Summary lecture 1

- Examined toy QM model with SUSY.
- Showed how naive discretization fails – lattice theory picks up new U.V contributions - fine tuning problem.
- Avoid by correcting action with counterterm computed at 1-loop.
- But better: Find modified action which is invariant under linear combination of original SUSYs - Q.
- Exact SUSY - fermion op. is derivative of *local* Nicolai map
- New (super)symmetry nilpotent $Q^2 = 0$ and action Q-exact $S = Q\Lambda$. Twisted formulation

Two dimensions

- How can we generalize previous construction to field theory ?
- In QM (1d FT) needed 2 SUSYs. Single scalar field and two fermions. Lift to 2d: find $\mathcal{N} = 2$ Wess-Zumino model - 1 Dirac fermion and 1 complex scalar.

$$
\begin{aligned}
S_{\mathrm{WZ}} &= \int d^2x\, \partial_\mu \phi \partial_\mu \overline{\phi} + W'(\phi) W'(\overline{\phi}) + \overline{\psi} \gamma_\mu \partial_\mu \psi + \\
&+ \overline{\psi} \left(\frac{1}{2} (1 + \gamma_5)\, W''(\phi) + \frac{1}{2} (1 - \gamma_5)\, W''(\overline{\phi}) \right) \psi
\end{aligned}
$$

Nicolai Map for 2d WZ

Fermion operator can be rewritten (chiral basis):

$$M_{\mathrm{F}} = \begin{pmatrix} W''(\phi) & \partial_{\overline{z}} \\ \partial_z & W''(\overline{\phi}) \end{pmatrix}$$

Associated Nicolai Map $\phi, \overline{\phi} \to \mathcal{N}, \overline{\mathcal{N}}$ is

$$\mathcal{N} = \partial_{\overline{z}}\overline{\phi} + W'(\phi)$$

Boson action $\mathcal{N}\overline{\mathcal{N}}$ again differs by cross term from continuum (total derivative)

Lattice action

$$S_L = \sum_x \mathcal{N}\overline{\mathcal{N}} + \overline{\omega}\left(\Delta^s_{\overline{z}}\lambda + W''_L(\phi)\omega\right) + \overline{\lambda}\left(\Delta^s_z\omega + W''_L(\overline{\phi})\lambda\right)$$

where $\psi = \begin{pmatrix} \omega \\ \lambda \end{pmatrix}$, $\overline{\psi} = \begin{pmatrix} \overline{\omega} \\ \overline{\lambda} \end{pmatrix}$.
Symmetric difference $\Delta^s_z = \Delta^s_1 + i\Delta^s_2$.
Doublers removed via Wilson term

$$W'_L(\phi) = W'(\phi) + \frac{1}{2}\Delta^+_z\Delta^-_{\overline{z}}\phi$$

Homework problem 2: check that fermion op is free of doubles by going to k-space.

Supersymmetries

The single exact SUSY follows by analogy to QM:

$$\begin{aligned} Q\phi &= \omega \\ Q\overline{\phi} &= \lambda \\ Q\lambda &= 0 \\ Q\omega &= 0 \\ Q\overline{\omega} &= \overline{\mathcal{N}} \\ Q\overline{\lambda} &= \mathcal{N} \end{aligned}$$

Note that $Q^2 = 0$ with E.O.M

Additional SUSYs

In continuum this model has 3 additional SUSY corresponding to 3 choices of $\mathcal{N}$ which leave fermion det unchanged .

$$\begin{aligned}
\mathcal{N} &= \Delta^S_{\overline{z}}\overline{\phi} + W''_L \\
\mathcal{N} &= \Delta^S_{\overline{z}}\overline{\phi} - W''_L \\
\mathcal{N} &= \Delta^S_z\phi + \overline{W}''_L \\
\mathcal{N} &= \Delta^S_z\phi - \overline{W}''_L
\end{aligned}$$

Corresponds to symmetries $\overline{\omega} \to \omega$, $\overline{\lambda} \to \lambda$ and rotations $\omega \to \lambda$ and $\phi \to \overline{\phi}$

Twisted/topological form

Not hard to verify that action can be rewritten (see QM) as

$$S_L = Q \sum_x \overline{\omega} \left(\mathcal{N} + \frac{1}{2} B \right) + \overline{\lambda} \left(\overline{\mathcal{N}} + \frac{1}{2} \overline{B} \right)$$

with additional $B, \overline{B}$

$$\begin{aligned} Q\overline{\omega} &= \overline{B} \\ Q\overline{\lambda} &= B \\ QB &= 0 \\ Q\overline{B} &= 0 \end{aligned}$$

Ward identities

Again, existence of exact SUSY leads to Ward identities:

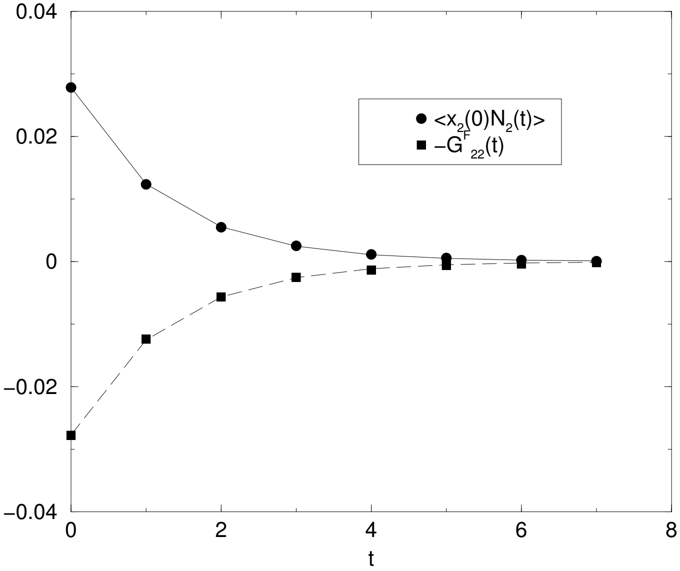
$$< S_B >= \frac{1}{2} N^f_{\rm d.o.f}$$

Monte Carlo:

L	$< S_B >$	$\frac{1}{2} N^f_{dof}$
4	31.93(6)	32
8	127.97(7)	128
16	512.0(3)	512
32	2046(3)	2048

Similarly for 2 pt functions ...

WZ Ward identity



Wess-Zumino and Kähler-Dirac fermions

If we choose

$$\begin{aligned} \omega &= \frac{1}{2}\eta - i\chi_{12} \\ \lambda &= \psi_1 + i\psi_2 \end{aligned}$$

Find (free part of) WZ action

$$S_{\rm WZ} = \omega^\dagger \partial_{\overline{z}} \lambda + {\rm h.c}$$

can be rewritten

$$\frac{1}{2}\eta\partial_\mu\psi_\mu + \chi_{\mu\nu}\partial_{[\mu}\psi_{\nu]}$$

All indices $\mu, \nu = 1, 2$. Kahler-Dirac action.
Homework problem 3: check this

Kahler-Dirac equation

- Introducing Kähler-Dirac fermion $\Psi = (\eta/2, \psi_\mu, \chi_{\mu\nu})$ can write EOM in compact form

$$(d - d^\dagger)\Psi = 0$$

  where d is exterior derivative – hence the tensor component form
- Note: $(d - d^\dagger)^2 = \Box$ following Dirac.
- Alternatively can regard Kähler-Dirac field as 2×2 matrix

$$\Psi = \frac{\eta}{2} I + \psi_\mu \gamma_\mu + \chi_{12} \gamma_5$$

Kahler-Dirac action=Dirac action

Using

$$\Psi = \frac{\eta}{2} I + \psi_\mu \gamma_\mu + \chi_{12} \gamma_5 = \sum_{a=1}^{4} \eta_a \Gamma_a$$

the property $Tr \Gamma_i \Gamma_j = 2\delta_{ij}$ and the constraint that components real find that the matrix action

$$S_{\mathrm{F}} = \mathrm{Tr}\left[\overline{\Psi} \gamma_\mu \partial_\mu \Psi\right]$$

yields

$$S_{\mathrm{F}} = \frac{1}{2} \eta \partial_\mu \psi_\mu + \chi_{12} \left(\partial_1 \psi_2 - \partial_2 \psi_1\right)$$

Kähler-Dirac action as before

Twisting mechanism in 2D

- Is there some way to see how matrix fermions are natural in these models ?
- Yes, this is the process of twisting ...
  Original fermions λ^i_α $i = 1, 2\,\mathrm{flavor}$, $\alpha = 1, 2\,\mathrm{spinor}$.
  Transform under rotations R and flavor F (both SO(2))

$$\lambda^i_\alpha = R^{\alpha\beta}\lambda^j_\beta(F^T)^{ji}$$

- Under diagonal subgroup - $R = F$ - fermions transform like matrix!
- Thus process of twisting=decomposing fields/ supercharges under a twisted rotation group leads naturally to a treatment of fermions as Kahler-Dirac fields.

Twisted supersymmetries in 2D

Fields transform as spinors under $SO(2)_R$ and $SO(2)_{\rm rot}$
Choose to decompose supercharges under same twisted symmetry

$$SO(2)' = {\rm Diag}\left(SO(2)_R \times SO(2)_{\rm rot}\right)$$

Subsequently expand on products of gamma matrices

$$q = QI + Q_\mu \gamma_\mu + Q_{12}\gamma_1\gamma_2$$

Scalars, vectors, tensor twisted supersymmetries.
Appearance of scalar supercharge is key for SUSY - it is same Q we had seen earlier ...

Twisted SUSY algebra

Original supersymmetry algebra $\{q, q\} = \gamma_\mu p_\mu$

$$\begin{aligned} \{Q, Q\} &= \{Q_{12}, Q_{12}\} = \{Q, Q_{12}\} = \{Q_\mu, Q_\nu\} = 0 \\ \{Q, Q_\mu\} &= p_\mu \qquad \{Q_{12}, Q_\mu\} = \epsilon_{\mu\nu} p_\nu \end{aligned}$$

Note that p_μ is Q-variation of something.
Makes it plausible that entire $T_{\mu\nu}$ is Q-exact.
(why?: $p_\mu p_\nu = Q\Lambda_\mu Q\Lambda_\nu = Q(\Lambda_\mu Q_\nu))$

Hence twisted theories have Q-exact actions! (remember $T_{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}}$)

Discretize twisted/KD equation

Can discretize such geometrical actions without introducing doubles (Rabin, Joos)

- Replace $\partial_\mu \rightarrow \Delta^+_\mu$ in curl
- Replace $\partial_\mu \rightarrow \Delta^-_\mu$ in div.

Theorem (homology theory) :Prohibits doubles!

Alternatively see

$$S_F = \begin{pmatrix} \eta/2 & \chi_{12} \end{pmatrix} \begin{pmatrix} \Delta^-_1 & \Delta^-_2 \\ -\Delta^+_2 & \Delta^+_1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$\det M_{\rm fermion} = \det \Delta^+\Delta^-$ – single zero

Relation to staggered fermions (2D)

- Natural to place scalars on sites, vectors on links and tensors on diagonal links.
- Introduce a lattice with half spacing – all fields now site fields.
- Forward/backward difference operators become symmetric differences on fine lattice. Free KD action reduces to free staggered action – staggered fermion phases now arise from antisymmetry of derivatives in KD equation.

Homework problem 4. Check this
Usual chiral U(1) symmetry preserved: rotate $\psi_{1,2} \to e^{\alpha}\, \psi_{1,2}$ and $\eta, \chi_{12} \to e^{-\alpha}\, \eta, \chi_{12}$.

Summary

- Learned that process of finding nilpotent super symmetries can be lifted to 2d.
- Lattice actions which preserve 1 susy possible in WZ case. This susy is nilpotent and action has Q-exact form as for QM.
- For WZ can add mass terms a la Wilson to remove doublers.
- This structure can be generalized by invoking idea of twisting: decompose fields under twisted rotation group which is diagonal subgroup of usual rotations/boosts with flavor (R) symmetry.
- KD action for fermions. Can discretize even in massless case without doubling. Same as staggered fermions.
- Gauge theories ? $D > 2$?