω into the expansion of large volume observables, e.g.

$$m_\mathrm{B} = \quad \omega_1 + E^\mathrm{stat}$$

from above: $m_\mathrm{B} = m_\mathrm{B}(M_\mathrm{b}) \longrightarrow$ determine M_b

Full strategy



$m_{\rm B} = m_{\rm B}(M_{\rm b}) \longrightarrow$ determine $M_{\rm b}$

explicitly in static approximation

$$m_{\rm B} =$$

$$\lim_{a\to 0}[E^{\rm stat} - \Gamma^{\rm stat}(L_2, a)] \qquad a = 0.1{\rm fm}\ldots 0.05{\rm fm} \qquad [S_4, S_5]$$

$$+ \lim_{a\to 0}[\Gamma^{\rm stat}(L_2, a) - \Gamma^{\rm stat}(L_1, a)] \qquad a = 0.05{\rm fm}\ldots 0.025{\rm fm} \qquad [S_2, S_3]$$

$$+ \frac{1}{L_1}\lim_{a\to 0}\Phi_1(L_1, M_{\rm b}, a) \qquad a = 0.025\,{\rm fm}\ldots 0.012\,{\rm fm} \qquad [S_1]\ .$$

Large volume

statistical and **systematic** precision

- statistical [Della Morte, Shindler, S., 2005; Hasenfratz & Knechtli, 2001]
  self energy: signal/noise becomes worse as $a \longrightarrow 0$
  change action: HYP-smearing

Large volume

statistical and **systematic** precision

- statistical [Della Morte, Shindler, S., 2005; Hasenfratz & Knechtli, 2001]
  self energy: signal/noise becomes worse as $a \to 0$
  change action: HYP-smearing
- systematic

$$F_{\rm B}\sqrt{m_{\rm B}} \quad = \quad \lim_{x_0\to\infty} \{2\exp(m_{\rm B}x_0)\, C_{\rm AA}^{\rm QCD}(x_0)\}^{1/2}$$

excited state contaminations typically non-negligible at $a \approx 0.7$fm

a common problem, e.g. nucleon structure

- the GEVP helps:
  an operator $\hat{\mathcal{Q}}_n^{\rm eff}$ can be constructed such that (rigorous [Blossier, Della Morte, von Hippel, Mendes & S., 2009]))

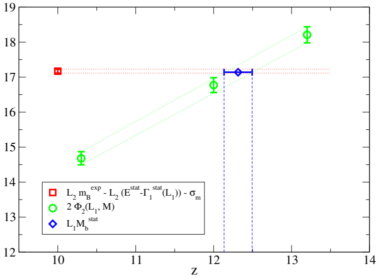
$$\begin{aligned}\langle 0|\hat{\mathcal{Q}}_n^{\rm eff}{\rm e}^{-\hat{H}t}\hat{P}{\rm e}^{-\hat{H}t}(\hat{\mathcal{Q}}_{n'}^{\rm eff})^\dagger|0\rangle &= \langle \mathcal{Q}_n^{\rm eff}(2t)P(t)(\mathcal{Q}_{n'}^{\rm eff}(0))^*\rangle \\ &= \langle n|\hat{P}|n'\rangle + {\rm O}({\rm e}^{-[E_{N+1}-E_n]\,t_0})\end{aligned}$$

e.g. $t = t_0 + a$

Examples of results: M_b [Blossier, Della Morte, Garron, Mendes, Papinutto, Simma, S.]

static approximation

$$
\begin{aligned}
m_\mathrm{B} =\; & \\
& \lim_{a\to 0}[E^\mathrm{stat} - \Gamma^\mathrm{stat}(L_2, a)] && a = 0.1\mathrm{fm} \ldots 0.05\mathrm{fm} && [S_4, S_5] \\
& + \lim_{a\to 0}[\Gamma^\mathrm{stat}(L_2, a) - \Gamma^\mathrm{stat}(L_1, a)] && a = 0.05\mathrm{fm} \ldots 0.025\mathrm{fm} && [S_2, S_3] \\
& + \frac{1}{L_1} \lim_{a\to 0} \Phi_1(L_1, M_\mathrm{b}, a) && a = 0.025\,\mathrm{fm} \ldots 0.012\,\mathrm{fm} && [S_1]\,.
\end{aligned}
$$



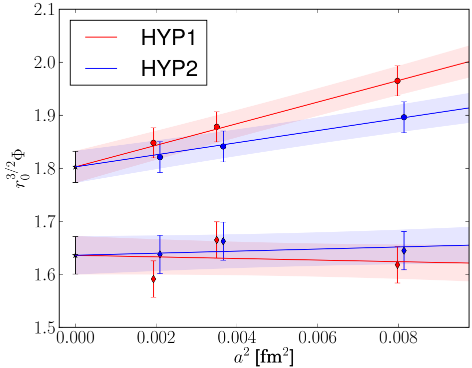
Examples of results: $M_{\rm b}$

	LO (static)	NLO (static + $\mathrm{O}(1/m)$)		
		$(\theta_1, \theta_2) = (0, 0.5)$	$(\theta_1, \theta_2) = (0.5, 1)$	$(\theta_1, \theta_2) = (0, 1)$
$\theta_0 = 0$	17.1 ± 0.2	17.1 ± 0.2	17.1 ± 0.2	17.1 ± 0.2
$\theta_0 = 0.5$	17.2 ± 0.2	17.2 ± 0.2	17.2 ± 0.2	17.1 ± 0.2
$\theta_0 = 1$	17.2 ± 0.2	17.3 ± 0.3	17.3 ± 0.3	17.3 ± 0.3

Table: Dimensionless b-quark mass, $r_0 M_{\rm b}$, obtained from the $B_{\rm s}$ meson mass, for different values of θ_i.

- small $1/m_{\rm b}$ corrections
- weak dependence on matching conditions

Examples of results: quenched $F_{\mathrm{B_s}}\sqrt{m_{\mathrm{B_s}}}$



static limit Φ_{RGI}

with $1/m_{\mathrm{b}}$: $\Phi^{\mathrm{HQET}} = F_{\mathrm{B_s}}\sqrt{m_{\mathrm{B_s}}}/C_{\mathrm{PS}}$

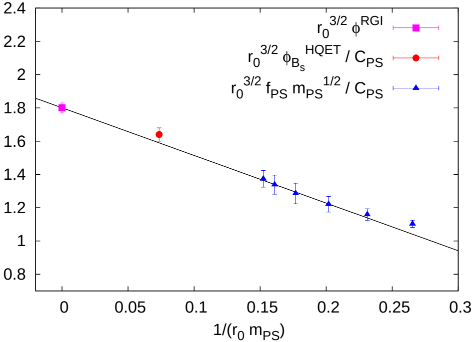
Examples of results: quenched $F_{\mathrm{B_s}}\sqrt{m_{\mathrm{B_s}}}$

	LO (static)	NLO (static + $\mathrm{O}(1/m)$)		
		$(\theta_1,\theta_2)=(0,0.5)$	$(\theta_1,\theta_2)=(0.5,1)$	$(\theta_1,\theta_2)=(0,1)$
$\theta_0=0$	233 ± 6	220 ± 9	218 ± 9	218 ± 9
$\theta_0=0.5$	229 ± 7	221 ± 9	219 ± 8	219 ± 9
$\theta_0=1$	219 ± 6	223 ± 9	221 ± 8	222 ± 8

Table: Pseudo-scalar heavy-light decay constant $f_{\mathrm{B_s}}$ in MeV, for different values of θ_i.

- small $1/m_{\mathrm{b}}$ corrections
- weak dependence on matching conditions

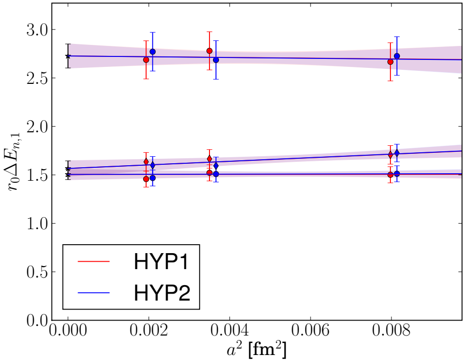
Comparison to relativistic (not so) heavy quarks



surprisingly consistent picture

$C_{\rm PS}$ inserted from perturbation theory (unclear theoretical status)

Examples of results: quenched level splittings



3s – 1s splitting static

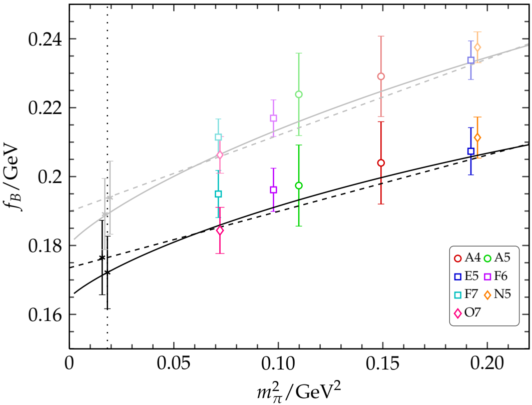
2s – 1s splitting static + $1/m_{\mathrm{b}}$

2s – 1s splitting static

Static results for splittings are in agreement with [T. Burch et al.]

Also ratio of ground state / excited state decay constant

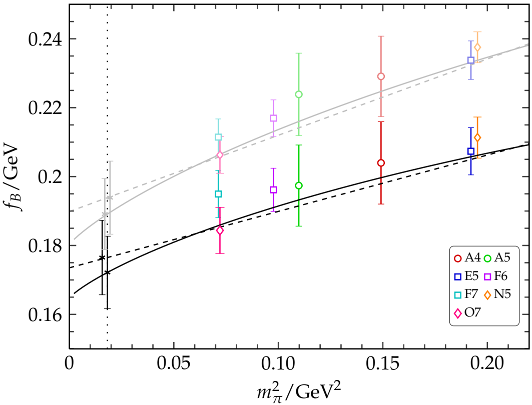
Example of a results for $N_\mathrm{f} = 2$ (Lattice 2011) [ALPHA Collaboration]



A: a=0.08fm
E,F: a=0.07fm
N,O: a=0.05fm

$$F_\mathrm{B} = F_\mathrm{B}|_{m_\pi^2=0} \times \left(1 - \frac{3}{4}\frac{1+3g^2_{B^*\,B\,\pi}}{16\pi^2 F_\pi^2} m_\pi^2 \log(m_\pi^2/F_\pi^2) + b\, m_\pi^2 + ca^2\right)$$

Example of a results for $N_\mathrm{f} = 2$ (Lattice 2011) [ALPHA Collaboration]



A: a=0.08fm
E,F: a=0.07fm
N,O: a=0.05fm

$$F_\mathrm{B} = F_\mathrm{B}|_{m_\pi^2=0} \times \left(1 - \frac{3}{4}\frac{1+3g^2_{B^*\,B\,\pi}}{16\pi^2 F_\pi^2} m_\pi^2 \log(m_\pi^2/F_\pi^2) + b\, m_\pi^2 + ca^2\right)$$

- V_{ub} becomes slightly more puzzling

- We are at the beginning of applications
- Lots of work remaining to be done ... and phenomenology waiting to be explored