Meson Form Factors in AdS/QCD

Lecture 2: pion form factors

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Based on papers written in collaboration with H.R. Grigoryan

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Action including χSB

- Full action of hard-wall model

$$S^B_{\rm AdS} = {\rm Tr} \int d^4x \int_0^{z_0} dz \left[\frac{1}{z^3}(D^M X)^\dagger (D_M X) + \frac{3}{z^5} X^\dagger X - \frac{1}{8 g_5^2 z} \left(B^{MN}_{(L)} B_{(L)MN} + B^{MN}_{(R)} B_{(R)MN} \right) \right]$$

- $DX = \partial X - iB_{(L)} X + iXB_{(R)}$, $B_{(L,R)} = V \pm A$,
  $X(x,z) = v(z)U(x,z)/2$,
  Chiral field: $U(x,z) = \exp\left[2it^a \pi^a(x,z)\right]$, $t^a = \sigma^a/2$
  Pion field: $\pi^a(x,z)$
  $v(z) = (m_q z + \sigma z^3)$ with $m_q \sim$ quark mass, $\sigma \sim$ condensate
- Longitudinal component of axial field

$$A^a_{\parallel\, M}(x,z) = \partial_M \psi^a(x,z)$$

gives another pion field $\psi^a(x,z)$

Equations of motion

- For transverse part of axial-vector gauge field $A^a_{\perp\mu}(x,z)$

$$\left[z^3 \partial_z \left(\frac{1}{z} \partial_z A^a_\mu \right) + p^2 z^2 A^a_\mu - g_5^2 v^2 A^a_\mu \right]_\perp = 0 \ ,$$

- Variation with respect to longitudinal part $\partial_\mu \psi^a$ gives

$$z^3 \partial_z \left(\frac{1}{z} \partial_z \psi^a \right) - g_5^2 v^2 \left(\psi^a - \pi^a \right) = 0 \ .$$

- Varying with respect to A_z produces

$$p^2 z^2 \partial_z \psi^a - g_5^2 v^2 \partial_z \pi^a = 0 \ .$$

- Taking $p^2 = m_\pi^2$ gives

$$\partial_z \pi = \left(m_\pi^2 z^2 / g_5^2 v^2 \right) \partial_z \psi \ . \qquad (1)$$

- $\partial_z \pi$ vanishes in $m_\pi^2 = 0$ limit

Pion wave function Ψ

- Model satisfies Gell-Mann–Oakes–Renner relation $m_\pi^2 \sim m_q$
- Chiral limit $m_q = 0$: analytic result for $\Psi(z) \equiv \psi(z) - \pi(z)$

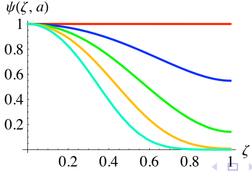
$$\Psi(z) = z\,\Gamma\left(2/3\right)\left(\frac{\alpha}{2}\right)^{1/3}\left[I_{-1/3}\left(\alpha z^3\right) - I_{1/3}\left(\alpha z^3\right)\frac{I_{2/3}\left(\alpha z_0^3\right)}{I_{-2/3}\left(\alpha z_0^3\right)}\right]$$

where $\alpha = g_5\sigma/3$

- $\Psi(z)$ satisfies $\Psi(0) = 1$, Neumann b.c. $\Psi'(z_0) = 0$ and

$$f_\pi^2 = -\frac{1}{g_5^2}\left(\frac{1}{z}\partial_z\Psi(z)\right)_{z=\epsilon\to 0}$$

$\Psi(z) \to \psi(\zeta, a)$
$\zeta \equiv z/z_0$
$a \equiv \alpha z_0^3$



$a = 0$
$a = 1$
$a = 2.26$
$a = 5$
$a = 10$

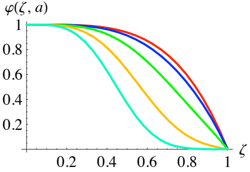
Pion wave function Φ

- Conjugate wave function

$$\Phi(z) = -\frac{1}{g_5^2 f_\pi^2}\left(\frac{1}{z}\,\partial_z\Psi(z)\right) = -\frac{2}{s_0}\,\left(\frac{1}{z}\,\partial_z\Psi(z)\right)$$

- Characteristic scale $s_0 = 4\pi^2 f_\pi^2 \approx 0.67\,\text{GeV}^2$
- $\Phi(z)$ satisfies $\Phi(0) = 1$ and Dirichlet b.c. $\Phi(z_0) = 0$

$\Phi(z) \to \phi(\zeta, a)$
$\zeta \equiv z/z_0$
$a \equiv \alpha z_0^3$



$a = 0$
$a = 1$
$a = 2.26$
$a = 5$
$a = 10$

Parameters of model

- z_0 is fixed through ρ-meson mass: $z_0 = z_0^\rho = (323 \text{ MeV})^{-1}$
- From $\Phi(0) = 1$, it follows that

$$g_5^2 f_\pi^2 = 3 \cdot 2^{1/3} \frac{\Gamma(2/3)}{\Gamma(1/3)} \frac{I_{2/3}\left(\alpha z_0^3\right)}{I_{-2/3}\left(\alpha z_0^3\right)} \alpha^{2/3}$$

- Experimental f_π is obtained for $\alpha = (424\,\text{MeV})^3$
- Then $a \equiv \alpha z_0^3$ equals $2.26 \equiv a_0$
- Note: $I_{2/3}(a)/I_{-2/3}(a) \approx 1$ for $a \gtrsim 1$
  $\Rightarrow$ value of f_π is basically determined by α alone

Pion Form Factor

- In terms of $\Psi(z)$:

$$F_\pi(Q^2) = \frac{1}{g_5^2 f_\pi^2} \int_0^{z_0} dz\, z\, \mathcal{J}(Q,z) \left[\left(\frac{\partial_z \Psi}{z} \right)^2 + \frac{g_5^2 v^2}{z^4} \Psi^2(z) \right]$$

- Normalization can be checked from

$$F_\pi(Q^2) = -\int_0^{z_0} dz\, \mathcal{J}(Q,z)\, \partial_z \Big(\Psi(z)\, \Phi(z) \Big)$$

that gives

$$F_\pi(0) = -\int_0^{z_0} dz\, \partial_z \Big(\Psi(z)\, \Phi(z) \Big) = \Psi(0)\, \Phi(0) = 1$$

Pion Charge Radius

- In terms of f_π:

$$\langle r_\pi^2 \rangle\Big|_{a \gtrsim 2} = \frac{3}{4\pi^2 f_\pi^2} + \frac{1}{2\pi^2 f_\pi^2} \ln\left(\frac{\alpha z_0^3}{0.566}\right) \approx 0.34 \mathrm{fm}^2$$

- Compare to Nambu-Jona-Lasinio model

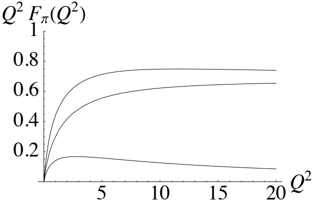
$$\langle r_\pi^2 \rangle_{\mathrm{NJL}} = \underbrace{\frac{3}{2\pi^2 f_\pi^2}}_{0.34 \mathrm{fm}^2} + \underbrace{\frac{1}{8\pi^2 f_\pi^2} \ln\left(\frac{m_\sigma^2}{m_\pi^2}\right)}_{0.11 \mathrm{fm}^2}$$

- Pion of hard-wall AdS/QCD model is too small (0.58 fm instead of 0.66 fm)

Pion Form Factor at Large Q^2

- Form factor in terms of $\Psi(z)$ and $\Phi(z)$:

$$F_\pi(Q^2) = \int_0^{z_0} dz\, z\, \mathcal{J}(Q,z) \left[g_5^2 f_\pi^2 \Phi^2(z) + \frac{9\alpha^2}{g_5^2 f_\pi^2}\, z^2\, \Psi^2(z) \right]$$



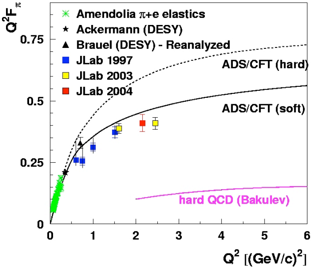
- Total (in GeV^2)
- Φ^2 term
- Ψ^2 term

- For large Q, only $z \sim 1/Q$ work:

$$F_\pi(Q^2) \to \frac{2\, g_5^2 f_\pi^2 \Phi^2(0)}{Q^2} = \frac{4\pi^2 f_\pi^2}{Q^2} \equiv \frac{s_0}{Q^2}$$

Pion Form Factor

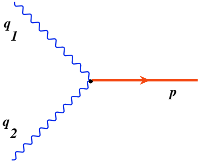
- Comparison with experiment



- Pion is too small again
- pQCD has $2\alpha_s/\pi$ factor due to one-gluon exchange:

$$F_\pi^{\rm pQCD}(Q^2) \to \frac{2\alpha_s}{\pi} \cdot \frac{s_0}{Q^2} \sim 0.2\, F_\pi^{\rm AdS/QCD}(Q^2)$$

Anomalous Amplitude



- $\pi^0\gamma^*\gamma^*$ form factor

$$\int \langle \pi, p | T\left\{ J^{\mu}_{\rm EM}(x)\, J^{\nu}_{\rm EM}(0)\right\} |0\rangle e^{-iq_1 x} d^4x$$

$$= \epsilon^{\mu\nu\alpha\beta} q_{1\,\alpha} q_{2\,\beta}\, \frac{N_c}{12\pi^2 f_\pi}\, K_{\gamma^*\gamma^*\pi^0}\left(Q_1^2, Q_2^2\right)$$

$p = q_1 + q_2$ and $q_{1,2}^2 = -Q_{1,2}^2$

- For real photons in QCD, K is fixed by axial anomaly

$$K_{\gamma^*\gamma^*\pi^0}(0,0) = 1$$

Extending AdS/QCD Model

- Need to have isoscalar fields $\Rightarrow$ gauging $U(2)_L \otimes U(2)_R$

$$\mathcal{B}_\mu = t^a B^a_\mu + \mathbf{1}\,\frac{\hat{B}_\mu}{2}$$

- Need Chern-Simons term

$$S^{(3)}_{\rm CS}[\mathcal{B}] = \frac{N_c}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\int d^4x\,dz\,(\partial_z \mathcal{B}_\mu)\left[\mathcal{F}_{\nu\rho}\mathcal{B}_\sigma + \mathcal{B}_\nu \mathcal{F}_{\rho\sigma}\right]$$

- Anomalous form factor conforming to QCD anomaly

$$\begin{aligned} K(Q_1^2, Q_2^2) &= \Psi(z_0)\mathcal{J}(Q_1, z_0)\mathcal{J}(Q_2, z_0) \\ &- \int_0^{z_0} \mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\,\partial_z\Psi(z)\,dz \end{aligned}$$

- Check:

$$K(0,0) = \Psi(z_0) - \int_0^{z_0} \partial_z\Psi(z)\,dz = \Psi(0) = 1$$

$\gamma^*\gamma^*\pi^0$ Form Factors

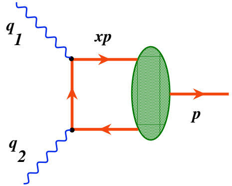
- For large Q_1 and/or Q_2

$$K(Q_1^2, Q_2^2) \simeq \frac{s_0}{2} \int_0^{z_0} \mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\, \Phi(z)\, z\, dz$$

- One real photon:

$$K(0, Q^2) \to \frac{\Phi(0)s_0}{2Q^2} \int_0^{\infty} d\chi\, \chi^2\, K_1(\chi) = \frac{s_0}{Q^2}$$

$\gamma^*\gamma\pi^0$ Form Factor in pQCD



- In pQCD:

$$K^{\rm pQCD}(0,Q^2) = \frac{s_0}{3Q^2}\int_0^1 \frac{\varphi_\pi(x)}{x}\,dx \equiv \frac{s_0}{3Q^2}\,I^\varphi$$

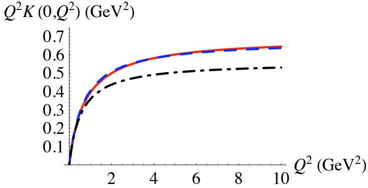
- Coincides with AdS/QCD model if $I^\varphi = 3$, e.g., for $\varphi_\pi(x) = 6x(1-x)$ (asymptotic DA)

Comparison with data

- Brodsky-Lepage interpolation

$$K^{\rm BL}(0,Q^2) = \frac{1}{1+Q^2/s_0}$$

- Our model (red) is very close to BL interpolation (blue)



- CLEO data represented by black dash-dotted line
- NLO pQCD fits data. Fits give DA's with $I^\varphi \approx 3$

Equal large photon virtualities

- For large Q_1 and/or Q_2

$$K(Q_1^2, Q_2^2) \simeq \frac{s_0}{2} \int_0^{z_0} \mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\, \Phi(z)\, z\, dz$$

- Equal photon virtualities:

$$K(Q^2, Q^2) \to \frac{\Phi(0) s_0}{Q^2} \int_0^{\infty} d\chi\, \chi^3\, [K_1(\chi)]^2 = \frac{s_0}{3Q^2}$$

- pQCD result does not depend on pion DA

$$K^{\rm pQCD}(Q^2, Q^2) = \frac{s_0}{3} \int_0^1 \frac{\varphi_\pi(x)\, dx}{xQ^2 + (1-x)Q^2} = \frac{s_0}{3Q^2}$$

- and coincides with AdS/QCD model!

Non-equal large photon virtualities

- Take $Q_1^2 = (1+\omega)Q^2$ and $Q_2^2 = (1-\omega)Q^2$
- Leading-order pQCD gives in this case

$$K^{\rm pQCD}(Q_1^2, Q_2^2) = \frac{s_0}{3Q^2} \int_0^1 \frac{\varphi_\pi(x)\, dx}{1+\omega(2x-1)} \equiv \frac{s_0}{3Q^2}\, I^\varphi(\omega)$$

- AdS/QCD model gives

$$\frac{\Phi(0) s_0}{2Q^2} \sqrt{1-\omega^2} \int_0^\infty d\chi\, \chi^3\, K_1(\chi\sqrt{1+\omega}) K_1(\chi\sqrt{1-\omega})$$
$$= \left(\frac{s_0}{3Q^2}\right) \left\{ \frac{3}{4\omega^3} \left[2\omega - (1-\omega^2)\, \ln\left(\frac{1-\omega}{1+\omega}\right)\right]\right\}$$

- $\{\ldots\}$ coincides with pQCD $I^\varphi(\omega)$ for $\varphi(x) = 6x(1-x)$

AdS/pQCD duality

- Use representation

$$\chi K_1(\chi) = \int_0^\infty e^{-\chi^2/4u - u}\, du \ ,$$

- And integrate over χ to get

$$K(Q_1^2, Q_2^2) \to \frac{s_0}{Q^2} \int_0^\infty \int_0^\infty \frac{u_1 u_2\, e^{-u_1 - u_2} du_1 du_2}{u_2(1+\omega) + u_1(1-\omega)} \ .$$

- Change $u_2 = x\lambda$, $u_1 = (1-x)\lambda$ and integrate over λ:

$$K(Q_1^2, Q_2^2) \to \frac{s_0}{3Q^2} \int_0^1 \frac{6\, x(1-x)\, dx}{1 + \omega(2x-1)}$$

- Coincides with the pQCD formula if $\varphi_\pi(x) = 6\, x(1-x)$

Bound-state decomposition

- GVMD for bulk-to-boundary propagator:

$$\mathcal{J}(Q,z) = \sum_{n=1}^{\infty} \frac{g_5 f_n \psi_n^V(z)}{Q^2 + M_n^2}$$

- Form factor $K(Q_1^2, Q_2^2)$ has double GVMD representation

$$K(Q_1^2, Q_2^2) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{A_{n,k}}{(1 + Q_1^2/M_n^2)(1 + Q_2^2/M_k^2)}$$

- But we know that $K(Q^2, Q^2) \sim 1/Q^2$!

How double GVMD gives $1/Q^2$

- Soft-wall model integral

$$K^{\mathrm{s}}(Q_1^2, Q_2^2) = 2\kappa^2 \int_0^\infty \mathcal{J}^{\mathrm{s}}(Q_1, z)\, \mathcal{J}^{\mathrm{s}}(Q_2, z)\, e^{-\kappa^2 z^2}\, z dz$$

- Gives ($a_i = Q_i^2/M^2$ and $M = 2\kappa$ is mass scale)

$$K^{\mathrm{s}}(Q_1^2, Q_2^2) = \sum_{n=0}^{\infty} \frac{a_1}{(a_1+n)(a_1+n+1)} \frac{a_2}{(a_2+n)(a_2+n+1)}$$

- Each term behaves like $1/Q_1^2 Q_2^2$, but

$$K^{\mathrm{s}}(Q^2, Q^2) \to a^2 \int_0^\infty \frac{dn}{(n+a)^4} = \frac{1}{3a} = \frac{M^2}{3Q^2}$$

- Higher resonances are important!

Summary

- Form Factors in AdS/QCD are given by QM-like formulas
- Only one mechanism $z \sim 1/Q$ for large Q
- Large-Q^2 asymptotics is s_0/Q^2 vs. pQCD $(2\alpha_s/\pi)s_0/Q^2$
- Overshoots data: AdS/QCD pion is too small
- Anomalous amplitude:
  1. Extension to $U(2)_L \otimes U(2)_R$ and Chern-Simons term
  2. Fixing normalization by conforming to QCD anomaly
  3. Large-Q^2 behavior coincides with pQCD calculations for asymptotic pion DA
  4. Double GVMD does not contradict to $1/Q^2$ asymptotics

Conclusion

- AdS/QCD provides instructive model
for what may happen
with form factors in QCD