Summary lecture 2

- WZ model: Exact lattice SUSY in 2D
- Appearance of twisted/KD fermions and relation to staggered.
- Today: Gauge symmetry $\mathcal{N} = 2$ SYM in 2D
- Lattice formulation of $\mathcal{N} = 4$ SYM in 4D

Ingredients of twisted gauge theory in 2D

- Expect to have twisted fermions $(\eta, \psi_\mu, \chi_{12})$
- Gauge field $A_\mu, \mu = 1 \ldots 2$
- 2 scalar fields B^1, B^2 - needed for SUSY to match dof.
- $\mathcal{Q}$ exact action with $\mathcal{Q}^2 = 0$
- But also twisting procedure acts on all fields charged under flavor group .. not just fermions.
- Specifically, scalars B are vectors under flavor. Expect they will transform as vectors under twisted rotation group

2D $\mathcal{N} = 2$ YM

- Fields: gauge field, 2 scalars, 2 Majorana fermions (dim red of $\mathcal{N} = 1$ YM in 4D)
- Twist: consider 2 fermions as matrix

$$\lambda^i_\alpha \to \Psi_{\alpha\beta}$$

Expand:

$$\Psi = \frac{\eta}{2} I + \psi_\mu \gamma_\mu + \chi_{12}\gamma_1\gamma_2$$

- $\eta, \psi_\mu, \chi_{\mu\nu}$ twisted fermions
- Scalar fermion - scalar supersymmetry $\mathcal{Q}$ with $\mathcal{Q}^2 = 0$

Twisted action and SUSY

Twisted form of action (adjoint fields with AH generators)

$$S = \frac{1}{g^2}\mathcal{Q}\int \mathrm{Tr}\left(\chi_{\mu\nu}\mathcal{F}_{\mu\nu} + \eta[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] - \frac{1}{2}\eta d\right)$$

$$\begin{aligned}
\mathcal{Q}\,\mathcal{A}_\mu &= \psi_\mu \\
\mathcal{Q}\,\psi_\mu &= 0 \\
\mathcal{Q}\,\overline{\mathcal{A}}_\mu &= 0 \\
\mathcal{Q}\,\chi_{\mu\nu} &= -\overline{\mathcal{F}}_{\mu\nu} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned}$$

Note: complexified gauge field $\mathcal{A}_\mu = A_\mu + iB_\mu$, $\mathcal{F}_{\mu\nu}(\mathcal{A})$

Untwisting

$\mathcal{Q}$-variation, integrate d:

$$S = \frac{1}{g^2}\int \mathrm{Tr}\left(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} - \eta\overline{\mathcal{D}}_\mu\psi_\mu\right)$$

Rewrite as

$$S = \frac{1}{g^2}\int \mathrm{Tr}\left(-F_{\mu\nu}^2 + 2B_\mu D_\nu D_\nu B_\mu - [B_\mu, B_\nu]^2 + L_F\right)$$

where

$$L_F = \begin{pmatrix} \chi_{12} & \frac{\eta}{2} \end{pmatrix}\begin{pmatrix} -D_2 - iB_2 & D_1 + iB_1 \\ D_1 - iB_1 & D_2 - iB_2 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Relation to conventional SYM model

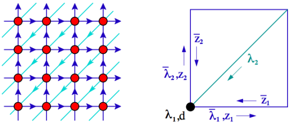
Homework 5. Use the decomposition

$$\begin{aligned} \mathrm{Re}\mathcal{F}_{\mu\nu} &= F_{\mu\nu} - [B_\mu, B_\nu]^2 \\ \mathrm{Im}\mathcal{F}_{\mu\nu} &= D_{[\mu} B_{\nu]} \end{aligned}$$

together with integration by parts to show that the bosonic action $\mathcal{F}^\dagger\mathcal{F}$ is indeed the usual one corresponding to a real YM term plus scalar kinetic and quartic terms
(D denotes the usual covariant derivative wrt to A)

2D Lattice construction

- Bosons: $\mathcal{A}_\mu(x) \rightarrow \mathcal{U}_\mu(n)$. Complexified Wilson links.
- Fermions: η on sites, ψ_μ same links as $\mathcal{U}_\mu$, χ_{12} diagonal links.



$$
\begin{aligned}
\eta(\mathbf{x}) &\rightarrow G(\mathbf{x})\eta(\mathbf{x})\mathbf{G}^\dagger(\mathbf{x}) \\
\psi_\mu(\mathbf{x}) &\rightarrow G(\mathbf{x})\psi_\mu(\mathbf{x})\mathbf{G}^\dagger(\mathbf{x}+\mu) \\
\chi_{\mu\nu}(\mathbf{x}) &\rightarrow G(\mathbf{x}+\mu+\nu)\chi_{\mu\nu}(\mathbf{x})\mathbf{G}^\dagger(\mathbf{x}) \\
\mathcal{U}_\mu(\mathbf{x}) &\rightarrow G(\mathbf{x})\mathcal{U}_\mu(\mathbf{x})\mathbf{G}^\dagger(\mathbf{x}+\mu)
\end{aligned}
$$

Orientation ensures G.I

Derivative technology

- Nilpotent SUSY transformations same as in continuum except $\mathcal{A}_\mu \to \mathcal{U}_\mu$
- Derivatives replaced with covariant differences compatible with lattice G.I eg.

$$\mathcal{D}^{(+)}_\mu \mathcal{U}_\nu = \mathcal{F}_{\mu\nu} = \mathcal{U}_\mu(x)\mathcal{U}_\nu(x+\mu) - \mathcal{U}_\nu(x)\mathcal{U}_\mu(x+\nu)$$

eg:

$$\mathcal{U}_\mu(x)\mathcal{U}_\nu(x+\mu) \to G(x)\mathcal{U}_\mu(x)G(x+\mu)^\dagger G(x+\mu)\mathcal{U}_\nu(x+\mu)G(x+\mu+\nu)^\dagger$$

transforms as lattice 2-form

Contract with $\chi_{\mu\nu}$ – gauge invariant loop.

Continuing

- Commutator term $[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]$ on lattice becomes

$$\overline{\mathcal{D}}^{(-)}_\mu \mathcal{U}_\mu(x) = \mathcal{U}_\mu(x)\overline{\mathcal{U}}_\mu(x) - \overline{\mathcal{U}}_\mu(x-\mu)\mathcal{U}_\mu(x-\mu)$$

  Transforms like site field – hence $\mathrm{Tr}\left(\eta\overline{\mathcal{D}}^{(-)}_\mu\mathcal{U}_\mu\right)$ G.I
- Technology for defining gauge covariant difference ops generalizes to arbitrary lattice p forms and lattice derivs in case of adjoint fields (Aratyn et al)
- All become *shifted* commutators. For $\mathcal{U} = 1$ curl-like derivs become forward difference ops. Divergence-like derivs replaced by backward derivs. Ensures no fermion doubling (Rabin, Joos, Becher)

Geometrical discretization

- Any SYM with $\mathcal{Q} \geq 2^D$ supercharges can be twisted and discretized this way.
- Lattice theories inherit 1 or more exact SUSYs
- Local, gauge invariant and free of fermion doubling.
- May also be derived by orbifolding a supersymmetric matrix theory - unique lattice theories ?
- Can be used as non-perturbative definition of theory .. discuss later for discussion of how to take limit $a \to 0$ (fine tuning question)

Twisting in D dimensions

- Consider (extended) SUSY theories possessing additional flavor (R) symmetries.
- Twist: decompose fields under $G = \mathrm{Diag}(SO_{\mathrm{Lorentz}}(D) \times SO_{\mathrm{R}}(D))$. Can only twist when flavor symmetry large enough ..
- Fermions: spinors under both factors – become integer spin after twisting.
- Scalars transform as vectors under R-symmetry – vectors after twisting.
- Gauge fields remain vectors – combine with scalars to make complex gauge fields. Still just $U(N)$ gauge symmetry.

Important: flat space: just a change of variable

Putting it all together

- Discussed 1D, 2D with and without gauge invariance
- How a non-abelian theory in 4D ?
- Constrained ... if we want to preserve at least 1 SUSY on lattice - need to have theory whose flavor (R) symmetry group contains SO(D) - Euclidean Lorentz group.
- In D=4: only $\mathcal{N} = 4$ picked out.
- Alternatively: A Kähler-Dirac field in 4D has $1 + 4 + 6 + 4 + 1 = 16$ components. Four Majorana fermions ...

Twisted continuum $\mathcal{N} = 4$ SYM

- $\mathcal{N} = 4$ contains 4 gauge fields A_μ, 4 Majorana fermions, and six scalars ($\mathcal{N} = 4$ arises by dim red of $\mathcal{N} = 1$ in 10 dims)
- Appropriate twist due to Marcus.
  $SO(4)' = SO_R(4) \times SO_{rot}(4)$
- After twisting expect:
  - Kähler-Dirac field $(\eta, \psi_\mu, \chi_{\mu\nu}, \theta_{\mu\nu\lambda}, \kappa_{1234})$
  - Gauge field $A_\mu, \mu = 1 \ldots 4$
  - 6 scalars decompose as 1 vector B_μ and 2 scalars ($SO(6) \to SO(4) \times SO(2)$)

5D description

Can package these fields more compactly as dimensional reduction of 5D theory!

- 16 fermions: $\Psi = (\eta, \psi_m, \chi_{mn}), m, n = 1 \ldots 5$
- 10 bosons as 5 complex gauge fields $\mathcal{A}_m, m = 1 \ldots 5$

Remarkably twisted action:

$$S = \mathcal{Q} \int \left(\chi_{ab} F_{ab} + \eta [\overline{\mathcal{D}}_a, \mathcal{D}_a] - 1/2 \eta d \right) + S_{\text{closed}}$$

(Almost) same as 2D example !

$$S_{\text{closed}} = \int \epsilon_{abcde} \chi_{ab} \overline{\mathcal{D}}_c \chi_{de}$$

$\mathcal{Q} S_{\text{closed}} = 0$ by Bianchi

Lattice $\mathcal{N} = 4$ theory

Usual fields	Twisted fields
$A_\mu, \mu = 1 \dots 4 \quad \phi_i, i = 1 \dots 6$	$\mathcal{U}_a, a = 1 \dots 5$
$\Psi^i, i = 1 \dots 4$	$\eta, \psi_a, \chi_{ab}, a, b = 1 \dots 5$

- Fields live on links of A_4^* lattice - 5 basis vectors correspond to vectors from center of hypertetrahedron to vertices.
  - $\eta \quad x \to x$
  - $\psi_a \quad x \to x + \mu_a$
  - $\chi_{ab} \quad x + \mu_a + \mu_b \to x$
- All fields transform like links: $X_p \to G(x)\, X_p\, G^\dagger(x + p)$
- Single exact lattice SUSY $\mathcal{Q}^2 = 0$

Lattice action

$$S = \beta(S_1 + S_2)$$

$$\begin{aligned} S_1 &= \sum_{\mathbf{x}} \mathrm{Tr}\left(\mathcal{F}_{ab}^{\dagger}\mathcal{F}_{ab} + \frac{1}{2}\left(\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a\right)^2\right. \\ &- \left.\chi_{ab}\mathcal{D}_{[a}^{(+)}\psi_{\,b]} - \eta\overline{\mathcal{D}}_a^{(-)}\psi_a\right) \\ S_2 &= -\frac{1}{2}\sum_{\mathbf{x}} \mathrm{Tr}\epsilon_{abcde}\chi_{de}(\mathbf{x}+\mu_{\mathbf{a}}+\mu_{\mathbf{b}}+\mu_{\mathbf{c}})\overline{\mathcal{D}}_{\mathbf{c}}^{(-)}\chi(\mathbf{x}+\mu_{\mathbf{c}}) \end{aligned}$$

Comments

- Bosonic action is just Wilson plaquette if $\mathcal{U}_a^\dagger \mathcal{U}_a = 1$. (Homework problem. Use formula for $\mathcal{F}$ to see this explicitly)
- Thus no boson doubling. Hence by exact SUSY no fermion doubling.
- $S_{\rm closed}$ supersymmetric ? Remarkably, *Bianchi identity holds also on lattice* !
  (Another homework problem: show that $\epsilon_{abcde}\mathcal{D}_c^{(+)}\mathcal{D}_b^{(+)}\mathcal{U}_a = 0$)
- Gauge invariance of $S_{\rm closed}$ term follows from fact that sum over 5 distinct (via epsilon symbol) basis vectors form closed loop. And sum over 5 basis links yields zero.

Some words about A_4^* lattice

- In 2D - A_2^* - triangular lattice
- Deformation of : hypercubic lattice plus body diagonal
- Key feature: 5 basis vectors which are therefore linearly dependent $\sum_a^5 \mathbf{e}_a = 0$
- High point group symmetry - important for counter term structure ...
- Fermions inhabit the A_4^* lattice *plus* additional 'face links' $\mathbf{e}_a + \mathbf{e}_b$.
- Introduce lattice with half lattice spacing. Can map fermion action onto that of reduced staggered quarks

Outstanding questions

Lattice theories are:

local, gauge invariant, doubler free and invariant under one SUSY

Two questions:

- Is rotational symmetry restored as $a \to 0$?
- What about restoration of full SUSY ?

Must understand how lattice theory renormalizes ...

Two approaches

- Examine at 1-loop using p. theory
- Attempt a non-perturbative tuning by measuring broken SUSY Ward identities