QCD Thermodynamics on the lattice

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Day I:

- Introduction: Dense Matter and Heavy Ion collisions
- Finite-T lattice QCD: Chiral symmetry and the hadron spectrum

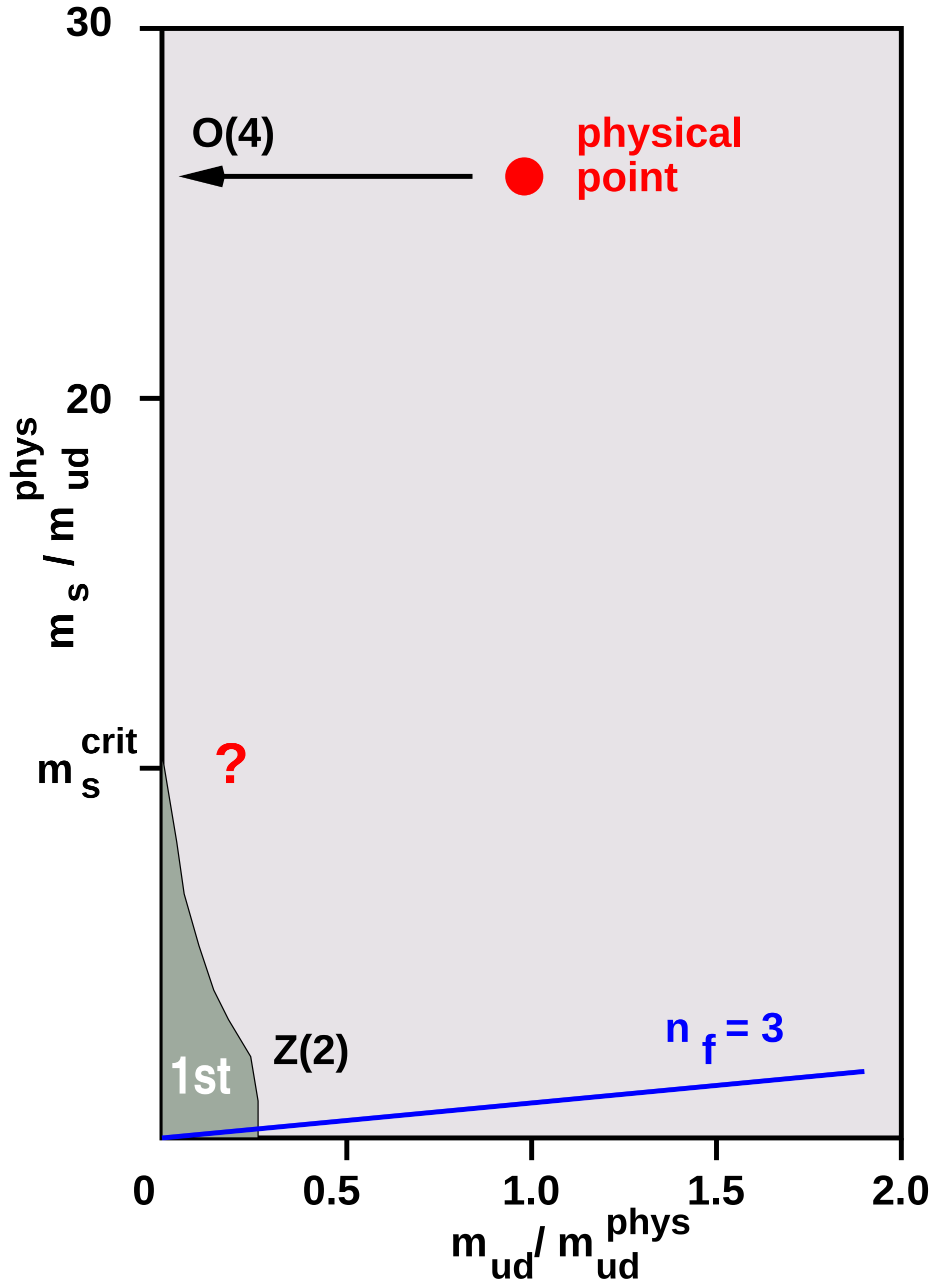
Day II:

- Chiral (phase) transition: O(4) scaling and T_c
- Deconfinement: Polyakov loop and $Z(3)$ symmetry, baryon number and electric charge fluctuations, the QCD equation of state
  - thermodynamics at $\mu_B \neq 0$ (lectures by C. Schmidt)

Helmholtz Summer School

Lattice QCD, Hadron Structure and Hadronic Matter
Dubna, Russia, 5-18 September, 2011

Phase diagram for $\mu_B = 0$



- drawn to scale

Is physics at the physical quark mass point sensitive to (universal) properties of the chiral phase transition?

physical point may be above m_s^{tric}

$N_\tau = 4, 6$; improved actions:

$\Rightarrow m_{ps}^{crit} \lesssim 70$ MeV

FK et al, NP(Proc.Suppl) 129 (2004) 614

G. Endrodi et al, PoS LAT 2007 (2007) 182

(also $N_\tau = 6$, unimp.)

2 (+1)-flavor QCD and O(N) spin models

physics of QCD at low energies as well as close to the chiral phase transition is described by effective, O(N) symmetric spin models

- $T = 0$: chiral symmetry breaking at $T = 0,\ m_q = 0$ as well as leading temperature and quark mass dependent corrections are related to universal properties of 4-dimensional, $O(4)$ symmetric spin models
- $T \simeq T_c$: chiral symmetry restoration at $T = T_c,\ m_q = 0$ as well as leading temperature and quark mass dependent corrections are related to universal properties of 3-dimensional, $O(4)$ symmetric spin models

R. Pisarski and F. Wilczek, PRD29 (1984) 338
K. Rajagopal and F. Wilczek, hep-ph/0011333
A. Pelissetto and E. Vicari, Phys. Rept 368 (2002) 549

Spontaneous Symmetry Breaking

$O(N)$ spin models in d-dimensions

- non-vanishing expectation value, M, of the scalar field, $\Phi_{||}$, parallel to the symmetry breaking field H
- $(N-1)$ transverse (Goldstone) modes give corrections for non-zero H (spin waves); controlled by M and the decay constant F for Goldstone modes

$$M_H = M_0 \left(1 - \frac{N-1}{32\pi^2} \frac{M_0 H}{F_0^4} \ln\left(\frac{M_0 H}{F_0^2 \Lambda_M}\right) + \mathcal{O}(H^2)\right) \quad , \quad d = 4$$

$$M_H = M_0 \left(1 + \frac{N-1}{8\pi} \frac{(M_0 H)^{1/2}}{F_0^3} + \mathcal{O}(H)\right) \quad , \quad d = 3$$

P. Hasenfratz and H, Leutwyler, NPB343, 241 (1990)

D.J. Wallace and R.K.P. Zia, PRB12, 5340 (1975)

Spontaneous Symmetry Breaking (cont.)

- (chiral) susceptibilities diverge below T_c for $H \to 0$

$$\chi_H = \frac{\mathrm{d}M_H}{\mathrm{d}H} \sim \langle \Phi_{||}^2 \rangle - \langle \Phi_{||} \rangle^2 \sim \begin{cases} H^{-1/2} & , \ d = 3 \\ -\ln H & , \ d = 4 \end{cases}$$

- divergence in the zero-field (chiral) limit

$$\chi_{H=0}(T) = \begin{cases} \infty & , \ T \le T_c \\ A(T - T_c)^{-\gamma} & , \ T > T_c \end{cases}$$

- divergence at T_c

$$\chi_H(T = T_c) = H^{1/\delta - 1} \ , \quad T = T_c$$

crit. exp. O(2) [O(4)]: $\gamma = 1.32$ [1.45], $1 - 1/\delta = 0.79$ [0.79]

Critical behavior & chiral limit of QCD

- Universal critical behavior: $f(T, \mu_q, m_q) = f_s + f_r$
  existence of (hyper-)scaling relations between critical exponents suggests that f_s is a homogenous function with a free 'scale parameter' b

$$f_s(T, \mu_q, m_q) \quad = \quad b^{-1} f_s(t b^{y_t}, h b^{y_h})$$

$$h = m_q/T \quad , \quad t = \left| \frac{T - T_c}{T_c} \right| + A\mu_q^2$$

- two relevant fields $t, \; h$;
  h couples to symmetry breaking operators;
  t depends (to leading order) on all couplings/parameters that do not break the symmetry

Critical behavior & chiral limit of QCD

- Universal critical behavior (thermal)

  $m_q \equiv 0$: $f_s(T, \mu_q, 0) = b^{-1} f_s(t b^{1/(2-\alpha)}) \sim t^{2-\alpha}$

  $\alpha < 0$ for $O(N)$ $\rightarrow$ specific heat has a cusp

  $t \equiv 0$: $f_s(0, \mu_q, m_q) = b^{-1} f_s(m_q b^{1/(1+1/\delta)}) \sim m_q^{1+1/\delta}$

- O(N) models:

  $\alpha = (2y_t - 1)/y_t$, $\beta = (1 - y_h)/y_t$ and $\delta = y_h/(1 - y_h)$

N	α	β	δ
2	-0.007(6)	0.3455(20)	4.808(7)
4	-0.19(6)	0.38(1)	4.82(5)

hyper-scaling relations: $\alpha + 2\beta + \gamma = 2$, $d\nu = 2 - \alpha$

Critical behavior & chiral limit of QCD

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  $t \equiv 0$: $f_s(0, \mu_q, m_q) = b^{-1} f_s(m_q b^{1/(1+1/\delta)}) \sim m_q^{1+1/\delta}$

- fluctuations of Goldstone modes influence behavior in the chiral limit also away from (thermal) criticality

$$\langle \bar{\psi}\psi \rangle \sim \begin{cases} c(T)\sqrt{m_q} + d(T) m_q + \text{regular} & T < T_c \\ c_\delta m_q^{1/\delta} + d(T_c) m_q + \text{regular} & T = T_c \\ d(T) m_q + \text{regular} & T > T_c \end{cases}$$

$$\Rightarrow \quad \chi_m \sim \left. \frac{\partial \langle \bar{\psi}\psi \rangle}{\partial m_q} \right|_{m_q = 0} \sim \begin{cases} \infty & T \leq T_c \\ t^{-\gamma} & T > T_c \end{cases}$$

O(N) scaling and chiral transition

- thermodynamics in the vicinity of a critical point:

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(V,T) = t^{2-\alpha} f_s(t/h^{1/\beta\delta}) + f_r(V,T)$$

with scaling fields, $t \equiv \frac{1}{t_0} \frac{T-T_c}{T_c} \quad , \quad h \equiv \frac{1}{h_0} H \quad , \quad (H \sim m_q)$

- In the vicinity of $(t,h) = (0,0)$ the chiral order parameter and its susceptibility are given in terms of scaling functions

$$M = h^{1/\delta} f_G(z) \quad , \quad \chi_M = \partial M/\partial h = h^{1/\delta-1} f_\chi(z)$$

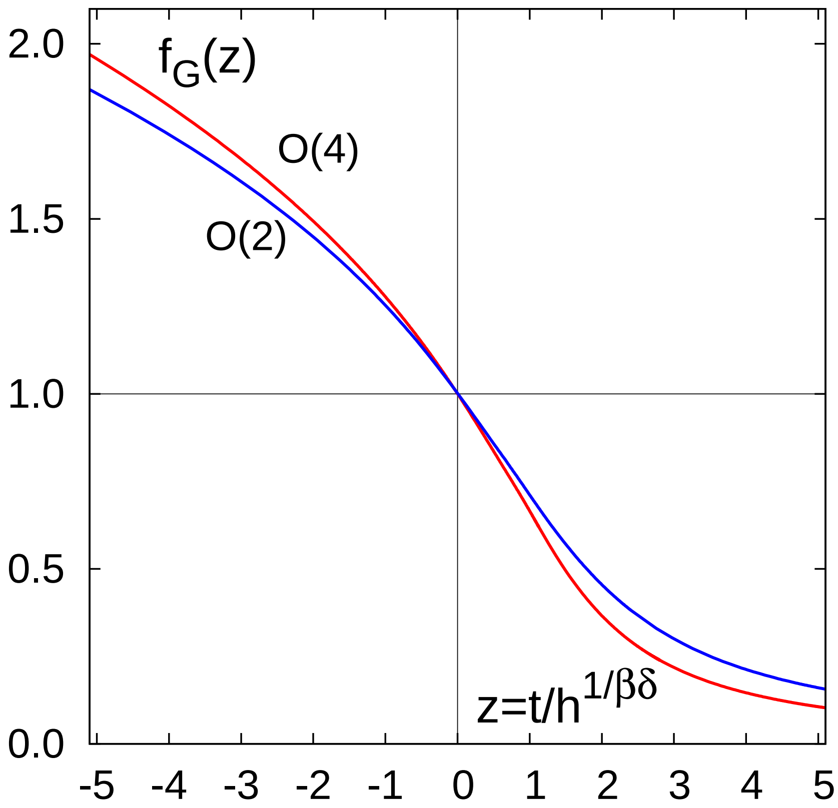
$$\chi_t = \partial M/\partial T = \frac{1}{t_0 T_c} h^{(\beta-1)/\delta\beta} f_G'(z)$$

$$f_\chi(z) = \frac{1}{\delta}\left(f_G(z) - \frac{z}{\beta} f_G'(z) \right)$$

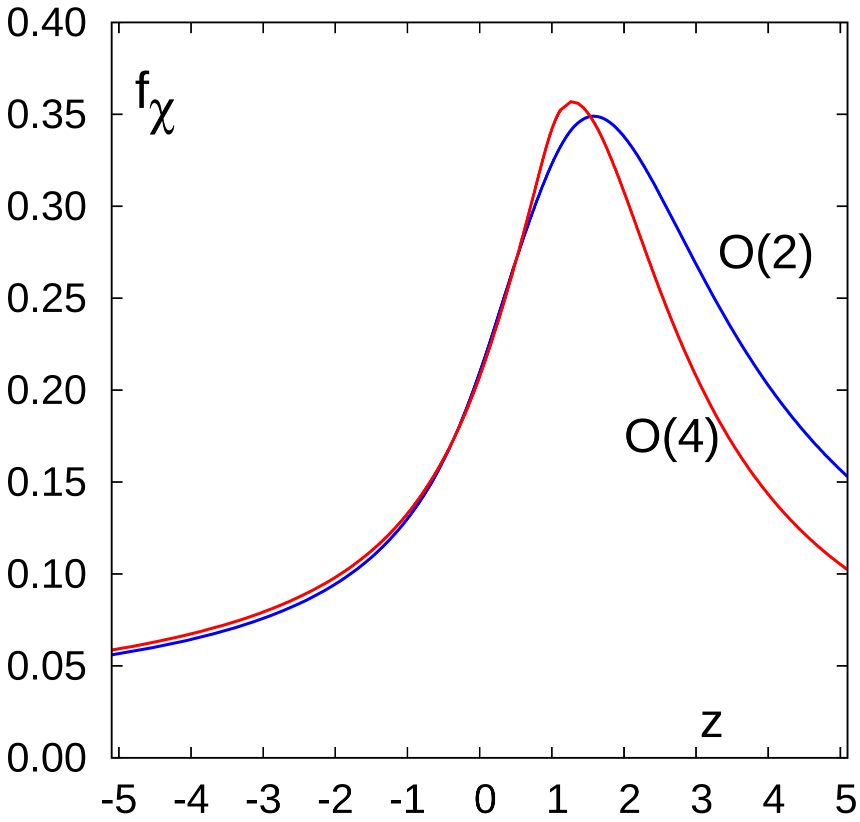
Scaling functions from studies of $O(N)$ spin models in 3-dimensions

3-d, O(N) scaling functions

$M = h^{1/\delta} f_G(z)$



$\chi_M = \partial M / \partial h = h^{1/\delta - 1} f_\chi(z)$

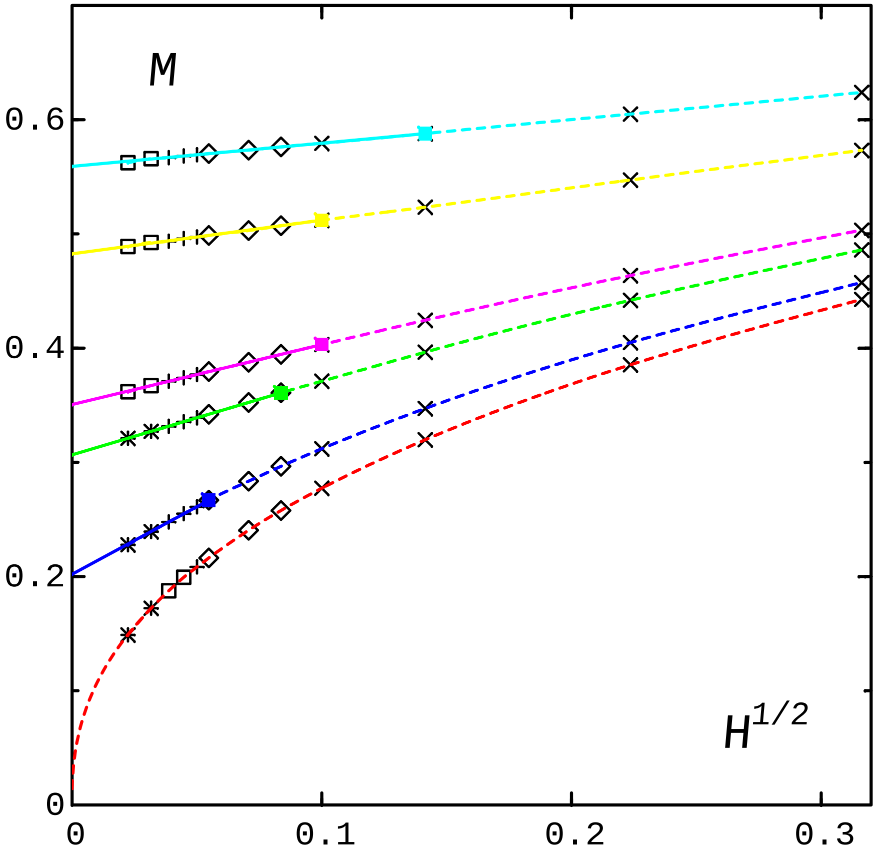


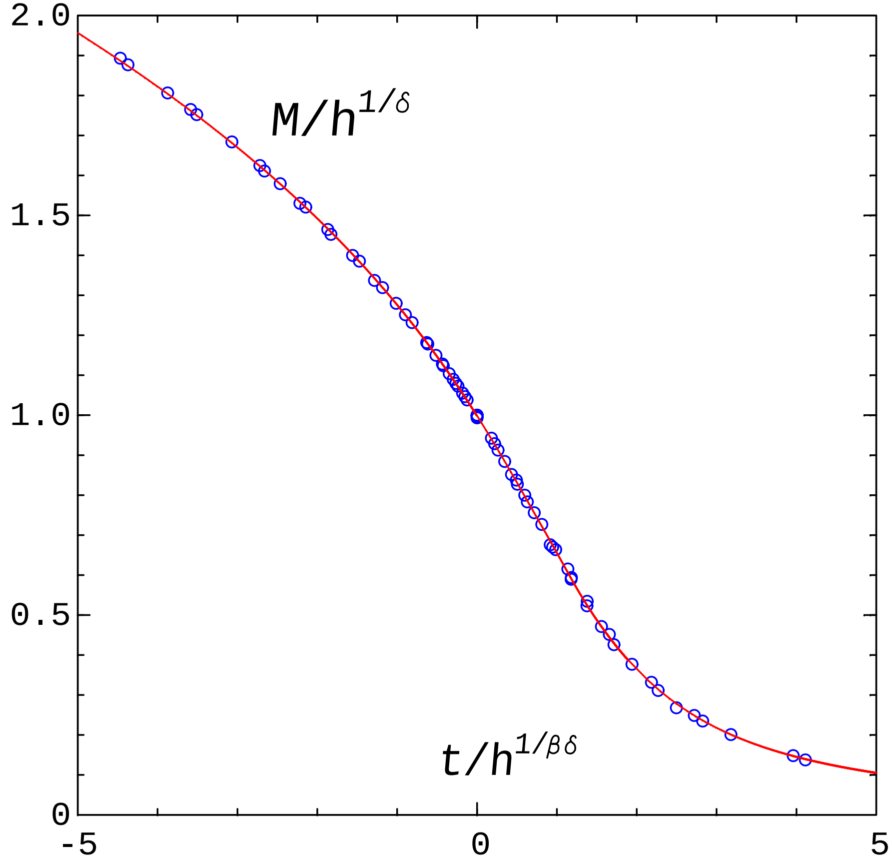
- 2-flavor QCD is in the same universality class as 3-d, O(4) spin models
- staggered fermions only have a O(2) symmetry at finite lattice spacing

O(2): J. Engels et al., 2001

O(4): J. Engels et al., 2003

3-d, O(4) models close to T_c





J. Engels and T. Mendes 2000

- condensate shows $\sqrt{H}$ dependence and $O(4)$ scaling
- magnetic equation of state reflects $O(4)$ scaling including Goldstone modes

Scaling in (2+1)-flavor QCD

$\mathcal{O}(a^2)$ improved staggered fermions

S. Ejiri et al. (BNL-Bielefeld-GSI), arXiv:0909.5122

A. Bazavov et al. (HotQCD), in preparation

chiral condensate

$$\langle\bar{\psi}\psi\rangle_l = \frac{1}{N_\sigma^3 N_\tau}\frac{1}{4}\frac{\partial \ln Z}{\partial m_l a}$$

order parameter: dimensionless; multiplicative renormalized

$$M_b \equiv \frac{m_s\langle\bar{\psi}\psi\rangle_l}{T^4}$$

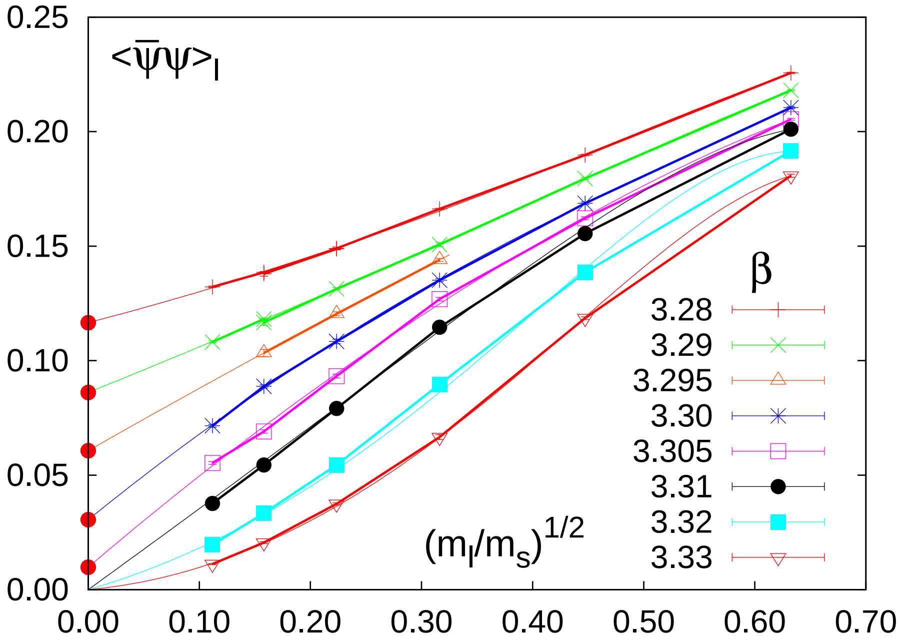
scaling fields: $h \equiv \frac{1}{h_0}\frac{m_l}{m_s}$; $t = \frac{1}{t_0}\frac{T-T_c}{T_c}$

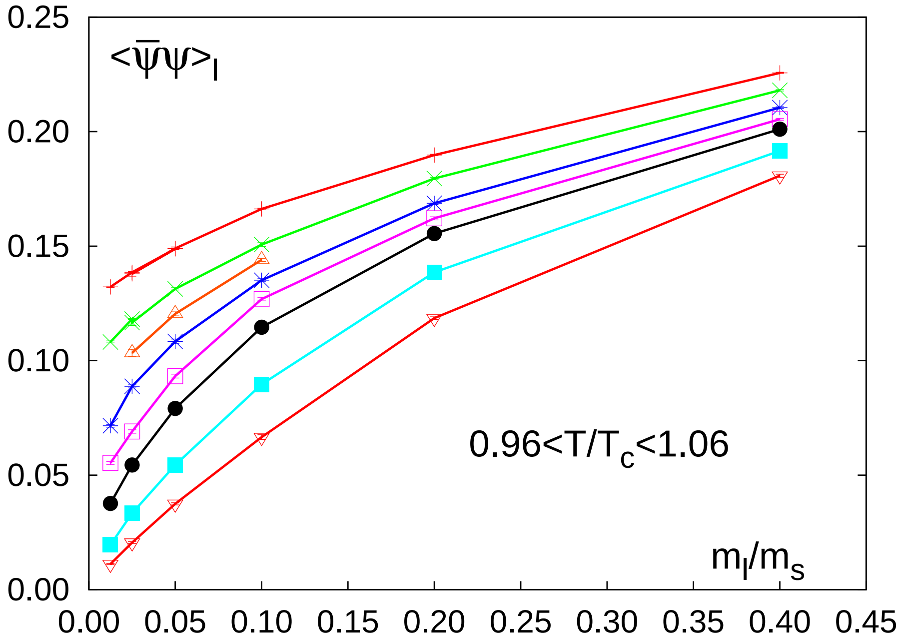
$$z = t/h^{1/\beta\delta}$$

Chiral condensate: $N_\tau = 4$:

(S. Ejiri et al. (BNL-Bielefeld-GSI), arXiv:0909.5122)

$$\langle\bar\psi\psi\rangle_l = \frac{1}{N_\sigma^3 N_\tau}\frac{1}{4}\frac{\partial \ln Z}{\partial m_l a}$$





- evidence for $\sqrt{m_l}$ term in $\langle\bar\psi\psi\rangle$

  for orientation: $\beta = 3.28$ $T \simeq 188$ MeV,
  $\beta = 3.30$ $T \simeq 196$ MeV

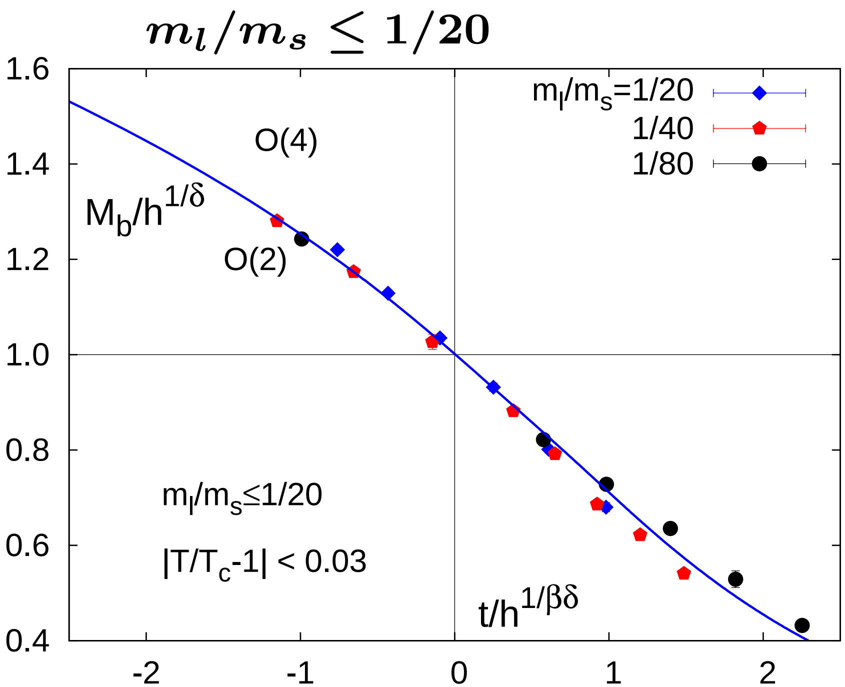
- Statistics:

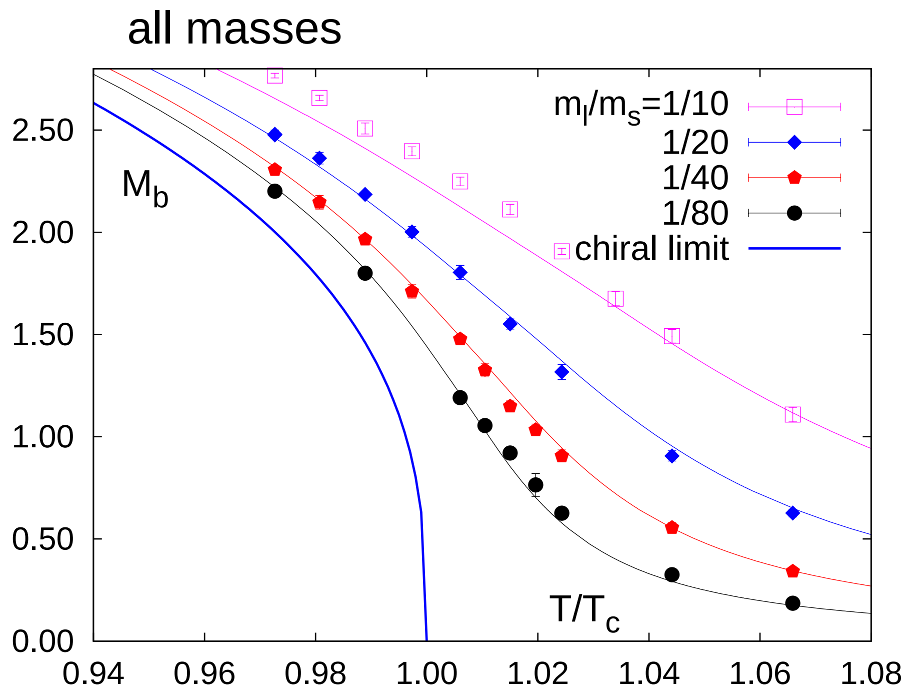
  20.000-40.000 trajectories per $(\beta,\ m_q)$

O(N) scaling analysis; p4-action

$$M \equiv h^{1/\delta} f_G(z) \quad ; \quad z = t/h^{1/\beta\delta}$$

- 3 parameter fit: t_0, h_0, T_c $\quad t = \frac{1}{t_0}\frac{T - T_c}{T_c}\ , \ h = \frac{1}{h_0}\frac{m_l}{m_s}$
- use only data for $m_l/m_s \leq 1/20$, $\beta \in [3.285, 3.31]$





$\Rightarrow$ t_0, h_0, T_c are non-universal parameters, characteristic to QCD

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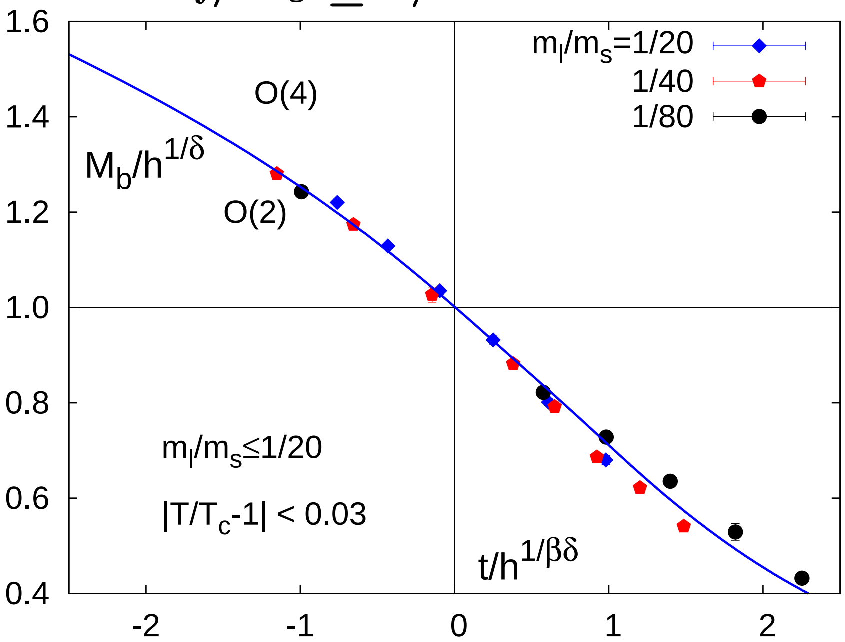
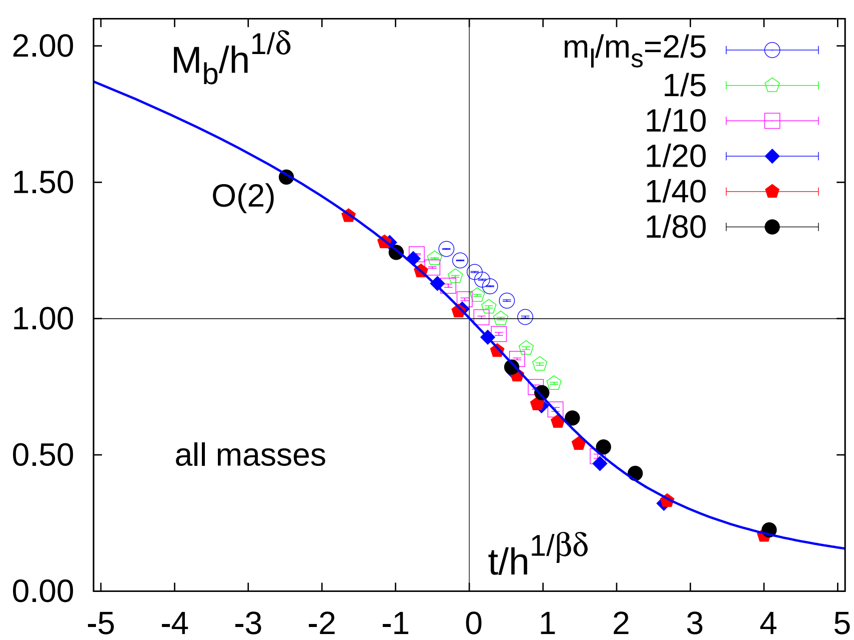
$m_l/m_s \leq 1/20$



all masses



$\Rightarrow$ t_0, h_0, T_c are non-universal parameters, characteristic to QCD

Scaling of the chiral susceptibility

$$M = h^{1/\delta} f_G(z) \quad , \quad \chi_M = \partial M/\partial h = h^{1/\delta - 1} f_\chi(z)$$

$$t = \frac{1}{t_0}\frac{T - T_c}{T_c} \; , \; h = \frac{1}{h_0}\frac{m_l}{m_s}$$

$$\chi_m/T^2 \;=\; N_\tau^3 \frac{\mathrm{d}\langle\bar\psi\psi\rangle_q}{\mathrm{d}(m_q/T)}$$

$$f_\chi(z) \;=\; \chi_m h_0/h^{1/\delta-1} \equiv h_0^{1/\delta}\left(\frac{m_l}{m_s}\right)^{1-1/\delta} \chi_m \; ,$$

$f_\chi(z)$ has maximum at $z \equiv z_p$ $\Rightarrow$ scaling of pseudo-critical temperatures:

$$z = z_p \;\Leftrightarrow\; t = h^{1/\beta\delta} \Leftrightarrow T(m_l) = T_c + A\left(\frac{m_l}{m_s}\right)^{1/\beta\delta}$$

Chiral Susceptibility

chiral condensate

$$\langle\bar\psi\psi\rangle_l = \frac{n_f}{4}\frac{1}{N_\sigma^3 N_\tau}\mathrm{Tr}\langle M_l^{-1}\rangle, \ ,$$

chiral susceptibility:

$$\chi_m(T) \quad = \quad \frac{\partial\langle\bar\psi\psi\rangle_l}{\partial m_l} = \chi_{disc} + \chi_{con} \ ,$$

$$\begin{aligned}
\chi_{disc} \quad &= \quad \frac{n_f^2}{16 N_\sigma^3 N_\tau}\left\{\langle\left(\mathrm{Tr} M_l^{-1}\right)^2\rangle - \langle\mathrm{Tr} M_l^{-1}\rangle^2\right\} \ , \\
\chi_{con} \quad &= \quad -\frac{n_f}{4}\mathrm{Tr}\sum_x\langle\, M_l^{-1}(x,0) M_l^{-1}(0,x)\,\rangle
\end{aligned}$$

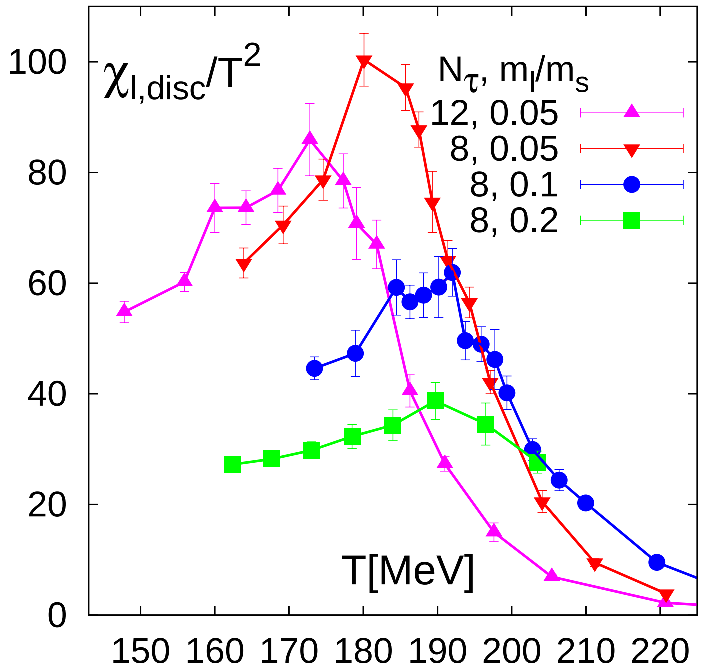
(staggered fermion normalization)

Scaling of the chiral susceptibility

$$\frac{\chi_m}{T^2} \sim \left(\frac{m_l}{m_s}\right)^{-1+1/\delta} \qquad (T = T_c)$$

$$\sim \left(\frac{m_l}{m_s}\right)^{-1/2} \qquad (T < T_c)$$

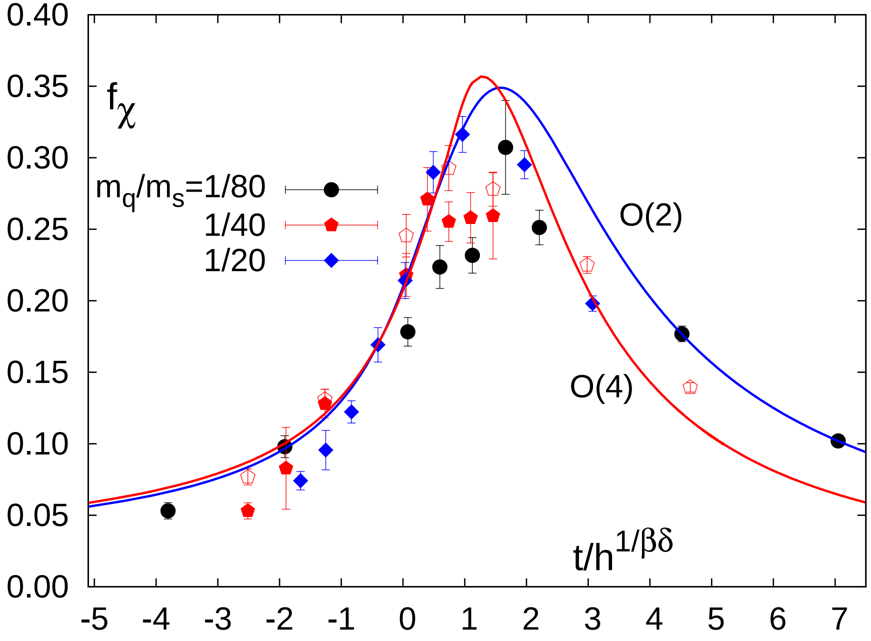
$$\sim \text{const.}(T - T_c)^{-\gamma} \quad (T > T_c)$$

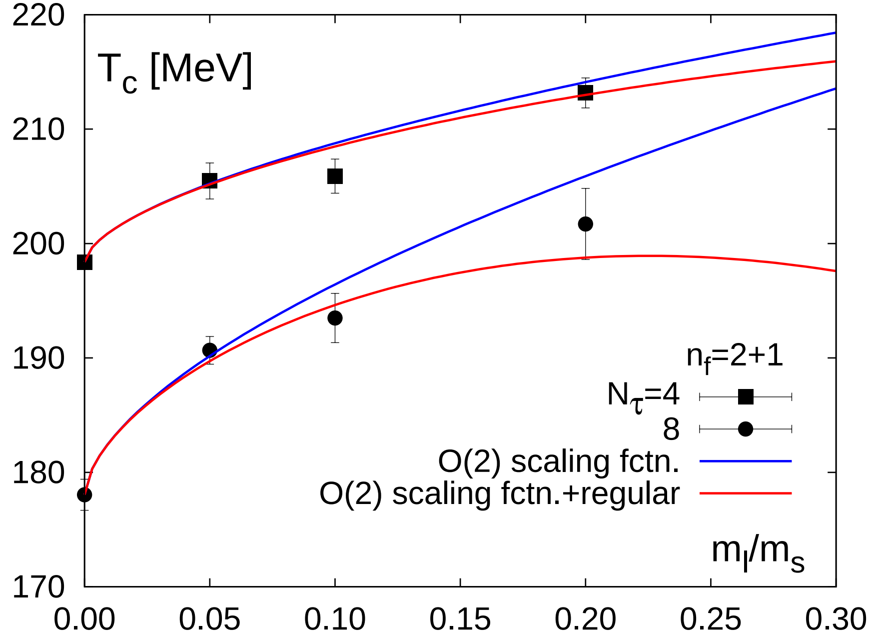


asqtad action

Scaling of the chiral susceptibility

$$f_\chi(z) \;=\; \frac{\chi_m}{T^2}\, h_0/h^{1/\delta-1} \equiv h_0^{1/\delta}\left(\frac{m_l}{m_s}\right)^{1-1/\delta}\frac{\chi_m}{T^2}\,,$$





p4-action

Pseudo-critical temperatures

- calculations with HISQ and asqtad actions at three values of the lattice cut-off: $N_\tau = 6,\ 8$ and 12
- pseudo-critical temperatures defined in terms of peaks in the chiral susceptibility
- control dependence on the choice of the temperature scale, e.g.

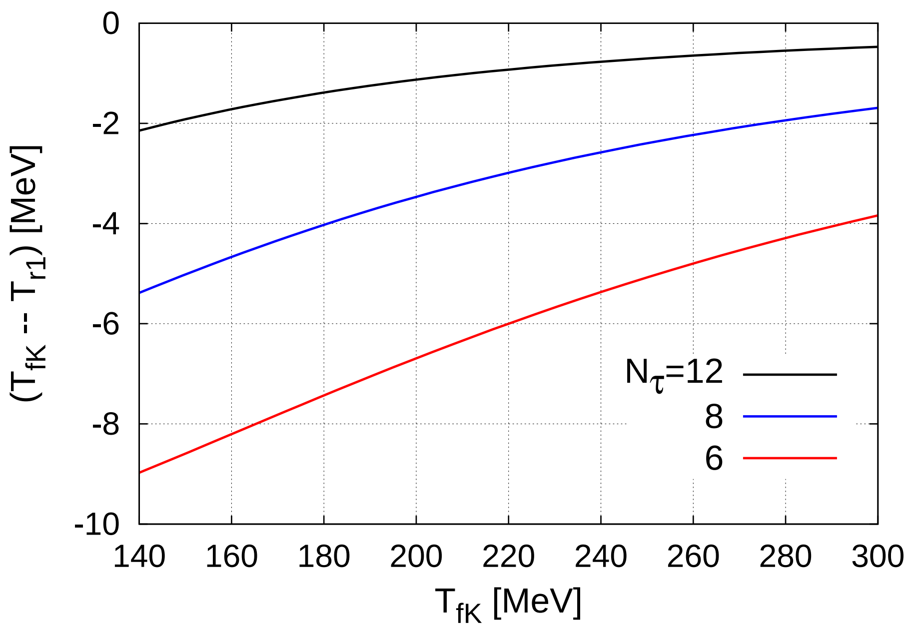
$$\frac{1}{T} = N_\tau a(g^2)$$

need to determine $a(g^2)$ through calculation of an "experimentally" known observable. For instance, a mass – $m_H a(g^2)$.

$$\Rightarrow \frac{m_H}{T} = m_H a(g^2) N_\tau$$

Pseudo-critical temperatures

- calculations with HISQ and asqtad actions at three values of the lattice cut-off: $N_\tau = 6,\ 8$ and 12
- pseudo-critical temperatures defined in terms of peaks in the chiral susceptibility
- control dependence on the choice of the temperature scale, e.g. r_1 or f_K or...



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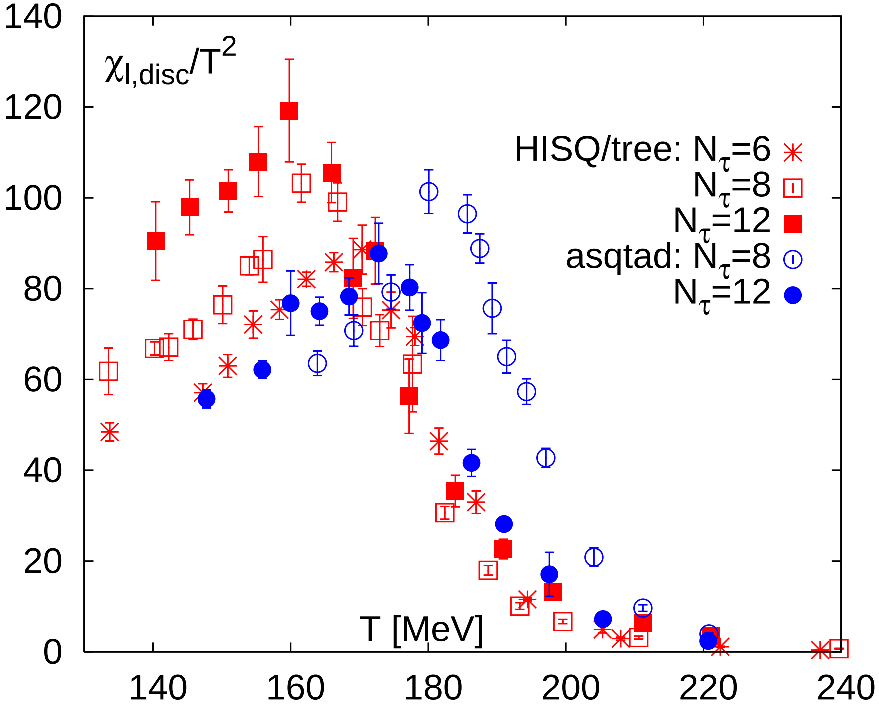
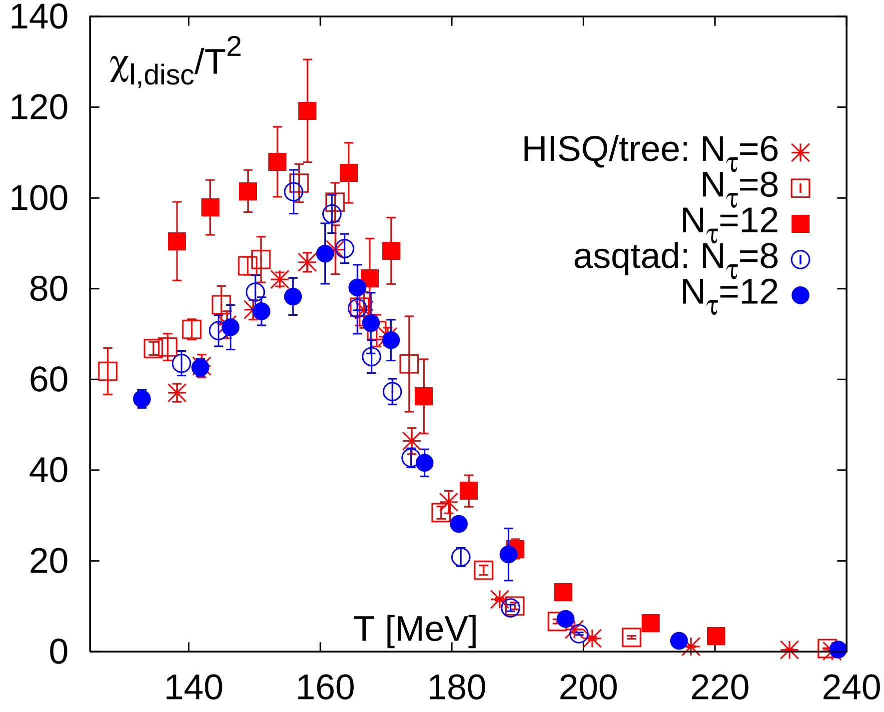


r_1 scale

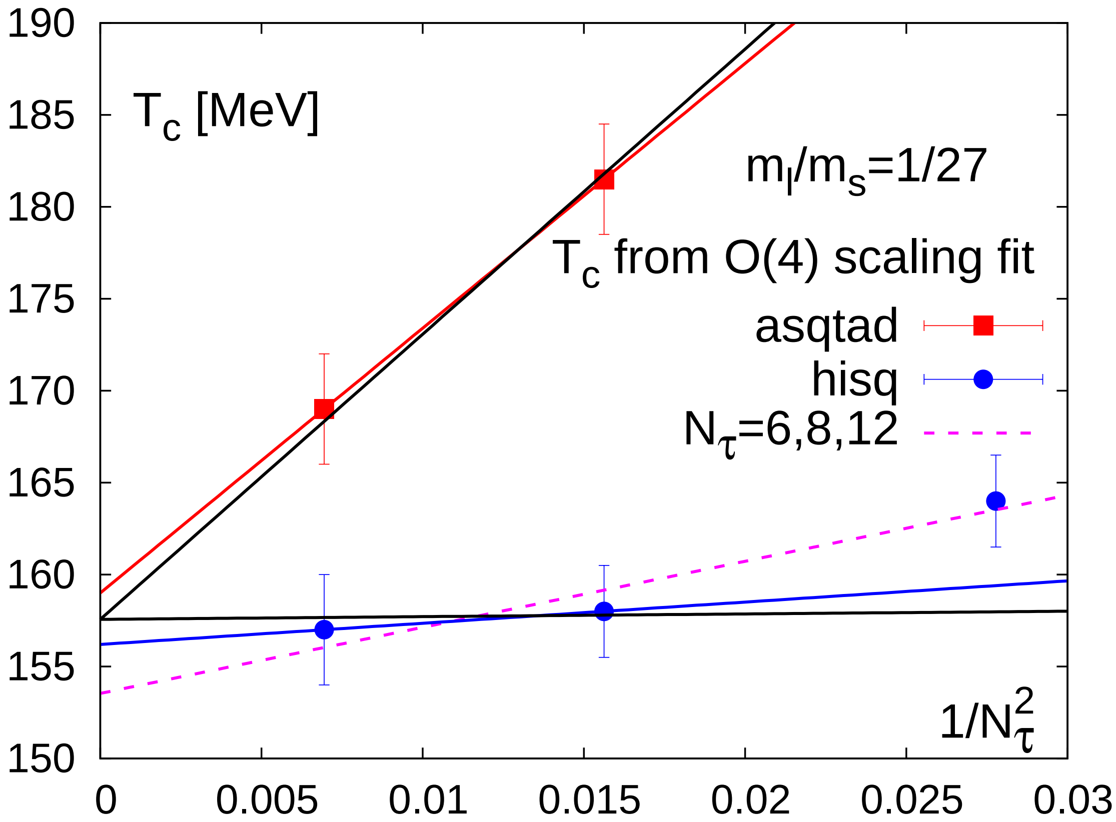


f_K scale

HotQCD preliminary

Pseudo-critical temperatures continuum extrapolation

$$T_{pc}(m_q^{phys}, m_s^{phys}) = (157 \pm 6) \text{ MeV}$$



HotQCD preliminary

consistent with: Y. Aoki et al, Phys. Lett. B643, 46 (2006)

Detecting the QCD phase transition on the lattice

Deconfinement

and

chiral symmetry restoration

Confinement and deconfinement

confinement

- stick together, find a comfortable separation
- controlled by confinement potential

$$V(r) = -\frac{4}{3}\frac{\alpha(r)}{r} + \sigma r$$

Confinement and deconfinement

confinement

- stick together, find a comfortable separation
- controlled by confinement potential

$$V(r) = -\frac{4}{3}\frac{\alpha(r)}{r} + \sigma r$$

$$\alpha(r) \equiv \frac{g^2(r)}{4\pi} \sim \frac{1}{\ln(1/r\Lambda)}$$

deconfinement

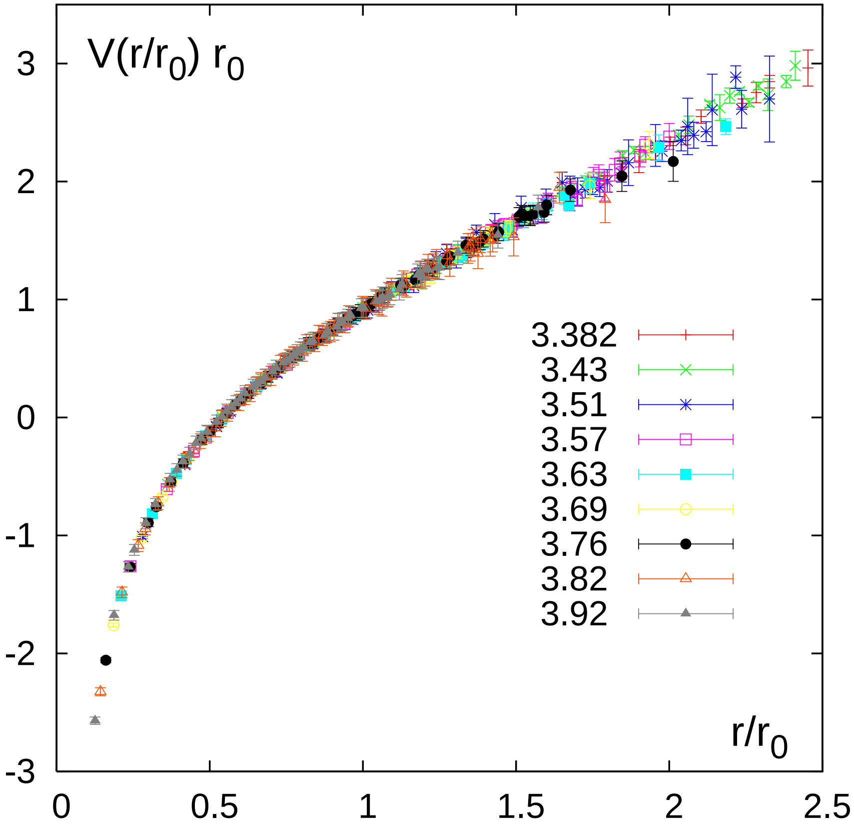
- free floating in the croud
- average distance always smaller than r_{af}:

$$r_{af} = \sqrt{\frac{4}{3}\frac{\alpha(r)}{\sigma}} \simeq 0.25\,\mathrm{fm}$$

Detecting the QCD phase transition on the lattice

Deconfinement

phase transition $\Leftrightarrow$ breaking/restoration of global symmetries



$\Updownarrow$

exist only for

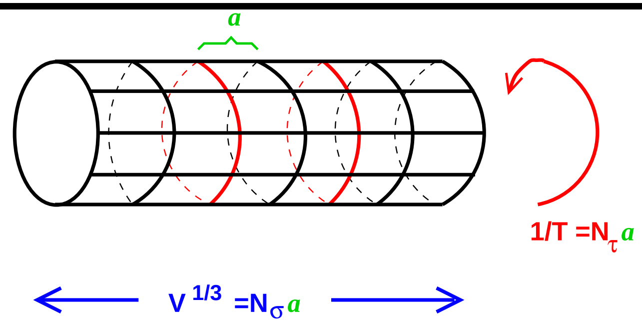
$m_q = 0$ and $m_q \to \infty$

confinement: $m_q \to \infty$

$$\lim_{r\to\infty} V_{\bar{q}q}(r) \to \infty$$

string tension $\sigma > 0$

Detecting the QCD phase transition on the lattice



Deconfinement

phase transition ⇔ breaking/restoration of global symmetries

⇕

exist only for

$m_q = 0$ and $m_q \to \infty$

heavy quark free energy:

$$\boxed{F_{\bar{q}q}(\vec{x},T) \equiv -T\ \ln G_L(\vec{x},T)}$$

$$G_L(\vec{x},T) = \langle \mathrm{Tr} L_{\vec{x}}\ \mathrm{Tr} L^\dagger_{\vec{0}} \rangle$$

Polyakov loop:

$$L_{\vec{x}} = \exp\left(i \int_0^{1/T} \mathrm{d}x_0 \mathcal{A}_0(x_0,\vec{x}) \right)$$

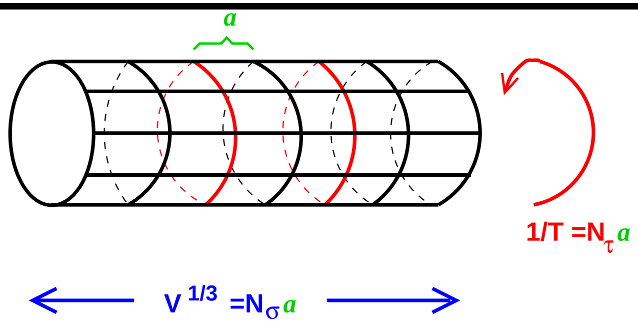
$$L = V^{-1} \sum_{\vec{x}} \mathrm{Tr} L_{\vec{x}}$$

confinement: $m_q \to \infty$

$$\lim_{r\to\infty} V_{\bar{q}q}(r) \to \infty$$

string tension $\sigma > 0$

Detecting the QCD phase transition on the lattice



Deconfinement

phase transition $\Leftrightarrow$ breaking/restoration of global symmetries

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$$\boxed{F_{\bar{q}q}(\vec{x}, T) \equiv -T \ \ln G_L(\vec{x}, T)}$$

$$G_L(\vec{x}, T) = \langle \mathrm{Tr} L_{\vec{x}} \ \mathrm{Tr} L^\dagger_{\vec{0}} \rangle$$

Polyakov loop expectation value:

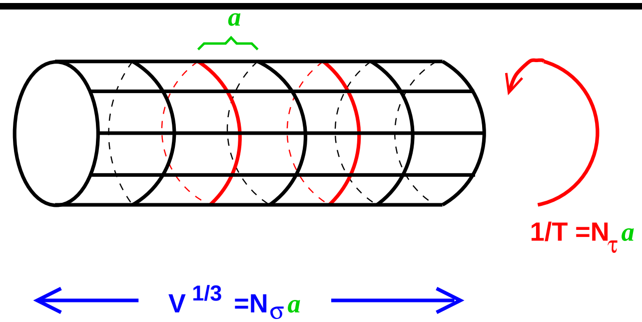
$$\boxed{\langle L \rangle \equiv \left(\lim_{|\vec{x}| \rightarrow \infty} G_L(\vec{x}, T) \right)^{1/2}}$$

confinement: $\underline{m_q \rightarrow \infty}$

$$\lim_{r \rightarrow \infty} V_{\bar{q}q}(r) \rightarrow \infty$$

string tension $\sigma > 0$

Detecting the QCD phase transition on the lattice



Deconfinement

phase transition $\Leftrightarrow$ breaking/restoration of global symmetries

$\Updownarrow$

exist only for

$m_q = 0$ and $m_q \to \infty$

global symmetry breaking

$(L_{\vec{x}} \to z L_{\vec{x}},\ z \in Z(3)) \Rightarrow (L \to zL)$

$\Rightarrow \langle L \rangle > 0$, if $Z(3)$ spontaneously broken

deconfinement:

$$\langle L \rangle > 0 \Leftrightarrow \lim_{|\vec{x}| \to \infty} F_{\bar{q}q}(\vec{x}, T) < \infty$$

confinement: $m_q \to \infty$

$$\lim_{r \to \infty} V_{\bar{q}q}(r) \to \infty$$

string tension $\sigma > 0$

Detecting the QCD phase transition on the lattice

Deconfinement

phase transition $\Leftrightarrow$ breaking/restoration of global symmetries

$\Updownarrow$

exist only for

$m_q = 0$ and $m_q \to \infty$

heavy quark free energy:

$$F_{\bar{q}q}(\vec{x}, T) \equiv -T \;\ln G_L(\vec{x}, T)$$

Polyakov loop expectation value:

$$\langle L \rangle \equiv \left(\lim_{|\vec{x}| \to \infty} G_L(\vec{x}, T) \right)^{1/2}$$

Polyakov loop susceptibility:

$$\chi_L \equiv \langle L^2 \rangle - \langle L \rangle^2$$

confinement: $m_q \to \infty$

$$\lim_{r \to \infty} V_{\bar{q}q}(r) \to \infty$$

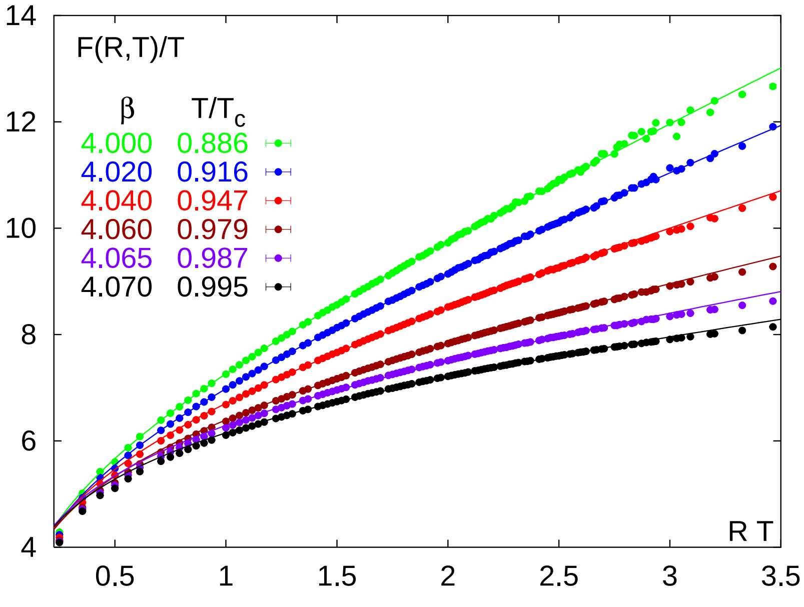
string tension $\sigma > 0$

Detecting the QCD phase transition on the lattice

Deconfinement $(m_q = \infty)$

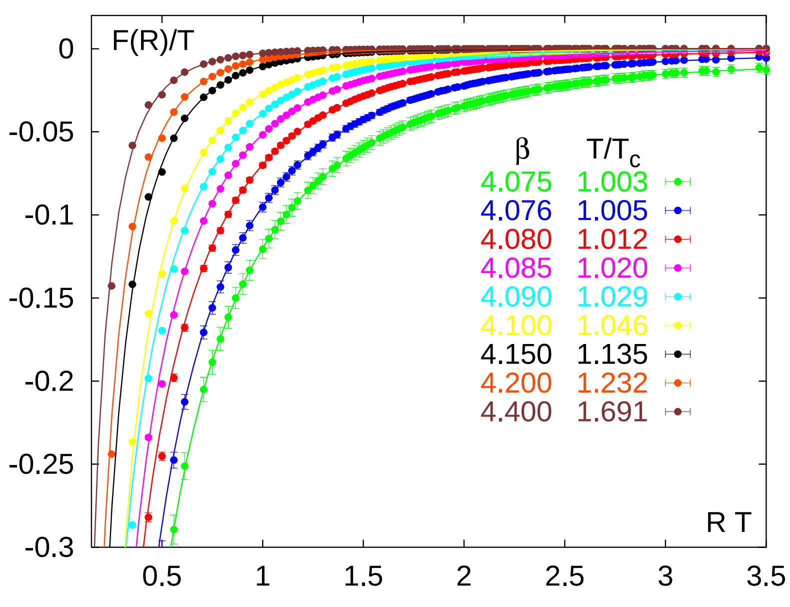
confined phase $(T < T_c)$:

$$\frac{F(R,T)}{T} = \frac{\sigma(T)}{T^2} RT + \ln(RT)$$



deconfined phase $(T > T_c)$:

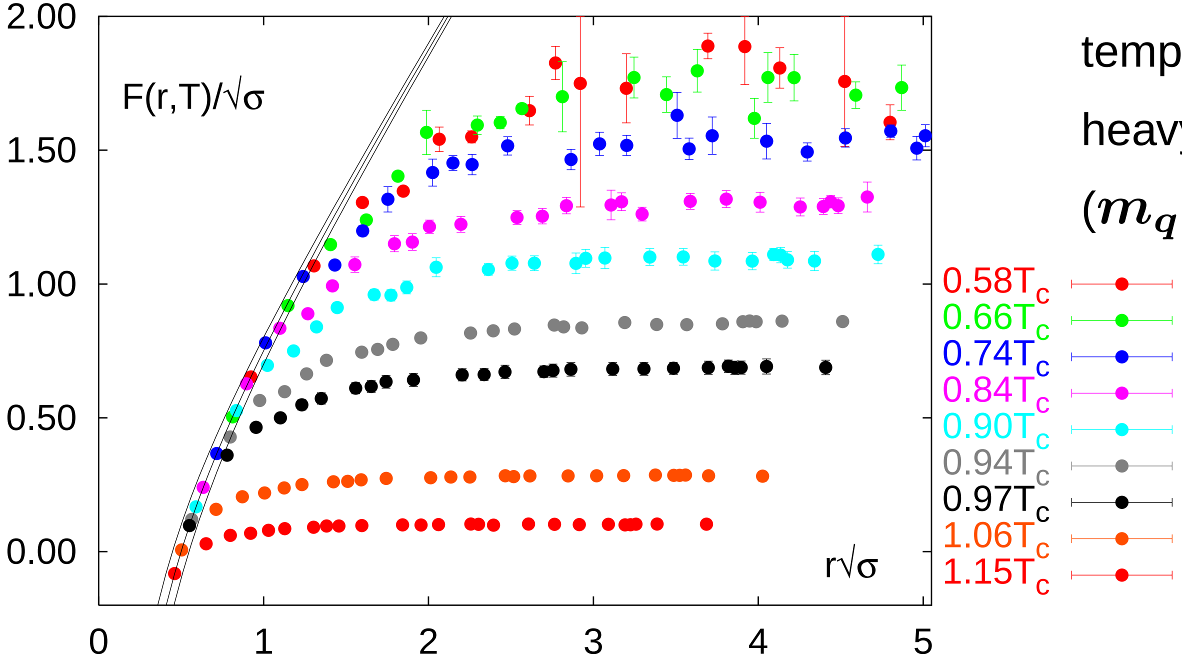
$$\frac{F(R,T)}{T} = \frac{\alpha}{(RT)^n} e^{-m_D R} + \text{const.}$$



Detecting the QCD phase transition on the lattice

Deconfinement $(m_q < \infty)$

2-flavour QCD simulation
on a $16^3 \times 4$ lattice



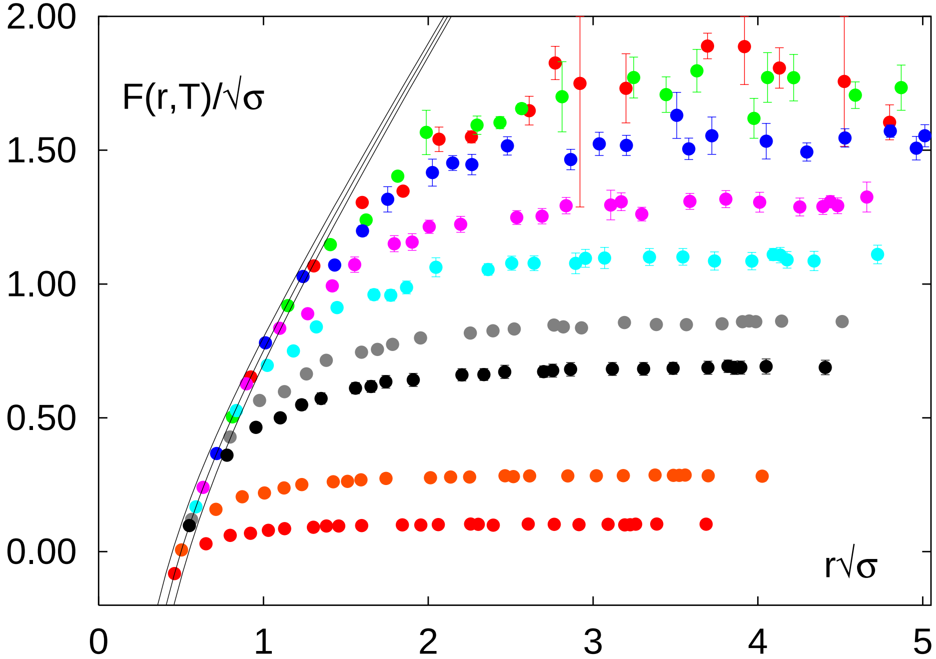
temperature dependence of heavy quark free energy $(m_q/T = 0.4)$

$\Uparrow \sim 1\ \mathrm{fm}$: string breaking

Detecting the QCD phase transition on the lattice

Deconfinement $(m_q < \infty)$

L not an order parameter; non-singular?



$\Leftarrow \ |\langle L \rangle|^2 > 0$

string breaking for $m_q < \infty$

shifts gradually to smaller distances

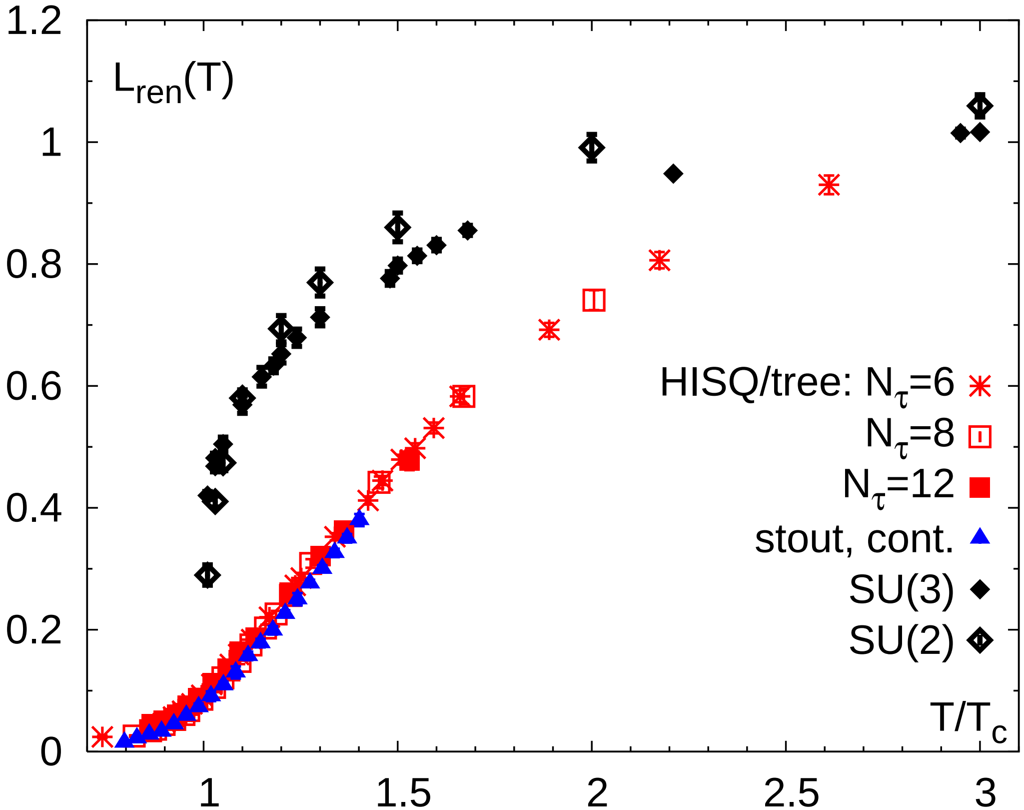
at higher temperatures

$\Uparrow \sim 1\ \mathrm{fm}$: string breaking

Remormalized Polyakov loop in QCD

(2+1)-flavor QCD: $24^3 \times 6$, $32^3 \times 8$, $48^3 \times 12$

$$L_{\rm ren} \equiv Z^{N_\tau}(\beta)\langle L \rangle$$



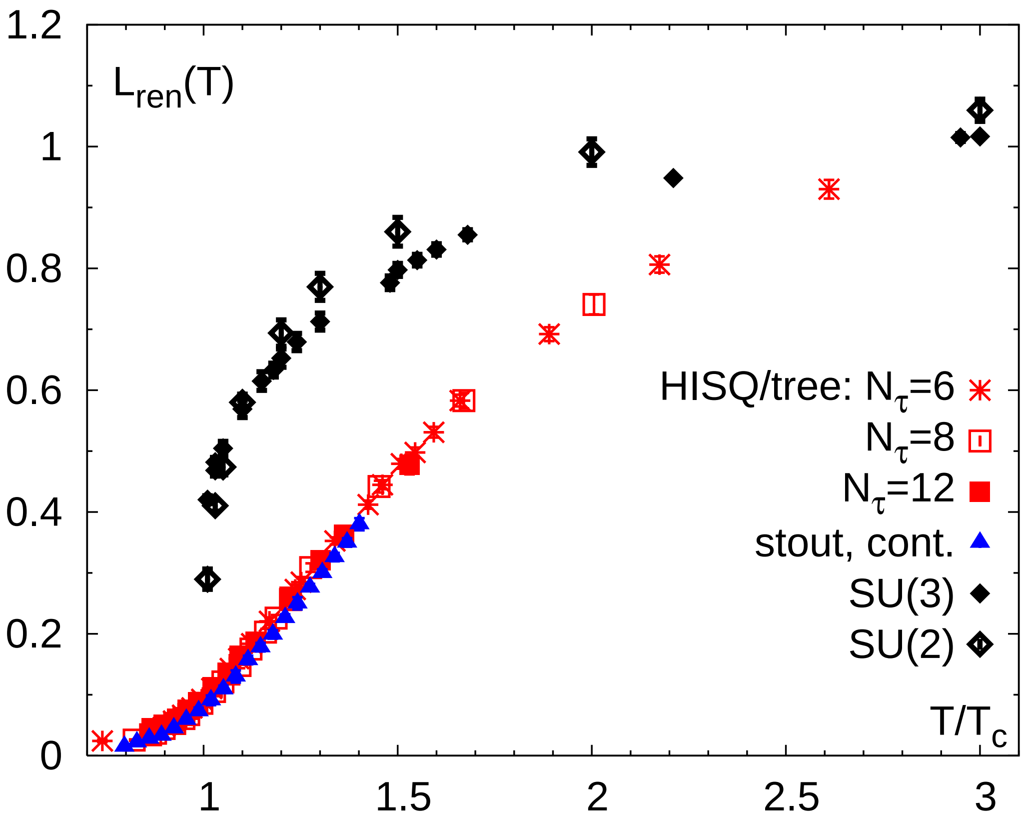
A. Bazavov et al. (HotQCD Collaboration)

arXiv:1107.5027

Remormalized Polyakov loop in QCD

(2+1)-flavor QCD: $24^3 \times 6,\ 32^3 \times 8,\ 48^3 \times 12$

$L_{\rm ren} \equiv Z^{N_\tau}(\beta)\langle L\rangle$ not a good order parameter for $m_q < \infty$



need a deconfinement criterion

that is linked to

critical behavior

also for $m_q \to 0$

A. Bazavov et al. (HotQCD Collaboration) arXiv:1107.5027

Quark number susceptibility...
...and its susceptibility

- rapid change in quark/baryon/strangeness number susceptibility reflects change in mass of the carrier of these quantum numbers ⇔ DECONFINEMENT
- quark number susceptibility feels nearby singular point just like the energy density

scaling field: $t = \left|\frac{T - T_c}{T_c}\right| + A\left(\frac{\mu_q}{T_c}\right)^2 \quad , \quad \mu_{crit} = 0$

singular part: $f_s(T, \mu_q) = b^{-1} f_s(tb^{1/(2-\alpha)}) \sim t^{2-\alpha}$

Y. Hatta, T. Ikeda, PRD67 (2003) 014028

$$c_2 \equiv \chi_q \sim \frac{\partial^2 \ln \mathcal{Z}}{\partial \mu_q^2} \sim t^{1-\alpha} \quad , \quad c_4 \sim \frac{\partial^4 \ln \mathcal{Z}}{\partial \mu_q^4} \sim t^{-\alpha} \quad (\mu = 0)$$

$$\epsilon \sim \frac{\partial \ln \mathcal{Z}}{\partial T} \sim t^{1-\alpha} \quad , \quad C_V \sim \frac{\partial^2 \ln \mathcal{Z}}{\partial T^2} \sim t^{-\alpha} \quad (\mu = 0)$$

⇒ 2^{nd} derivative w.r.t μ_q ”looks like energy density”

⇒ 4^{th} derivative w.r.t μ_q ”looks like specific heat”

Hadronic fluctuations at $\mu > 0$ from
Taylor expansion coefficients at $\mu = 0$

$n_f = 2,\ m_\pi \simeq 770$ MeV: S. Ejiri, FK, K.Redlich, PLB633 (2006) 275
$n_f = 2+1,\ m_\pi \simeq 220$ MeV: RBC-Bielefeld, preliminary

- quadratic and quartic fluctuations

$$\chi_2^x = \frac{\partial^2 p/T^4}{\partial(\mu_x/T)^2} = \frac{1}{VT^3}\langle(\delta N_x)^2\rangle_{\mu=0} = \frac{1}{VT^3}\langle N_x^2\rangle_{\mu=0}$$

$$\chi_4^x = \frac{\partial^4 p/T^4}{\partial(\mu_x/T)^4} = \frac{1}{VT^3}\left(\langle(\delta N_x)^4\rangle - 3\langle(\delta N_x)^2\rangle^2\right)_{\mu=0}$$

$$= \frac{1}{VT^3}\left(\langle N_x^4\rangle - 3\langle N_x^2\rangle^2\right)_{\mu=0}$$

with $x = u,\ d,\ s$ or $B,\ Q,\ S$

Hadronic resonance gas $\Rightarrow$ Boltzmann approximation

heavy resonances, $T << m_H \quad \Rightarrow$ Boltzmann statistics

$\mu_B \equiv B\mu_q$

thermodynamics: $p(T,\mu_B) = \frac{T}{V}\ln Z(T,\mu_B,V) = \sum_m p_m(T,\mu_B)$

$$\ln Z(T,\mu_B,V) = \sum_{i\,\in\,\mathrm{mesons}} \ln Z^B_{m_i}(T,V) + \sum_{i\,\in\,\mathrm{baryons}} \ln Z^F_{m_i}(T,\mu_B,V)$$

contribution of baryons (fermions, -) or mesons (bosons, +) with mass m

$$\frac{p_m}{T^4} = \frac{d}{\pi^2}\left(\frac{m}{T}\right)^2 \sum_{\ell=1}^{\infty} (\pm 1)^{\ell+1}\,\ell^{-2}\,K_2(\ell m/T)\,\cosh(\ell\mu_B/T)$$

$\Uparrow$

$$K_2(x) \simeq \sqrt{\pi/2x}\,\exp(-x)\,,\quad x >> 1$$

baryons:

$\mu_3 = 3\mu_q$

diquarks:

$\mu_2 = 2\mu_q$

quasi-part.:

$\mu_1 = \mu_q$

- only $\ell = 1$ contributes for $(m_H - \mu_B) \gtrsim T$

$\Rightarrow$ Boltzmann approximation: $\frac{p_m}{T^4} = \frac{d}{\pi^2}\left(\frac{m}{T}\right)^2 K_2(m/T)\,\cosh(\mu_B/T)$

Quark number in Boltzmann approximation

- baryonic sector of pressure in a hadron resonance gas;

  $m_B \gg T \;\Rightarrow$ Boltzmann approximation: $p_B/T^4 = \sum_{m \le m_{max}} p_m/T^4$

with $$p_m/T^4 = F(T,m,V)\cosh(B\mu_B/T)$$

$$\chi_2^B \equiv \frac{\partial^2 p_m/T^4}{\partial(\mu_B/T)^2} = B^2 F(T,m,V)\cosh(B\mu_B/T)$$

$$\chi_4^B \equiv \frac{\partial^4 p_m/T^4}{\partial(\mu_B/T)^4} = B^4 F(T,m,V)\cosh(B\mu_B/T)$$

ratio of fourth (χ_4^B) and second (χ_2^B) cumulant of quark number fluctuation gives "unit of charge" carried by the particle with mass "m":

$$m \gg T \quad \Rightarrow \quad R_{4,2}^B \equiv \frac{\chi_4^B}{\chi_2^B} = B^2$$

Charge fluctuations in Boltzmann approximation

- hadronic resonance gas: contributions from neutral ($G^{(1)}: \pi^0, ...$) and charged ($G^{(2)}: \pi^{\pm}, ...$) mesons and baryons as well as doubly charged baryons ($G^{(3)}: \Delta^{++}, ...$)

$$\frac{p(T, \mu_B = 0, \mu_Q)}{T^4} \simeq G^{(1)}(T) + G^{(2)}(T) \cosh\left(\mu_Q/T\right) + G^{(3)}(T) \cosh(2\mu_Q/T)$$

- charge fluctuations at $\mu_B = 0$;
enhanced contribution from $G^{(3)}$ i.e. from doubly charged baryons

$$R^Q_{4,2} \equiv \frac{\chi^Q_4}{\chi^Q_2} = \frac{G^{(2)} + 16 G^{(3)}}{G^{(2)} + 4 G^{(3)}} \rightarrow \text{ 1 for } T \rightarrow 0$$

contribution of doubly charged baryons increases quartic relative to quadratic fluctuations

Cumulant ratios

- ratios of cumulants reflect carriers of baryon number and charge

$$R^x_{4,2} = \chi^x_4/\chi^x_2 \quad , \quad x = B,\, Q$$

$$R^q_{4,2} = \begin{cases} 1 & ,\mathrm{HRG} \\ \dfrac{2}{3\pi^2} + \mathcal{O}(g^3) & ,\mathrm{high}-T \end{cases}$$

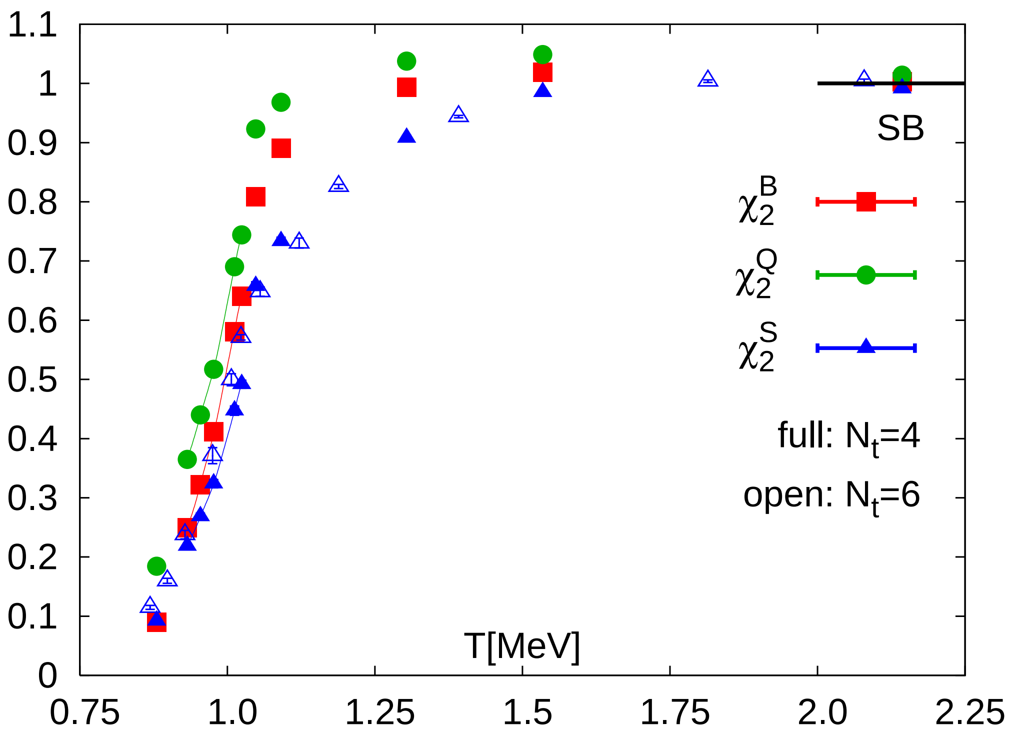
$$R^Q_{4,2} = \begin{cases} 1 & ,\mathrm{HRG}, T \to 0 \\ \dfrac{34}{15\pi^2} + \mathcal{O}(g^3) & ,\mathrm{high}-T \end{cases}$$

Quadratic fluctuations of baryon number charge & strangeness in (2+1)-flavor QCD

RBC-Bielefeld, arXiv:0811.1006

p4-action, coarse lattices, strong taste violations

vanishing chemical potentials:



$$\chi_2^Q = \frac{1}{VT^3}\langle Q^2\rangle$$

$$\chi_2^B = \frac{1}{VT^3}\langle N_B^2\rangle$$

$$\chi_2^S = \frac{1}{VT^3}\langle N_S^2\rangle$$

rapid approach to SB limit

continuum limit??

$\Rightarrow$ smooth change of quadratic fluctuations across transition region

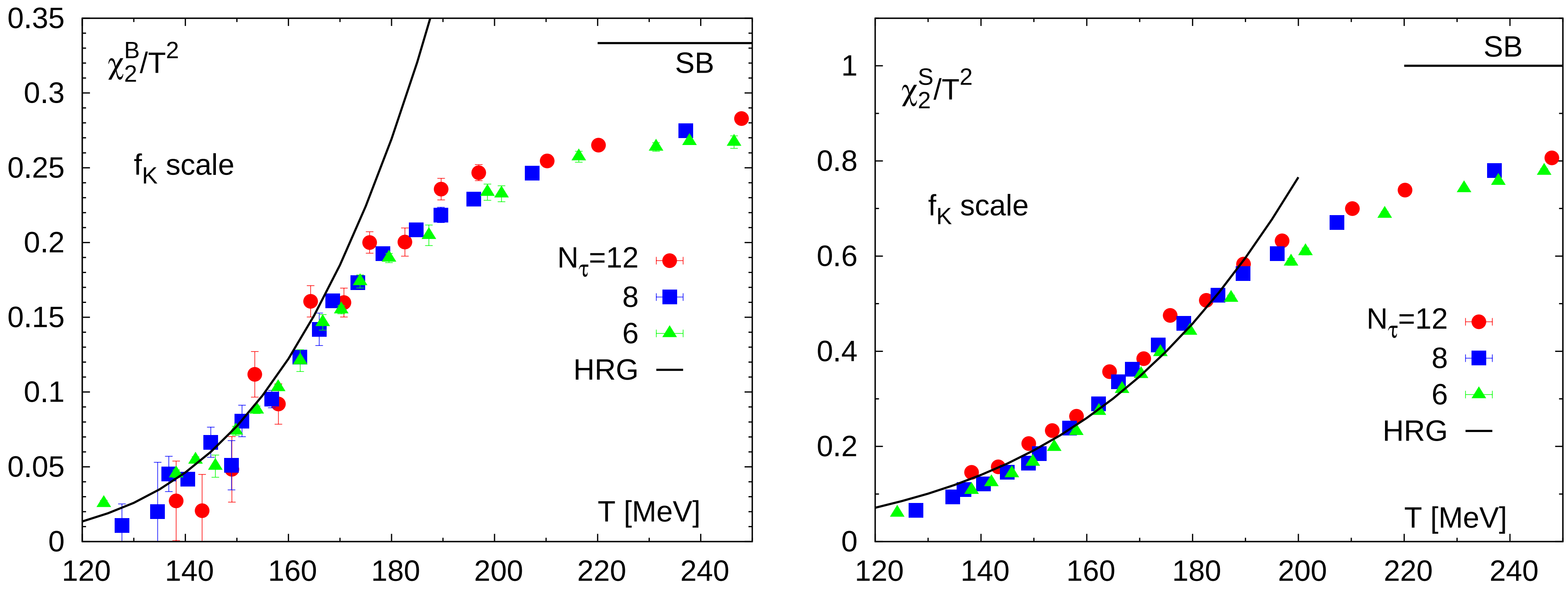
chiral limit: $\chi_2^B,\ \chi_2^Q \sim |T - T_c|^{1-\alpha} + \mathrm{regular}$

Quadratic fluctuations of baryon number charge & strangeness in (2+1)-flavor QCD

HotQCD preliminary

HISQ-action, $N_\tau = 6,\ 8,\ 12$

vanishing chemical potentials:



$\Rightarrow$ smooth change of quadratic fluctuations across transition region

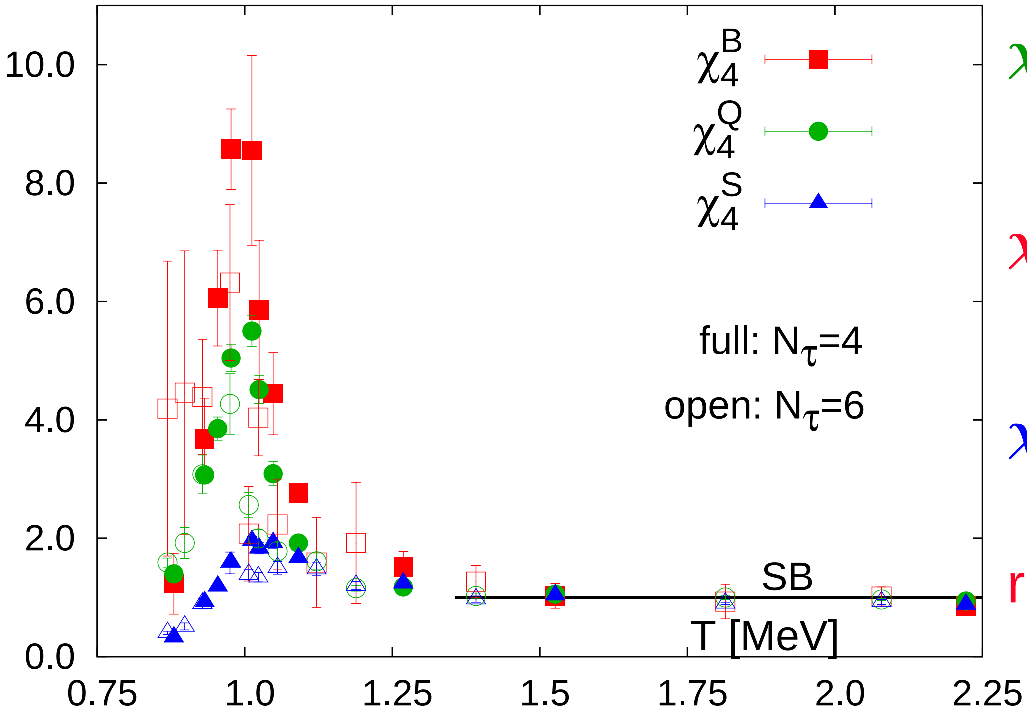
good agreement with HRG model at low T

Quartic fluctuations of baryon number charge & strangeness in (2+1)-flavor QCD

RBC-Bielefeld, arXiv:0811.1006

p4-action, coarse lattices, strong taste violations

vanishing chemical potentials:



$$\chi_4^Q = \frac{1}{VT^3}\left(\langle Q^4\rangle - 3\langle Q^2\rangle^2\right)$$

$$\chi_4^B = \frac{1}{VT^3}\left(\langle N_B^4\rangle - 3\langle N_B^2\rangle^2\right)$$

$$\chi_4^S = \frac{1}{VT^3}\left(\langle N_S^4\rangle - 3\langle N_S^2\rangle^2\right)$$

rapid approach to SB limit

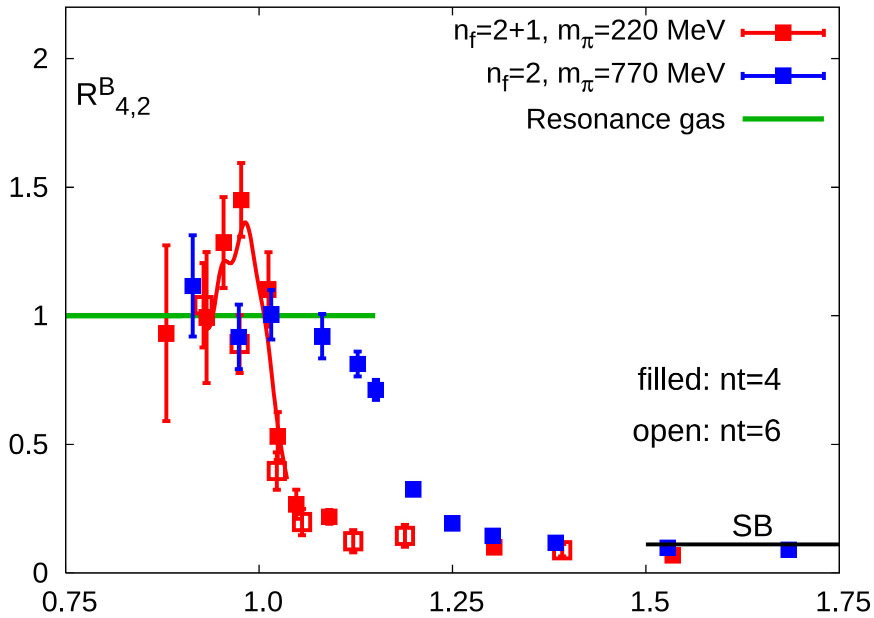
$\Rightarrow$ large light quark number & charge fluctuations across transition region

chiral limit: $\chi_4^B,\ \chi_4^Q \sim |T-T_c|^{-\alpha} + \text{regular}$

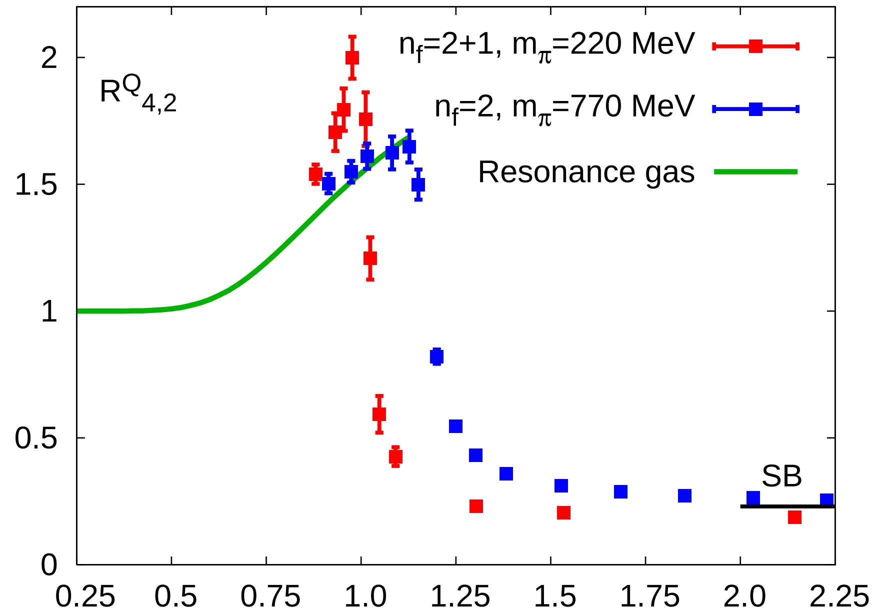
Ratios of quartic and quadratic fluctuations of charges in (2+1)-flavor QCD

$n_f = 2$: S. Ejiri, FK, K.Redlich, PLB633 (2006) 275
$n_f = 2+1$: RBC-Bielefeld, arXiv:0811.1006

baryon number fluctuation



charge fluctuation



chiral limit: ratios $\sim |T - T_c|^{-\alpha} + \mathbf{regular}$

$\Rightarrow$ enhancement over resonance gas values? (need to improve $N_\tau = 6$)

$\Rightarrow$ may be observable in event-by-event fluctuations

quark sector quickly ($T \gtrsim 1.5 T_c$) behaves perturbative

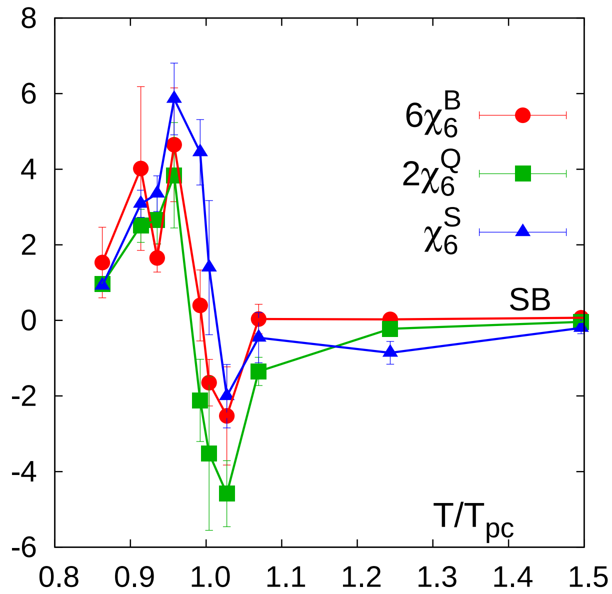
Higher moments of charge fluctuations at RHIC and LHC

- higher moments (e.g. 6th order) are drastically different in QCD close to criticality and in a hadron resonance gas, e.g.

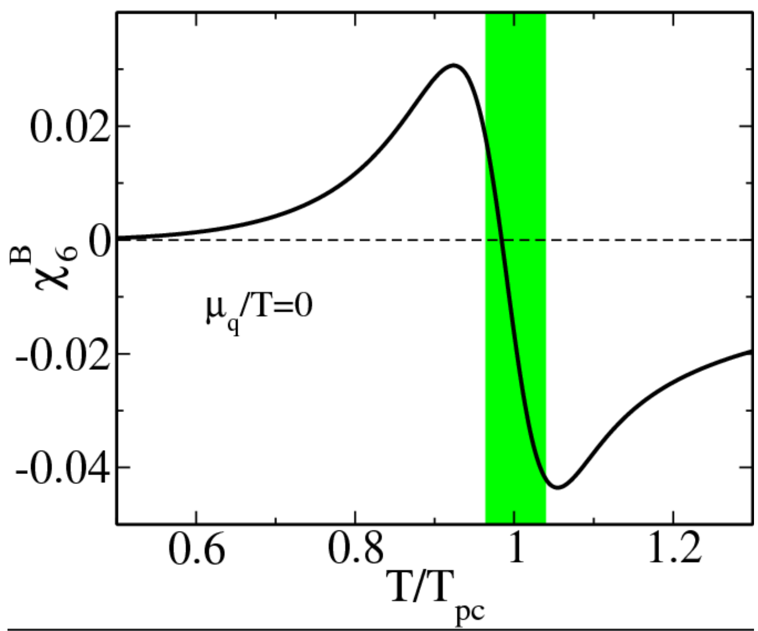
$\mu_B = 0$

$$\frac{\chi^{(6)}_{B,0}}{\chi^{(2)}_{B,0}} = \begin{cases} = 1 & \text{, hadron resonance gas} \\ < 0 & \text{, QCD at the crossover transition} \end{cases}$$

LGT: $16^3 \times 4$ (p4)



PQM model

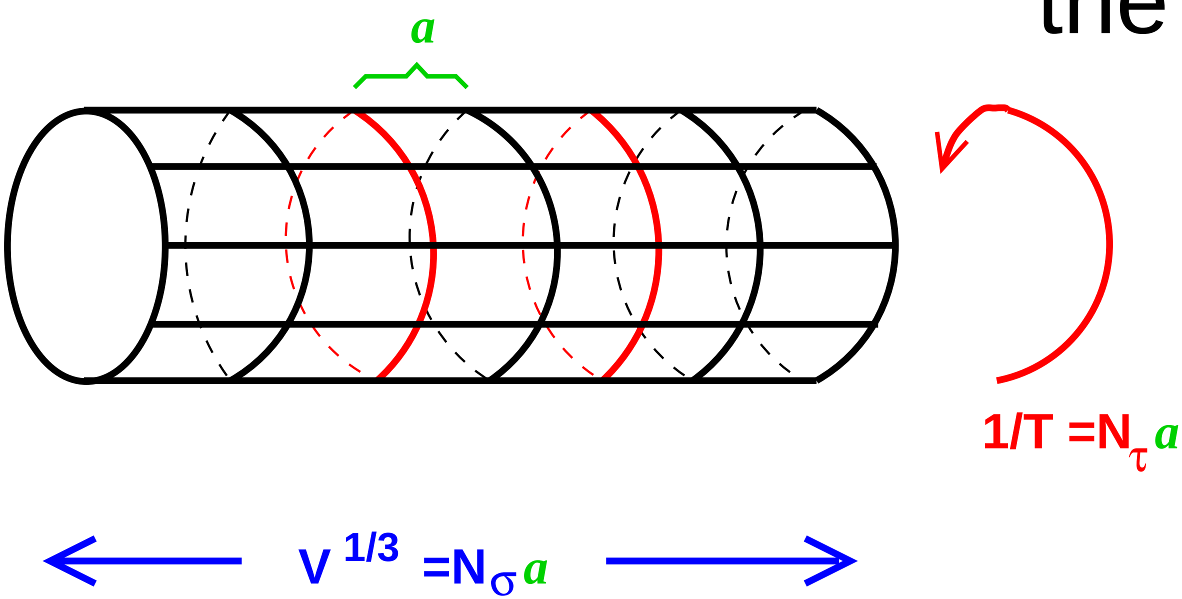


PQM model and LGT calculations reproduce expected O(4) scaling structure

B. Friman et al, arXiv:1103.3511

QCD Thermodynamics:
Simulating hot and dense matter

the lattice: $\mathbf{N_\sigma^3} \times \mathbf{N_\tau}$



lattice spacing : $\mathbf{a_\sigma}$, $\mathbf{a_\tau}$

gauge coupling : $\beta = 6/g^2$

partition function:

$$Z(V,T,\mu) = \int \mathcal{D}\mathcal{A}\, Det M(\mathcal{A},\mu)\, e^{-S_G}$$

$$S_E = \int_0^{1/T} dx_0 \int_V d^3x\, \mathcal{L}_E(\mathcal{A},\psi,\bar{\psi},\mu)$$

temperature volume chemical potential

bulk thermodynamics:

$$\frac{p}{T^4} = -\frac{1}{VT^3}\ln Z$$

$$\frac{\epsilon}{T^4} = -\frac{1}{VT^4}\frac{\partial}{\partial T^{-1}}\ln Z$$

$$\frac{n_q}{T^3} = \frac{1}{VT^3}\frac{\partial}{\partial \mu_q/T}\ln Z$$

$$\frac{\chi_q}{T^2} = \frac{1}{VT^3}\frac{\partial^2 \ln Z}{\partial(\mu_q/T)^2} = \frac{T}{V}\left(\langle N_q^2\rangle - \langle N_q\rangle^2\right)$$

Calculating the EoS on lines of constant physics (LCP)

- The interaction measure for $N_f = 2+1 \quad \Leftrightarrow \quad$ Trace Anomaly

$$\frac{\epsilon - 3p}{T^4} = T\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{p}{T^4}\right) = \left(a\frac{\mathrm{d}\beta}{\mathrm{d}a}\right)_{LCP} \frac{\partial p/T^4}{\partial \beta}$$
$$= \left(\frac{\epsilon - 3p}{T^4}\right)_{gluon} + \left(\frac{\epsilon - 3p}{T^4}\right)_{fermion} + \left(\frac{\epsilon - 3p}{T^4}\right)_{\hat{m}_s/\hat{m}_l}$$

- The pressure

$$\left.\frac{p}{T^4}\right|_{\beta_0}^{\beta} = \int_{T_0}^{T} \mathrm{d}T\, \frac{1}{T}\left(\frac{\epsilon - 3p}{T^4}\right)$$

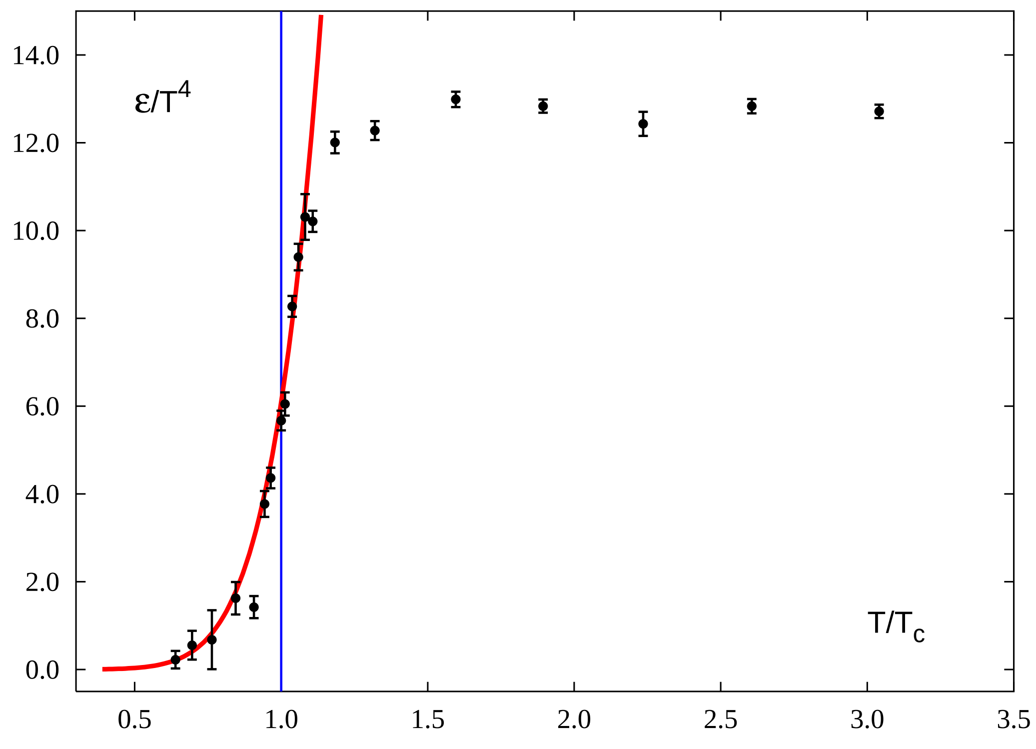
- need T-scale, $aT = 1/N_\tau$ and its relation to the gauge coupling $a \equiv a(\beta)$

N.B.: $a(\beta)$ is only defined through physical observables
$\Rightarrow$ choose a simple one

Critical temperature, equation of state and the resonance gas

$$\textbf{Hagedorn spectrum}: \quad \rho(m_H) \sim c\, m_H^a\, \mathrm{e}^{m_H/T_H}$$

$$\ln Z(T, \mu_B) = \int \mathrm{d}m_H\, \rho(m_H)\, \ln Z_{m_H}(T, \mu_B)$$



resonance gas:

$\sim$ 1500 d.o.f. from

$\sim$ 300 exp. known resonances

vs.

lattice calculation:

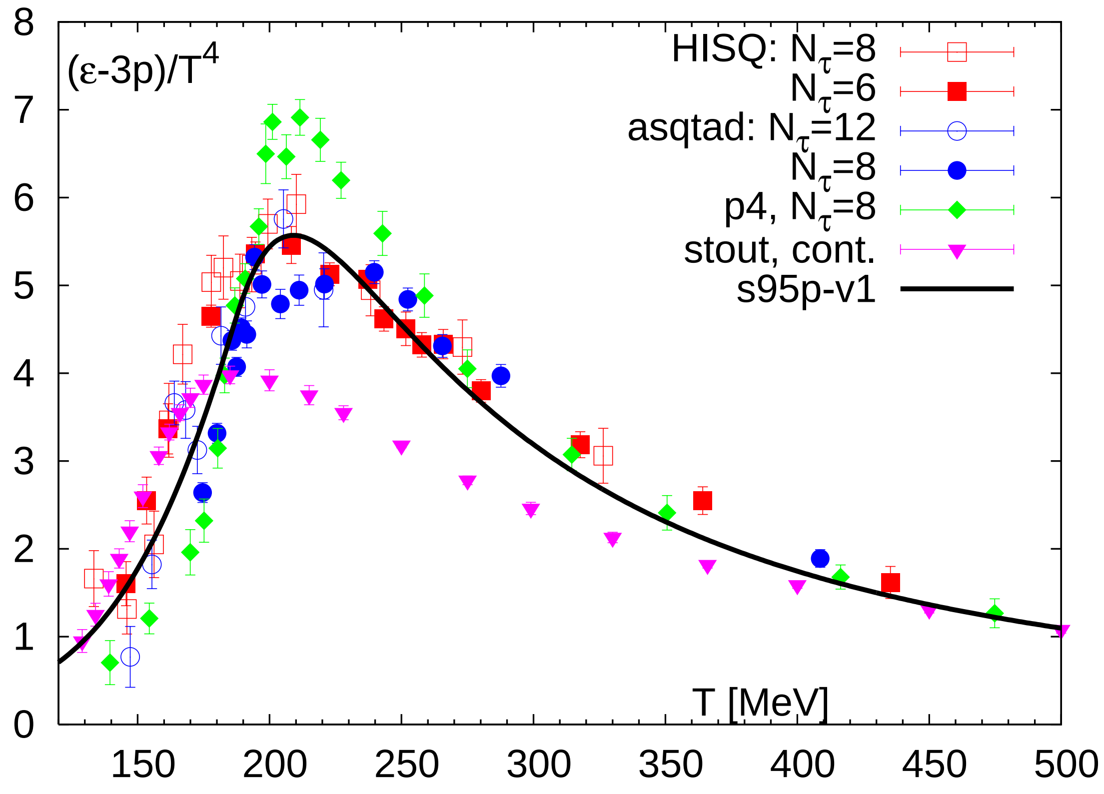
(2+1)-flavor QCD, $m_q/T = 0.4$

resonances give large contribution at T_c

- explain eos for $T \leq T_c$;

Trace anomaly: $(\epsilon - 3p)/T^4$

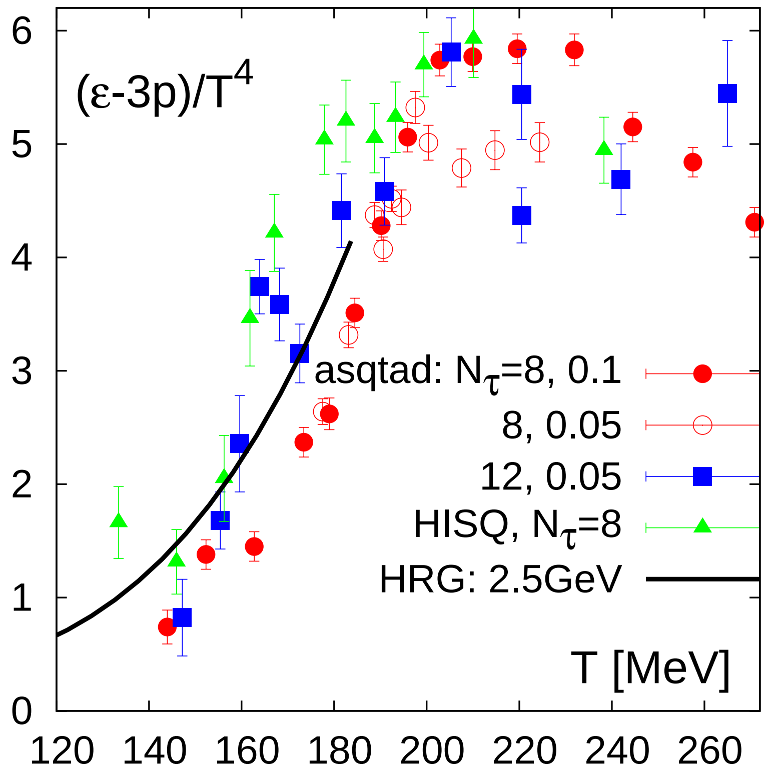
..towards the cont. limit: $N_\tau = 6,\ 8,\ 12$



- basic input for the calculation of all bulk thermodynamic observables
- cut-off effects in the peak region seem to be under control within HISQ and asqtad calculations
- origin of differences with stout calculations at present 'unclear'

Trace anomaly: $(\epsilon - 3p)/T^4$

..comparison with HRG model at low T



- the HRG model is a good approximation to the QCD EoS at low temperature

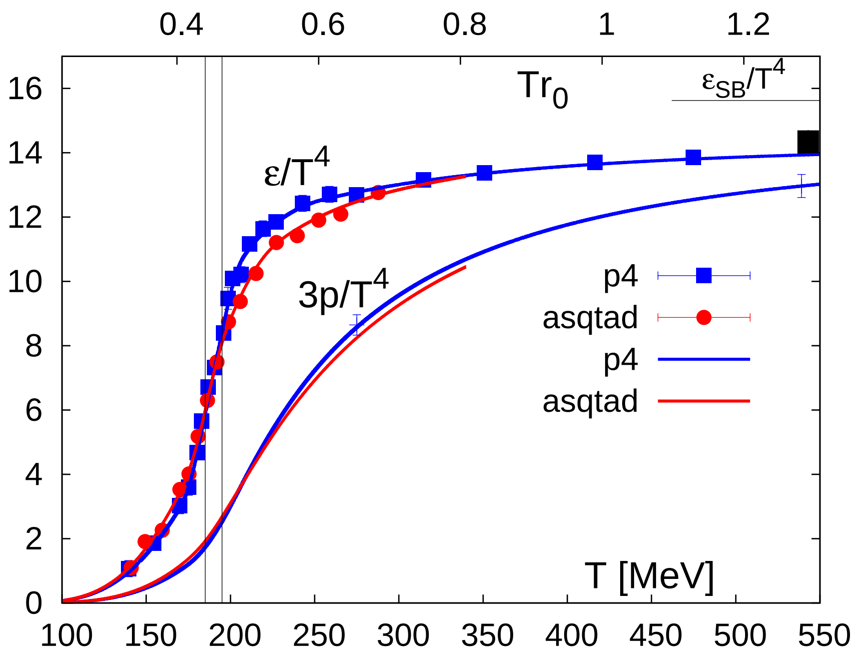
Energy Density and Light Quark Susceptibility

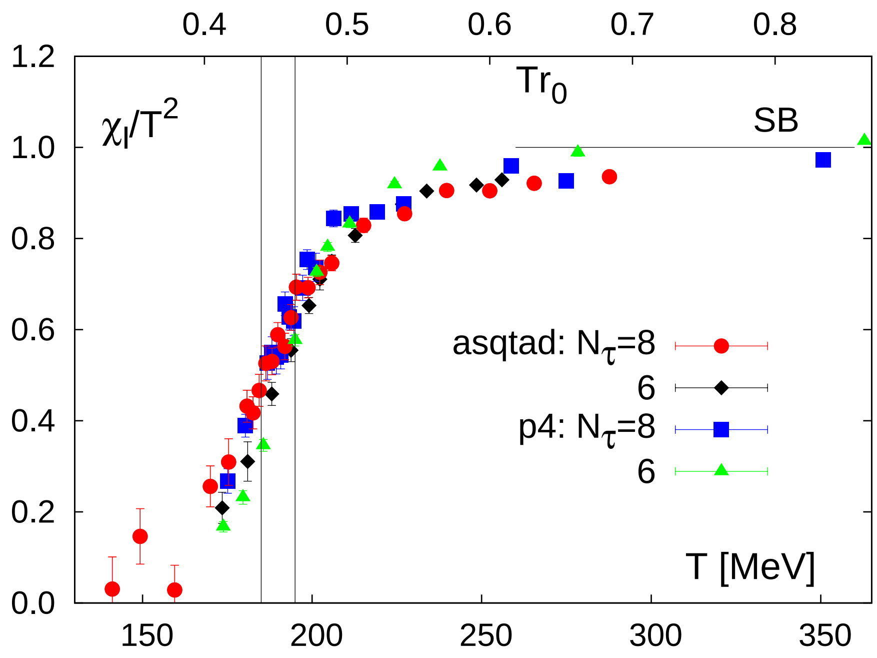
- singular parts of ϵ/T^4 and χ_l/T^2 have identical T-dependence
- ϵ/T^4 and χ_l/T^2 couple to pions at low temperature

  $$\chi_l,\ \epsilon \sim \exp(-m_\pi/T)$$

- ϵ/T^4 and χ_l/T^2 sensitive to change in light degrees of freedom

  $\rightarrow$ deconfinement





Summary:
Phase transition and bulk thermodynamics

- The scaling of thermodynamic quantities in the vicinity of the QCD (crossover) transition is consistent with the expected $O(4)$ scaling close to the chiral limit of 2-flavor QCD.
  Many details still need to be settled
- The QCD (phase) transition temperature (T_c) is about 160 MeV at $\mu_B = 0$, which is close to the experimentally determined freeze-out temperature (T_{freeze})
  Maybe a bit too low?
- Higher moments of fluctuations of conserved charges became incresingly sensitive to critical behavior and may 'keep memory' of a nearby chiral phase transition even if $T_{freeze} \neq T_c$?.
- The QCD transition shows many features expected from a 'deconfinement transition'.