Exotic mesons from lattice QCD

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Overview

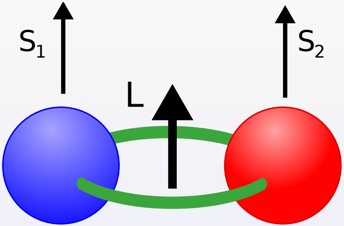
Non-exotic

vs

Exotic

A simple quark model of mesons

- Combine quark and anti-quark and find J^{PC} values



- Total spin, $J = L + (S_1 + S_2)$
- Two spin-$1/2$ quarks in two possible combinations:

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

Singlet	$S = 0$	$PC = -+$
Triplet	$S = 1$	$PC = --$

- Combine with angular momentum around centre. L
- Odd L wavefunctions have $PC = --$

	$L = 0$	$L = 1$	$L = 2$	$L = 3$	$\cdots$
Singlet:	0^{-+}	1^{+-}	2^{-+}	3^{+-}	$\cdots$
Triplet:	1^{--}	$\{0, 1, 2\}^{++}$	$\{1, 2, 3\}^{--}$	$\{2, 3, 4\}^{++}$	$\cdots$
	S-wave	P-wave	D-wave	F-wave	$\cdots$

Exotic quantum numbers

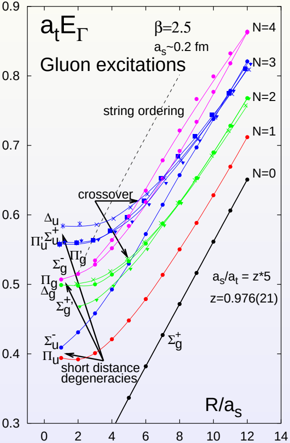
- On inspection, some J^{PC} values are missing **from this simple quark model**:

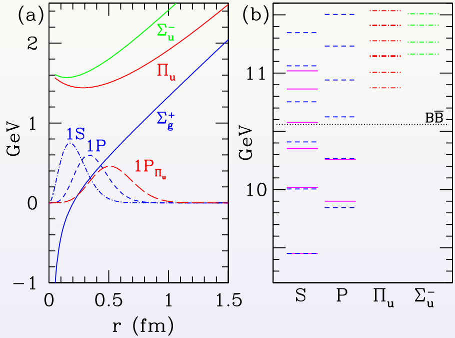
Exotic quantum numbers

$J^{PC} = 0^{--}, 0^{+-}, 1^{-+}, 2^{+-}, 3^{-+}, \ldots, \text{even}^{+-}, \text{odd}^{-+}$

- Finding a meson with these quantum numbers would be a “smoking gun” for something beoynd the quark model
- Are these the only signatures of exotic states?
  - Extra states in the spectrum?
  - States with decays that seem unusual in the model?

Gluonic excitations of the QCD potential





- Heavy quarks: Solve Schrödinger in adiabatic potential

KJ. Juge, J. Kuti and C. Morningstar

hep-lat/0312019, nucl-th/0307116

A constituent picture of hadrons from QCD

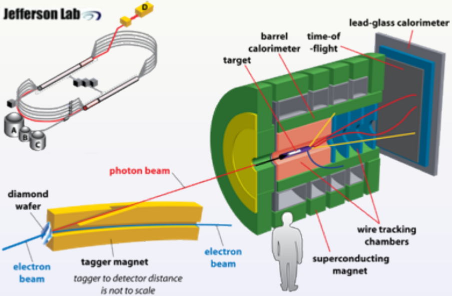
- QCD has **quarks** and **gluons**
- **The confinement conjecture:** fields of the QCD lagrangian combine into colourless combinations: the **mesons** and **baryons**

A constituent model

constituents			quark model label
$3 \otimes \bar{3}$	$=$	$\mathbf{1} \oplus 8$	meson
$3 \otimes 3 \otimes 3$	$=$	$\mathbf{1} \oplus 8 \oplus 8 \oplus 10$	baryon
$8 \otimes 8$	$=$	$\mathbf{1} \oplus 8 \oplus 8 \oplus 10 \oplus 10 \oplus 27$	glueball
$\bar{3} \otimes 8 \otimes 3$	$=$	$\mathbf{1} \oplus 8 \oplus 8 \oplus 8 \oplus 10 \oplus 10 \oplus 27$	hybrid
$\bar{3} \otimes \bar{3} \otimes 3 \otimes 3$	$=$	$\mathbf{1} \oplus \mathbf{1} \oplus 8 \oplus 8 \oplus 8 \oplus 8 \oplus 10 \oplus 10 \oplus 27$	tetraquark/ molecule
$\vdots$		$\vdots$	$\vdots$

- QCD does not always respect this constituent labelling! There can be strong mixing.

The GlueX experiment at JLab



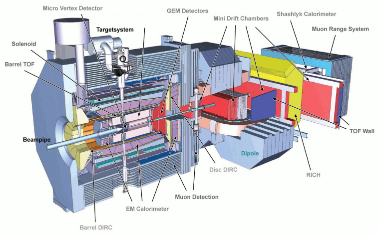
- 12 GeV upgrade to CEBAF ring
- New experimental hall: Hall D
- New experiment: GlueX

- Aim: photoproduce mesons, in particular the hybrid mesons (with intrinsic gluonic excitations)
- Expected to start taking data 2014

- Extensive new construction at GSI Darmstadt
- Expected to start operation 2014

PANDA: Anti-Proton ANnihilation at DArmstadt

- Anti-proton beam from FAIR on fixed-target.
- Physics goals include searches for hybrids and glueballs (as well as charm and baryon spectroscopy).



A renaissance in spectroscopy

- Early in the noughties, new narrow structures were seen by Belle and BaBar above the open-charm threshold.
- This led to substantial renewed interest in spectroscopy. Were these more quark-anti-quark states, or something more?
    - $X(3872)$: very close to $D\bar{D}$ threshold - a molecule?
    - $Y(4260)$: a 1^{--} hybrid?
    - $Z^{\pm}(4430)$: charged, can't be $\bar{c}c$.
- Very little is known and no clear picture seems to be emerging...

The PDG view

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from the 2010 Review of Particle Physics.
Please use this **CITATION**: K. Nakamura *et al.* (Particle Data Group), J. Phys. G **37**, 075021 (2010).

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LIGHT UNFLAVORED MESONS ($S = C = B = 0$)

For I=1 (π, b, ρ, a): $u\bar{d}$, $(u\bar{u}-d\bar{d})/\sqrt{2}$, $d\bar{u}$;
for I=0 (η, η', h, h', ω, φ, f, f'): $c_1(u\bar{u}+d\bar{d})+c_2(s\bar{s})$

	$\pi^\pm$	$1^-(0^-)$		$\eta(1475)$	$0^+(0^{-+})$	•	$f_2(1910)$	$0^+(2^{++})$
	π^0	$1^-(0^{-+})$		$f_0(1500)$	$0^+(0^{++})$		$f_2(1950)$	$0^+(2^{++})$
	η	$0^+(0^{-+})$	•	$f_1(1510)$	$0^+(1^{++})$	•	$\rho_3(1990)$	$1^+(3^{--})$
	$f_0(600)$ or σ	$0^+(0^{++})$		$f_2'(1525)$	$0^+(2^{++})$		$f_2(2010)$	$0^+(2^{++})$
	$\rho(770)$	$1^+(1^{--})$	•	$f_2(1565)$	$0^+(2^{++})$	•	$f_0(2020)$	$0^+(0^{++})$
	$\omega(782)$	$0^-(1^{--})$	•	$\rho(1570)$	$1^+(1^{--})$		$a_4(2040)$	$1^-(4^{++})$
	$\eta'(958)$	$0^+(0^{-+})$	•	$h_1(1595)$	$0^-(1^{+-})$		$f_4(2050)$	$0^+(4^{++})$
	$f_0(980)$	$0^+(0^{++})$		$\pi_1(1600)$	$1^-(1^{-+})$	•	$\pi_2(2100)$	$1^-(2^{-+})$
	$a_0(980)$	$1^-(0^{++})$	•	$a_1(1640)$	$1^-(1^{++})$	•	$f_0(2100)$	$0^+(0^{++})$
	$\varphi(1020)$	$0^-(1^{--})$	•	$f_2(1640)$	$0^+(2^{++})$	•	$f_2(2150)$	$0^+(2^{++})$
	$h_1(1170)$	$0^-(1^{+-})$		$\eta_2(1645)$	$0^+(2^{-+})$	•	$\rho(2150)$	$1^+(1^{--})$
	$b_1(1235)$	$1^+(1^{+-})$		$\omega(1650)$	$0^-(1^{--})$		$\varphi(2170)$	$0^-(1^{--})$
	$a_1(1260)$	$1^-(1^{++})$		$\omega_3(1670)$	$0^-(3^{--})$	•	$f_0(2200)$	$0^+(0^{++})$
	$f_2(1270)$	$0^+(2^{++})$		$\pi_2(1670)$	$1^-(2^{-+})$	•	$f_J(2220)$	$0^+(2^{++}$ or $4^{++})$
	$f_1(1285)$	$0^+(1^{++})$		$\varphi(1680)$	$0^-(1^{--})$	•	$\eta(2225)$	$0^+(0^{-+})$
	$\eta(1295)$	$0^+(0^{-+})$		$\rho_3(1690)$	$1^+(3^{--})$	•	$\rho_3(2250)$	$1^+(3^{--})$
	$\pi(1300)$	$1^-(0^{-+})$		$\rho(1700)$	$1^+(1^{--})$		$f_2(2300)$	$0^+(2^{++})$
	$a_2(1320)$	$1^-(2^{++})$	•	$a_2(1700)$	$1^-(2^{++})$	•	$f_4(2300)$	$0^+(4^{++})$
	$f_0(1370)$	$0^+(0^{++})$		$f_0(1710)$	$0^+(0^{++})$	•	$f_0(2330)$	$0^+(0^{++})$
•	$h_1(1380)$	$?^-(1^{+-})$	•	$\eta(1760)$	$0^+(0^{-+})$		$f_2(2340)$	$0^+(2^{++})$
	$\pi_1(1400)$	$1^-(1^{-+})$		$\pi(1800)$	$1^-(0^{-+})$	•	$\rho_5(2350)$	$1^+(5^{--})$
	$\eta(1405)$	$0^+(0^{-+})$	•	$f_2(1810)$	$0^+(2^{++})$	•	$a_6(2450)$	$1^-(6^{++})$
	$f_1(1420)$	$0^+(1^{++})$	•	$X(1835)$	$?^?(?^{-+})$	•	$f_6(2510)$	$0^+(6^{++})$
	$\omega(1420)$	$0^-(1^{--})$		$\varphi_3(1850)$	$0^-(3^{--})$			
•	$f_2(1430)$	$0^+(2^{++})$	•	$\eta_2(1870)$	$0^+(2^{-+})$			
	$a_0(1450)$	$1^-(0^{++})$		$\pi_2(1880)$	$1^-(2^{-+})$			
	$\rho(1450)$	$1^+(1^{--})$	•	$\rho(1900)$	$1^+(1^{--})$			

• — OMITTED FROM SUMMARY TABLE

What are these states? $\bar{q}q$ mesons?

Lattice Hadron Spectroscopy

- Significant experimental effort hoping to understand light hadron and charm spectroscopy
  - Are there resonances that don't fit in the quark model?
  - Are there gluonic excitations in this spectrum?
  - What structure does confinement lead to?
  - How do exotic resonances decay?
- To use LQCD to address these questions means:
  - finding continuum properties of states accurately
  - computing scattering and resonance widths
- To acheive this we need
  - Techniques that give **statistical precision**
  - **Spin identification**
  - **A framework for decay physics**
  - Control over extrapolations ($m_q \to 0, V \to \infty, a \to 0$).

Variational techniques

Variational techniques (1)

[C. Michael and I. Teasdale. NPB215 (1983) 433]
[M. Lüscher and U. Wolff. NPB339 (1990) 222]

- Crucial for precision spectroscopy, and for studying excitations.
- Needed to fully understand states above threshold.

A variational basis

Consider a set of creation operators $\{\phi_i\}, i = 1 \ldots N$ with the same quantum numbers. Suppose we can measure

$$C_{ij}(t) = \langle 0| \phi_i(t)\phi_j(0) |0\rangle$$

for a range of values of t

- Define a new operator $\Phi = \sum_i \alpha_i \phi_i$ as linear combination of basis operators, what combination $\{\alpha_i\}$ would be "best" to make ground-state?
- Details : [Blossier *et.al.* JHEP 0904:094,2009]

Variational techniques (2)

The correlation function

$$\begin{aligned} C_{ij}(t) &= \langle 0|\,\phi_i(t)\phi_j(0)\,|0\rangle \\ &= \langle 0|\,\phi_i\, e^{-Ht}\,\phi_j\,|0\rangle \\ &= \sum_{k,k'} \langle 0|\,\phi_i\,|k\rangle\langle k|\,e^{-Ht}\,|k'\rangle\langle k'|\,\phi_j\,|0\rangle \\ &= \sum_{k} \langle 0|\,\phi_i\,|k\rangle\langle k|\,\phi_j\,|0\rangle\; e^{-E_k t} \end{aligned}$$

- Correlation function gets contributions from all states
- Light states have longest-range correlations
- **Idea:** minimise the fall-off of the correlation function of Φ over some range $[t_0, t]$

Variational techniques (3)

An optimal choice for Φ

- Choose Φ to maximise

$$\lambda(t,t_0) = \frac{\langle 0|\Phi(t)\Phi^\dagger(0)|0\rangle}{\langle 0|\Phi(t_0)\Phi^\dagger(0)|0\rangle}$$

with $t > t_0$

- Since $\Phi^\dagger = \sum_i \alpha_i \phi_i^\dagger$ we get

$$\lambda(t,t_0) = \frac{\alpha_i^* C_{ij}(t)\alpha_j}{\alpha_k^* C_{kl}(t_0)\alpha_l}$$

- and differentiation w.r.t α^* yields the **generalised eigenvalue problem:**

A generalised eigenvalue problem

$$C(t)\alpha = \lambda C(t_0)\alpha$$

Fermionic correlation functions

Fermions in the path integral

- In path integral, fermions are represented using **Grassmann** algebra.

$$\int d\eta = 0, \quad \int d\eta\, \eta = 1, \quad \eta^2 = 0$$

- Higher dimensions - anticommutation rule:

$$\eta_i \eta_j = -\eta_j \eta_i$$

- Expensive to manipulate directly by computer ...

Exercise 1

Find 3 4×4 matrices, α_1, α_2, μ such that for any f,

$$\int d\eta_1 d\eta_2 \;\; f(\eta_1, \eta_2) = \mathrm{Tr}\left\{\mu\, f(\alpha_1, \alpha_2)\right\}$$

Fermions in the path integral

- In QCD the action is (usually) bilinear.
- Consider computing a correlation function for the ρ-meson in 2-flavour QCD:

$$C_\rho(t_1,t_0) = \frac{\int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi\ \bar{\psi}_u\gamma_i\psi_d(t_1)\ \bar{\psi}_d\gamma_i\psi_u(t_0)\ e^{-S_G[U]+\bar{\psi}_f M_f[U]\psi_f}}{\int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{-S_G[U]+\bar{\psi}_f M_f[U]\psi_f}}$$

- Integrate the grassmann fields analytically, giving:

$$C_\rho(t_1,t_0) = \frac{\int \mathcal{D}U\ \mathrm{Tr}\ \gamma_i M_d^{-1}(t_1,t_0)\gamma_i M_u^{-1}(t_0,t_1)\ \det M^2[U]\ e^{-S_G[U]}}{\int \mathcal{D}U\ \det M^2[U]\ e^{-S_G[U]}}$$

- Fermions in lagrangian → fermion determinant
- Fermions in measurement → propagators

Fermions in the path integral

- With more insertions, need **Wick's theorem**
- Example — four field insertions:

$$\langle \psi_i \bar{\psi}_j \psi_k \bar{\psi}_l \rangle$$

- and the pairwise contraction can be done in two ways:

$$\psi_i \bar{\psi}_j \psi_k \bar{\psi}_l \quad \text{and} \quad \psi_i \bar{\psi}_j \psi_k \bar{\psi}_l$$

- ...giving the propagator combination

$$M^{-1}_{ij} M^{-1}_{kl} - M^{-1}_{jk} M^{-1}_{il}$$

- the minus-sign comes from the anti-commutation needed in the second term.
- More fields means more combinations
- This is important in (eg.) isoscalar meson spectroscopy.

Exercise 2

For a system with six degrees of freedom, $\{\bar{\eta}_i, \eta_i\}, i = 1, 2, 3$, evaluate the grassmann integral

$$I_4 = \int \prod_{i=1}^{3} d\bar{\eta}_i d\eta_i \;\; \eta_1 \bar{\eta}_2 \eta_2 \bar{\eta}_1 \;\; e^{-\bar{\eta} M \eta}$$

and compare this answer to the prediction of Wick's theorem.

The lattice propagator

Handling lattice propagators

- On a finite lattice, the propagator is the inverse of a very large matrix.
- It is **impractical to compute** all elements of the propagator directly using a standard elimination method.
- The action $Ma = b$ for vectors a, b in the space of quark fields is practical. Can store lattice quark fields but not matrices.
- Given χ, can **solve the linear system**

$$M\psi = \chi$$

Handling lattice propagators

- Krylov space solver: the Krylov space $\mathcal{K}_n(M,\chi)$ is defined by

$$\mathcal{K}_n(M,\chi) = \text{Span}\left\{\chi, M\chi, M^2\chi, \ldots M^n\chi\right\}$$

- Examples include CG, MinRes, BiCG, ...
- As the physical quark mass is approached, so the convergence of these algorithm slows rapidly.
- Newer algorithms use **deflation**: simultaneously build an approximation to the low-modes of M
- Algebraic multi-grid is re-emerging too

Handling lattice propagators

- Most lattice fermions obey γ_5-hermiticity:

$$M^\dagger(x,y) = \gamma_5 M(y,x)\gamma_5$$

- QCD vacuum is translationally invariant. Solving $M\psi = \eta$ gives access to **one row** of M^{-1}

The point-to-all propagator

- Choose an origin y
- For all spin, colour combinations $\{\alpha, a\}$
  - construct a source, $\eta_{x,\beta,b} = \delta_{x,y}\delta_{\beta,\alpha}\delta_{b,a}$
  - solve $M\psi^{(y,\alpha,a)} = \eta$ with this rhs
- Now have a block-row (at y) of M^{-1}

- Simple isovector meson and baryon creation operators can be constructed from this data

Hadronic physics

Isovector meson correlation functions

- To create a meson, we need to build functions that couple to quarks.
- Meson can be created by a quark bilinear. Appropriate gauge invariant creation operator (for isospin $I = 1$) would be

$$\Phi_{\text{meson}}(t) = \sum_{x} \bar{u}(\underline{x}, t)\Gamma U_{\mathcal{C}}(\underline{x}, \underline{y}; t) d(\underline{y}, t)$$

  where Γ is some appropriate Dirac structure, and $U_{\mathcal{C}}$ a product of (smeared) link variables.
- Operators that transform irreducibly under the lattice rotation group O_h are needed.

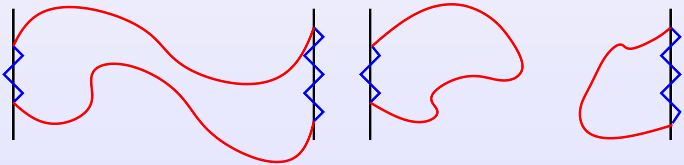
Isoscalar meson correlation functions

- If we are interested in measuring isoscalar meson masses, extra diagrams must be evaluated, since four-quark diagrams become relevant. The Wick contraction yields extra terms, since

$$\langle \psi_i \bar{\psi}_j \psi_k \bar{\psi}_l \rangle = M^{-1}_{ij} M^{-1}_{kl} - M^{-1}_{il} M^{-1}_{jk}$$

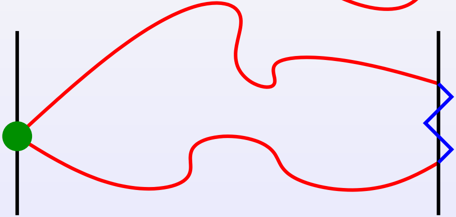
- Now

$$\langle 0|\Phi_{I=0}(t)\Phi^\dagger_{I=0}(0)|0\rangle =$$
$$\langle 0|\Phi_{I=1}(t)\Phi^\dagger_{I=1}(0)|0\rangle - \langle 0|\text{Tr } M^{-1}\Gamma U_{\mathcal{C}}(t)\text{Tr } M^{-1}\Gamma U_{\mathcal{C}}(0)|0\rangle$$



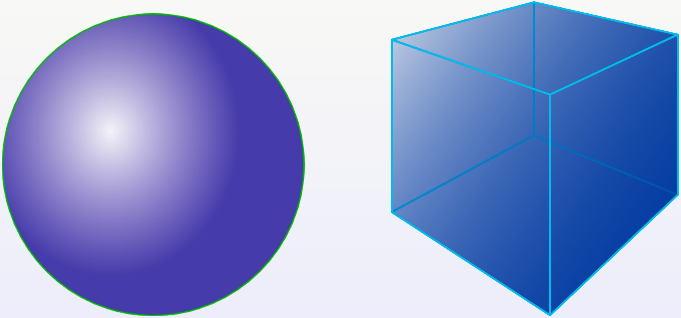
Isovector meson correlation functions (2)

The most general operator.



A restricted correlation function accessible to one point-to-all computation.

- Continuum: states classified by J^P irreducible representations of $O(3)$.



$O(3)$ $\qquad$ O_h

- Lattice regulator breaks $O(3) \to O_h$
- Lattice: states classified by R^P **“quantum letter” labelling irrep of** O_h

Irreps of O_h

- O has 5 conjugacy classes (so O_h has 10)
- Number of conjugacy classes = number of irreps
- Schur: $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$
- These irreps are labelled A_1, A_2, E, T_1, T_2

	E	$8C_3$	$6C_2$	$6C_4$	$3C_2$
A_1	1	1	1	1	1
A_2	1	1	-1	-1	1
E	2	-1	0	0	2
T_1	3	0	-1	1	-1
T_2	3	0	1	-1	-1

Spin on the lattice

- O_h has 10 irreps: $\{A_1^{g,u}, A_2^{g,u}, E^{g,u}, T_1^{g,u}, T_2^{g,u}, \}$, where $\{g, u\}$ label even/odd parity.
- Link to continuum: subduce representations of $O(3)$ into O_h

	A_1	A_2	E	T_1	T_2
$J = 0$	1				
$J = 1$				1	
$J = 2$			1		1
$J = 3$		1		1	1
$J = 4$	1		1	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

- Enough to search for degeneracy patterns in the spectrum? $4 \equiv 0 \oplus 1 \oplus 2$!

Example: $J^{PC} = 2^{++}$ meson creation operator

- Need more information to discriminate spins. Consider continuum operator that creates a 2^{++} meson:

$$\Phi_{ij} = \bar{\psi}\left(\gamma_i D_j + \gamma_j D_i - \frac{2}{3}\delta_{ij}\gamma \cdot D\right)\psi$$

- Lattice: Substitute gauge-covariant lattice finite-difference $D_{\rm latt}$ for D
- A reducible representation:

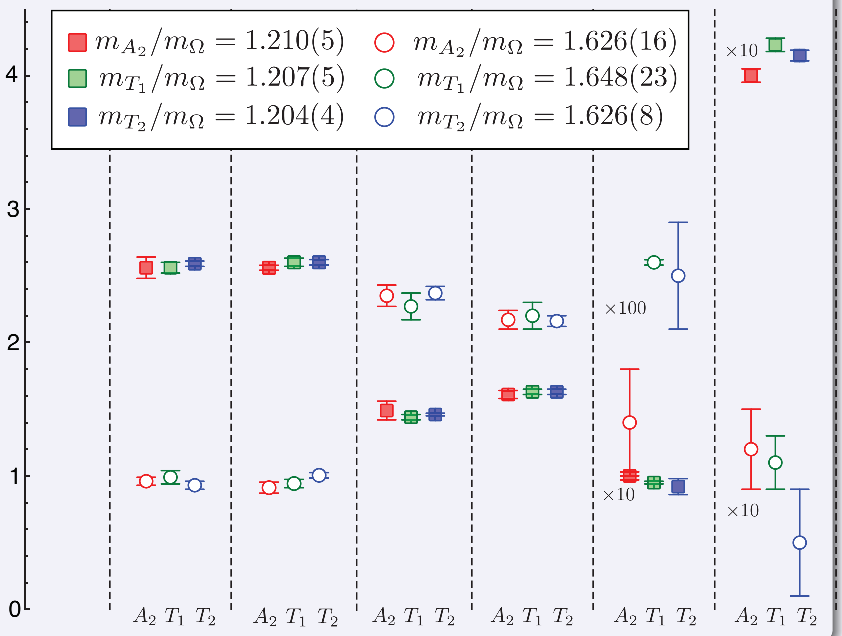
$$\Phi^{T_2} = \{\Phi_{12}, \Phi_{23}, \Phi_{31}\}$$

$$\Phi^{E} = \left\{\frac{1}{\sqrt{2}}(\Phi_{11} - \Phi_{22}), \frac{1}{\sqrt{6}}(\Phi_{11} + \Phi_{22} - 2\Phi_{33})\right\}$$

- Look for signature of continuum symmetry:

$$\langle 0|\Phi^{(T_2)}|2^{++(T_2)}\rangle = \langle 0|\Phi^{(E)}|2^{++(E)}\rangle$$

Spin-3 identification: J. Dudek *et.al.*, Hadron Spectrum Collab.



Quark-field smearing

Smearing - an essential ingredient for precision

- To build an operator that projects effectively onto a low-lying hadronic state need to use smearing
- Instead of the creation operator being a direct function applied to the fields in the lagrangian first smooth out the UV modes which contribute little to the IR dynamics directly.
- A popular gauge-covariant smearing algorithm; Jacobi/Wuppertal smearing: Apply the linear operator

$$\Box_J = \exp(\sigma \Delta^2)$$

- Δ^2 is a lattice representation of the 3-dimensional gauge-covariant laplace operator on the source time-slice

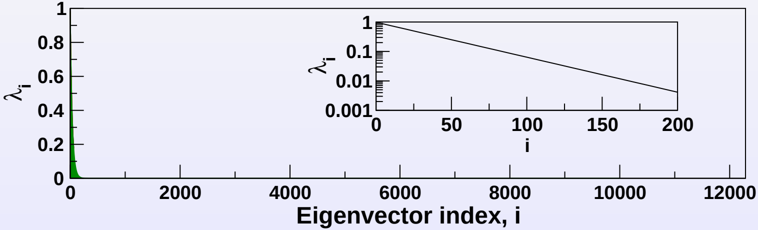
$$\Delta^2_{x,y} = 6\delta_{x,y} - \sum_{i=1}^{3} U_i(x)\delta_{x+\hat{\imath},y} + U_i^\dagger(x-\hat{\imath})\delta_{x-\hat{\imath},y}$$

- Correlation functions look like $\mathrm{Tr}\ \Box_J M^{-1} \Box_J M^{-1} \Box_J \ldots$

- **Gaussian** smearing:

$$\lim_{n\to\infty}\left(1+\frac{\sigma\nabla^2}{n}\right)^n = \exp(\sigma\nabla^2)$$

- This acts in the space of coloured scalar fields on a time-slice: $N_s \times N_c$



- Data from $a_s \approx 0.12\text{fm}\ 16^3$ lattice: $16^3 \times 3 = 12288$.

Can redefining smearing help?

- Computing quark propagation in configuration generation and observable measurement is expensive.
- Objective: extract as much information from correlation functions as possible.

Two problems:

1. Most correlators: signal-to-noise falls exponentially
2. Making measurements can be costly:
   - Variational bases
   - Exotic states using more sophisticated creation operators
   - Isoscalar mesons
   - **Multi-hadron states**

- Good operators are **smeared**; helps with problem 1, can it help with problem 2?

- **Smeared field:** $\tilde{\psi}$ from ψ, the “raw” quark field in the path-integral:

$$\tilde{\psi}(t) = \square[U(t)] \;\; \psi(t)$$

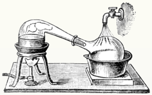
- Extract the essential degrees-of-freedom.
- Smearing should preserve symmetries of quarks.
- Now form creation operator (e.g. a meson):

$$O_M(t) = \bar{\tilde{\psi}}(t)\Gamma\tilde{\psi}(t)$$

- Γ: operator in $\{\underline{s}, \sigma, c\} \equiv \{\text{position,spin,colour}\}$
- Smearing: overlap $\langle n|O_M|0\rangle$ is large for low-lying eigenstate $|n\rangle$

Distillation

"*distill*: to extract the quintessence of" [OED]



- Distillation: **define** smearing to be explicitly a very low-rank operator. Rank is $N_{\mathcal{D}} (\ll N_s \times N_c)$.

Distillation operator

$$\Box(t) = V(t)V^{\dagger}(t)$$

with $V^{a}_{\underline{x},c}(t)$ a $N_{\mathcal{D}} \times (N_s \times N_c)$ matrix

- Example (used to date): $\Box_{\nabla}$ the **projection operator into $\mathcal{D}_{\nabla}$, the space spanned by the lowest eigenmodes of the 3-D laplacian**
- Projection operator, so idempotent: $\Box_{\nabla}^2 = \Box_{\nabla}$
- $\lim_{N_{\mathcal{D}} \to (N_s \times N_c)} \Box_{\nabla} = I$
- Eigenvectors of ∇^2 not the only choice...

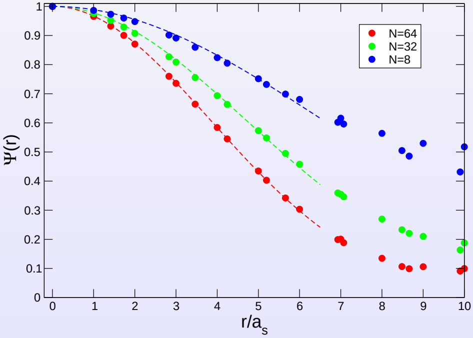
Distillation: preserve symmetries

- Using eigenmodes of the gauge-covariant laplacian **preserves lattice symmetries**

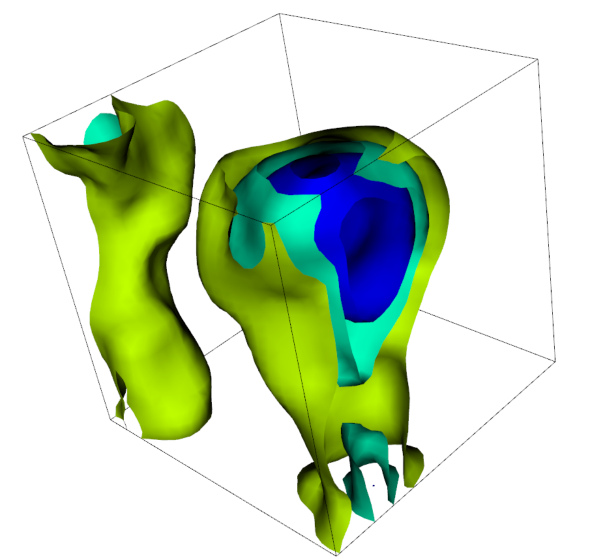
$$U_i(\underline{x}) \xrightarrow{g} U_i^g(\underline{x}) = g(\underline{x})U_i(\underline{x})g^\dagger(\underline{x}+\hat{\underline{\imath}})$$

$$\Box_\nabla(\underline{x},\underline{y}) \xrightarrow{g} \Box_\nabla^g(\underline{x},\underline{y}) = g(\underline{x})\Box_\nabla(\underline{x},\underline{y})g^\dagger(\underline{y})$$

- Translation, parity, charge-conjugation symmetric
- O_h symmetric
- Close to $SO(3)$ symmetric
- “local” operator



Eigenmodes of the laplacian



- Lowest mode on a $32^3 \equiv (3.8\ \text{fm})^3$ lattice.

- Consider an isovector meson two-point function:

$$C_M(t_1-t_0) = \langle\langle \bar{u}(t_1)\Box_{t_1}\Gamma_{t_1}\Box_{t_1}d(t_1) \quad \bar{d}(t_0)\Box_{t_0}\Gamma_{t_0}\Box_{t_0}u(t_0)\rangle\rangle$$

- Integrating over quark fields yields

$$C_M(t_1-t_0) = \langle \mathrm{Tr}_{\{\underline{s},\sigma,c\}}\left(\Box_{t_1}\Gamma_{t_1}\Box_{t_1}M^{-1}(t_1,t_0)\Box_{t_0}\Gamma_{t_0}\Box_{t_0}M^{-1}(t_0,t_1)\right)\rangle$$

- Substituting the low-rank distillation operator $\Box$ reduces this to a **much smaller** trace:

$$C_M(t_1-t_0) = \langle \mathrm{Tr}_{\{\sigma,\mathcal{D}\}}\left[\Phi(t_1)\tau(t_1,t_0)\Phi(t_0)\tau(t_0,t_1)\right]\rangle$$

- $\Phi^{\alpha,a}_{\beta,b}$ and $\tau^{\alpha,a}_{\beta,b}$ are $(N_\sigma \times N_{\mathcal{D}}) \times (N_\sigma \times N_{\mathcal{D}})$ matrices.

$$\Phi(t) = V^\dagger(t)\Gamma_t V(t)$$

$$\tau(t,t') = V^\dagger(t)M^{-1}(t,t')V(t')$$

The “perambulator”

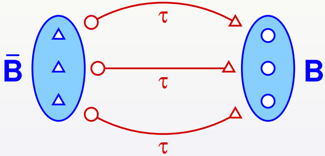
Isovector meson correlation functions

- To create a meson, we need to build functions that couple to quarks.
- Meson can be created by a quark bilinear. Appropriate gauge invariant creation operator (for isospin $I = 1$) would be

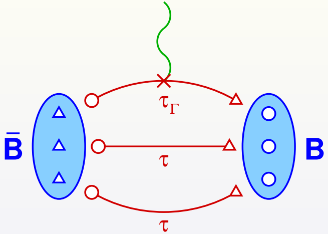
$$\Phi_{\text{meson}}(t) = \sum_{x} \bar{u}(\underline{x}, t)\Gamma U_{\mathcal{C}}(\underline{x}, \underline{y}; t) d(\underline{y}, t)$$

  where Γ is some appropriate Dirac structure, and $U_{\mathcal{C}}$ a product of (smeared) link variables.
- Operators that transform irreducibly under the lattice rotation group O_h are needed.

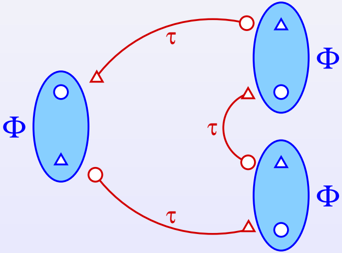
More diagrams



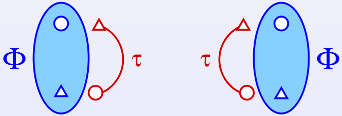
$\bar{B}_{abc}\tau_{aa'}\tau_{bb'}\tau_{cc'}B_{a'b'c'}$



$\bar{B}_{abc}\tau_{aa'}\tau^{\Gamma}{}_{bb'}\tau_{cc'}B_{a'b'c'}$

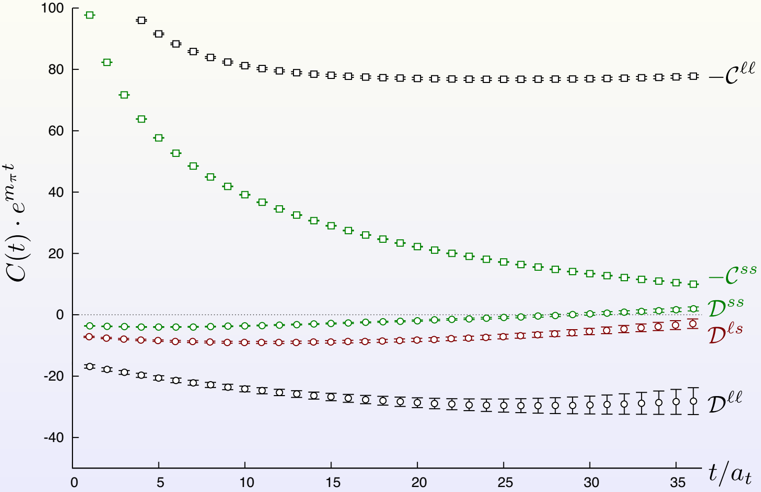


$\mathrm{Tr}\,[\Phi\tau\Phi\tau\Phi\tau]$



$\mathrm{Tr}\,[\Phi\tau]\ \mathrm{Tr}\,[\Phi\tau]$

Isoscalar meson (η') correlation function



- Correlation functions for $\bar{\psi}\gamma_5\psi$ operator, with different flavour content (s, l).
- 16^3 lattice (about 2 fm).

[arXiv:1102.4299]

Bad news - the price tag

- So far - good results on modest lattice sizes $N_s = 16^3 \equiv (1.9\text{fm})^3$.
- Used $N_{\mathcal{D}} = 64$ for mesons, $N_{\mathcal{D}} = 32$ for baryons

The problem:

- To maintain constant resolution, need $N_{\mathcal{D}} \propto N_s$
- **Budget:**

Fermion solutions	construct τ	$\mathcal{O}(N_s^{\mathbf{2}})$
Operator constructions	construct Φ	$\mathcal{O}(N_s^{\mathbf{2}})$
Meson contractions	$\mathrm{Tr}[\Phi\tau\Phi\tau]$	$\mathcal{O}(N_s^{\mathbf{3}})$
Baryon contractions	$\bar{B}\tau\tau\tau B$	$\mathcal{O}(N_s^{\mathbf{4}})$

- 32^3 lattice: $64 \times (\frac{32}{16})^3 = 512$ — too expensive.
- Some benefits in reduction in variance with N_s
- **Can stochastic estimation technology help?**

Stochastic estimation in the distillation space

- Construct a **stochastic identity matrix in $\mathcal{D}$**: introduce a vector η with $N_{\mathcal{D}}$ elements and with

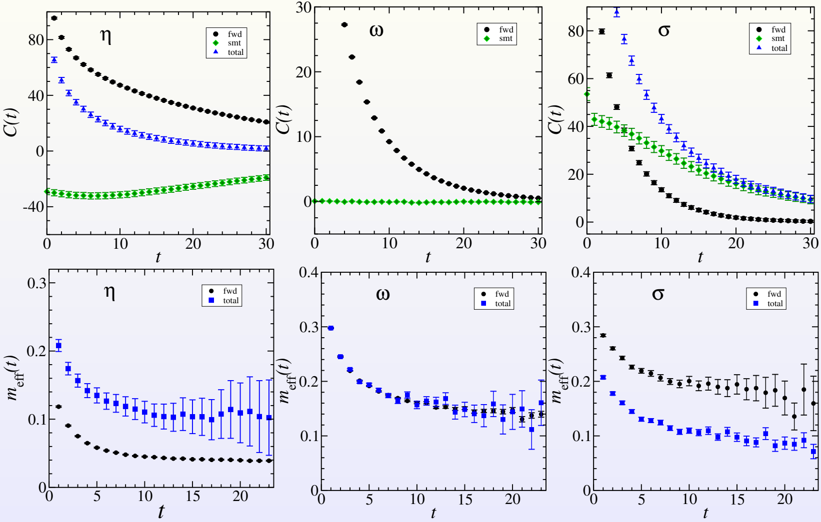
$$E[\eta_i] = 0 \text{ and } E[\eta_i \eta_j^*] = \delta_{ij}$$

- Now the distillation operator is written

$$\square = E[V\eta\eta^\dagger V^\dagger] = E[WW^\dagger]$$

- Introduces noise into computations
- **Dilution:** “thin out” the stochastic noise with N_η orthogonal projectors to make a variance-reduced estimator of $I_{\mathcal{D}} = E[WW^\dagger] = \sum_{k=1}^{N_\eta} E[V\mathcal{P}_k\eta\eta^\dagger\mathcal{P}_k V^\dagger]$, with $W_k = V\mathcal{P}_k\eta$, a $N_\eta \times (N_s \times N_c)$ matrix

Stochastic estimation: $I = 1, 0$ mesons



- propagation from $t - t$ is estimated differently from $t - t'$

[arXiv:1101.5398v1]

Summary

- QCD allows for **Exotic** mesons, absent in a simple quark model.
- There is new experimental interest in these objects.
- Studying exotic states on the lattice is a challenge:
  - **Statistical precision** needs good Monte Carlo technology
  - Quark fields are expensive to manipulate numerically
  - **Spin identification** on the lattice needs care
  - Exotics are **resonances**
- Progress recently on some of these challenges
- Timeline: new data from experiments in $\approx$ 2015