Introduction to Lattice Supersymmetry

Simon Catterall

Syracuse University

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Motivation

- Motivation: SUSY theories - improved U.V behavior. Light Higgs natural in SUSY ...
- More tractable analytically – toy models for understanding confinement and chiral symmetry breaking
- Key component of string theory – remove tachyon of bosonic string.
- AdSCFT – super YM theories may tell us about gravity ..
- Realistic theories must break SUSY nonperturbatively at low energies – lattice.

Problems with lattice SUSY

- Extension of Poincare symmetry: $\{Q, \overline{Q}\} = \gamma.P$. Broken by lattice.
- Equivalently: Leibniz rule does not hold for difference operators
- Fermion doubling - $n_B \neq n_F$. Wilson terms break SUSY.
- Consequence: Naively discretized classical action breaks SUSY. Effective action picks up (many) SUSY violating operators. Generically some of these relevant.
- SUSY violating couplings generated via divergent Feynman graphs. Couplings must be fine tuned as $a \to 0$. Unnatural and impractical.

Solutions

- Just do it.
  - Certain simple cases eg. $\mathcal{N} = 1$ SYM in 4D – single counterterm using Wilson fermions. Or use DWF where gluino mass is zero as $L_s \to \infty$
  - For $D < 4$ finite number of divergences occuring at small numbers of loops - calc using (lattice) perturbation theory and subtract with counter terms
- For special class of theories can find novel discretizations which preserve one or more SUSY's exactly. Discretize a reformulation of continuum theory in twisted variables. Connections to topological field theory, orbifold constructions, ...

Plan

- Motivation/Problems.
- Witten's SUSYQM. Naive discretization. Fine tuning. Modification to maintain exact SUSY.
- Nicolai maps. Topological/twisted field theory interpretation.
- Lifting to 2D. Wess Zumino models. Twisting in 2D. Kähler-Dirac fermions.
- Gauge theories. Example: $\mathcal{N} = 2$ SYM.
- Lifting to 4D. $\mathcal{N} = 4$ SYM.
- Simulations: tools and problems

Witten's SUSYQM

$$S = \int dt \, \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}P'(\phi)^2 + \frac{1}{2}\psi_i \frac{d\psi_i}{dt} + i\psi_1\psi_2 P''(\phi)$$

Invariant under 2 SUSYs:

$$\begin{array}{rclrcl}
\delta_A\phi & = & \psi_1\epsilon_A & \delta_B\phi & = & \psi_2\epsilon_B \\
\delta_A\psi_1 & = & \frac{d\phi}{dt}\epsilon_A & \delta_B\psi_1 & = & -iP'\epsilon_B \\
\delta_A\psi_2 & = & iP'\epsilon_A & \delta_B\psi_2 & = & \frac{d\phi}{dt}\epsilon_B
\end{array}$$

Homework problem 1: verify these invariances

Continuum variation

Find:

$$\delta_A S = \int dt \; i\epsilon \left(P' \frac{d\psi_2}{dt} + \frac{d\phi}{dt} P'' \psi_2 \right)$$

Need to integrate by parts and use Leibniz to get zero. Problem for lattice.

Notice that $\delta_A^2 = \delta_B^2 = \frac{d}{dt}$ acting on any field.

Example of SUSY algebra since $H = \frac{d}{dt}$.

Naive discretization

Place fields on sites of (periodic) 1D lattice. Replace $\int dt \to \sum_t a$ and replace $\frac{d}{dt}$ by symmetric difference (fermion doubling ?)

$$a\Delta^S f_x = \frac{1}{2}(f(x+a) - f(x-a))$$

Now find:

$$\delta_A S_L = \sum_t \, i\epsilon \left(P' \Delta^S \psi_2 + \Delta^S \phi P'' \psi_2 \right)$$

Leibniz rule *does not hold for lattice difference ops* - susy breaking term O(a).
Naively goes away as $a \to 0$.
But radiative quantum corrections can change that ...

Aside on fermion doubling

- On lattice replace fermion derivative by suitable finite difference op.
- Natural choice $\partial \to \Delta^s$ where $\Delta^s f(x) = \frac{1}{2}\left(f(x+a) - f(x-a)\right)$
- In k-space: sin(ka)=0 for $ka = 0$ *and* $ka = \pi$ - doubler mode
- Additional mode *does not* decouple in continuum limit.
- Add (irrelevant) Wilson op. $\Delta_W f = -\frac{1}{2}\left(f(x+a) + f(x-a) - 2f(x)\right)$ - in k space lifts mass of additional modes to cut-off.
- Modify $P' \to P' + \Delta_W \phi$

Specific model

- Consider $P' = m\phi^2 + g\phi^3$
- Add Wilson mass term to P' - no doubles
- Action breaks both supersymmetries by terms O(a) (Leibniz broken)
- Test restoration of SUSY by extracting boson and fermion masses from 2pt functions and examining as $a \to 0$.
- See that SUSY is not restored even in QM (a finite theory in continuum!) What is going on ?

Boson/fermion masses - naive

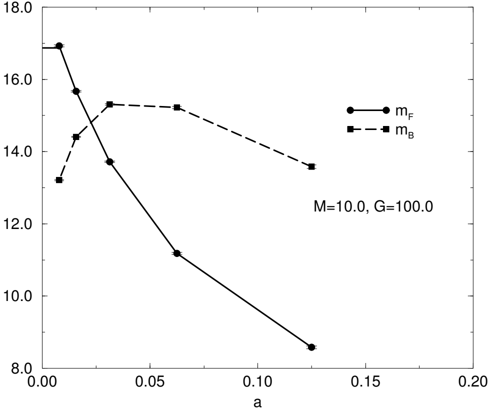

Figure: $P' = m\phi + g\phi^3,\ m = 10.0,\ g = 100.0$

Radiative corrections

Points to note:

- Any Feynman graph which is convergent in U.V can be discretized naively (Reisz theorem).
- Restrict attention to superficially divergent continuum graphs.
- In previous example only one of these. One loop fermion contribution to boson mass.
- Homework Problem 2. Convince yourself of this!
- Notice in continuum this diagram is actually finite because of additional symmetries - hence continuum theory is finite as expected. Not true on lattice ...

Radiative corrections II

Continuum:

$$\Sigma_{\rm cont} = 6g \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dp}{2\pi} \frac{-ip+m}{p^2+m^2}$$

Actually convergent ($p \to -p$ symmetry)

$$\Sigma_{\rm cont} = 6g \left(\frac{1}{\pi} \tan^{-1} \frac{\pi}{2ma} \right) \sim 6g \left(\frac{1}{2} + \mathcal{O}(ma) \right)$$

Lattice result is

$$\Sigma_{\rm latt} = \frac{6g}{L} \sum_{k=0}^{L-1} \frac{-2i \sin\left(\frac{\pi k}{L}\right) e^{i\left(\frac{\pi k}{L}\right)} + ma}{\sin^2\left(\frac{\pi k}{L}\right) + (ma)^2} \to 6g!$$

Intuitive argument

Consider using lattice derivative:

$$\Delta^s + \frac{r}{2} m_W$$

Doubler mass $M = m + 2r/a$. Consider limit where $r << 1$. Then $ma << Ma << 1$. Lattice integral is approx:

$$\begin{aligned}
\Sigma &= \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dp}{2\pi} \frac{m}{p^2 + m^2} + \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dp}{2\pi} \frac{M}{p^2 + M^2} \\
&= \frac{1}{\pi} \left(\tan^{-1} \frac{\pi}{2ma} + \tan^{-1} \frac{\pi}{2Ma} \right)
\end{aligned}$$

Radiative corrections III

- If take $a \to 0$ after doing sum get twice the result!
- Would be doublers have mass $O(1/a)$ and make an additional contribution to integral (don't decouple from small loops)
- Lattice p theory can be different from continuum!
- Restore SUSY need to add counterterm

$$S_L \to S_L + \sum_t 3g\phi^2$$

SUSY broken but regained now as $a \to 0$. Example of general approach: naive discretizations of SUSY actions require fine tuning of counterterms to achieve SUSY continuum limit. For superrenormalizable theories can be done in (lattice) p theory.

Masses - counterterm corrected

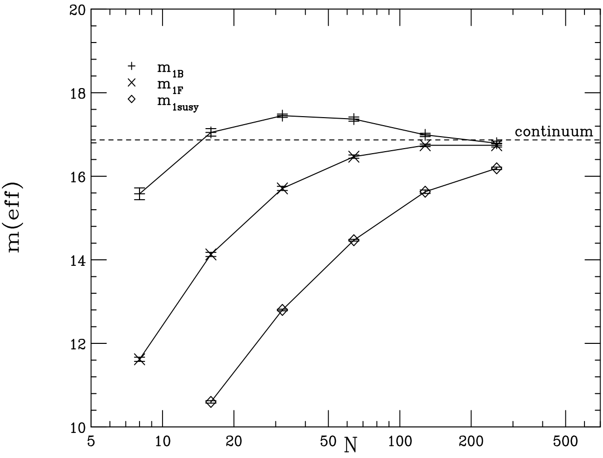

Figure: $P' = m\phi + g\phi^3, \; m = 10.0, \; g = 100.0$

Exact SUSY

For SUSYQM can do better. Find combination of SUSY's that can be preserved on lattice.
Notice that:

$$\delta_A S_L = -i\delta_B \sum_x P' \Delta^S \phi \qquad \delta_B S_L = i\delta_A \sum_x P' \Delta^S \phi$$

Thus

$$\left(\delta_A + i\delta_B\right) S_L = -\left(\delta_A + i\delta_B\right) O \qquad \text{where } O = \sum_t P' \Delta^S \phi$$

So can find $\delta S_{\rm exact}$ of form

$$S_L^{\rm exact} = \sum_t \frac{1}{2}(\Delta^S \phi)^2 + \frac{1}{2}P'^2 + P'\Delta^S\phi + \overline{\psi}(\Delta^S + P'')\psi$$

Exact SUSY II

Where

$$\begin{aligned} \psi &= \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \\ \overline{\psi} &= \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \end{aligned}$$

and the new supersymmetry acts:

$$\begin{aligned} \delta\phi &= \psi\epsilon \\ \delta\psi &= 0 \\ \delta\overline{\psi} &= (\Delta^S\phi + P'(\phi))\epsilon \end{aligned}$$

Notice: $\delta^2 = 0$ now. No translations.

$$S_L^b = \sum_x \left(\Delta^S\phi + P'(\phi)\right)^2$$

Masses - exact SUSY

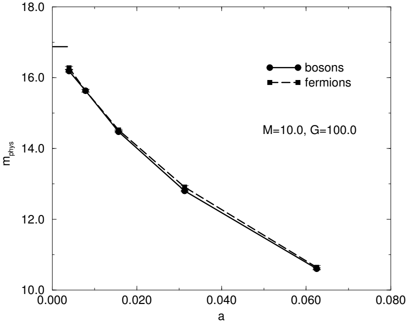

Figure: Boson and fermion masses vs lattice spacing for supersymmetric action

Exact SUSY action

- Exact lattice SUSY possible with $S_B = \frac{1}{2}N^2$ where $N = \Delta^S\phi + P' = \Delta^-\phi + P'_{\rm local}$
- Corresponding fermion operator $\Delta^- + P''$
- Notice that additional operator that is added to action is a total derivative in continuum. Needed to restore single SUSY on lattice
- This structure not a coincidence ...

Nicolai map

Partition function:

$$Z = \int D\phi D\psi D\overline{\psi}\ e^{-S} = \int D\phi \det\left(\Delta^- + P''\right) e^{-S_B}$$

Change variables to $\mathcal{N} = \Delta^-\phi + P'(\phi)$ Jacobian is $\det\left(\frac{\partial \mathcal{N}_i}{\partial \phi_j}\right)$. Cancels fermionic determinant!

$$Z = \int \prod_i d\mathcal{N}_i e^{-\mathcal{N}_i^2}$$

Details of $P(\phi)$ disappeared! Z is a topological invariant. Simple argument: $< S_B >= \frac{1}{2} N_{\rm d.o.f}$

Ward identities

Classical invariance of action replaced by relationships between correlation functions of form

$$< \delta O >= 0$$

Choosing $O = \overline{\psi}_x \phi_y$ we find

$$< \overline{\psi}_x \psi_y > + < (\Delta^- \phi + P')_x \phi_y >= 0$$

Expect other SUSY $\bar{\delta} = \frac{1}{\sqrt{2}}(\delta_A - i\delta_B)$ broken.
Restored in continuum limit without fine tuning.

More on 2nd SUSY ?

- Replace $\mathcal{N} \rightarrow -\Delta^{+}\phi + P'(\phi)$.
- Interchanges ψ, $\overline{\psi}$ and leaves fermion det unchanged.
- Bosonic action changes by term that is total deriv in continuum. Nonzero on lattice.
- However existence of 1 exact susy prohibits appearance of counter term $g\phi^2$ in effective action.
- Thus 2nd susy restored automatically as $a \rightarrow 0$ or equivalently long distances ..

Ward identities II

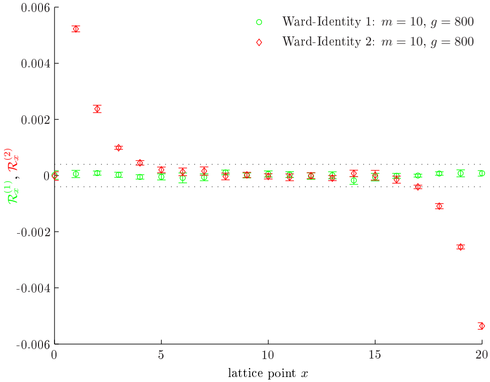

Figure: $m = 10.0$, $g = 800.0$, from Kaestner et al.

Topological field theory

Notice that $\delta^2 = 0$ for all fields using EOM.
Can render symmetry nilpotent off-shell by introducing auxiliary field

$$\begin{array}{rcl} Q\phi & = & \psi \\ Q\psi & = & 0 \\ Q\overline{\psi} & = & B \\ QB & = & 0 \end{array}$$

Note: absorbed ϵ into variation δ and renamed it Q. Also

$$S_L^b = \sum_x -B(\Delta^-\phi + P') - \frac{1}{2}B^2$$

TQFT II

Remarkably:

$$S_L = Q \sum_x \overline{\psi}(-\Delta^- \phi - P' - \frac{1}{2} B)$$

The action is Q-exact. Like BRST ?
Consider bosonic model with $S(\phi) = 0$. Invariant under $\phi \rightarrow \phi + \epsilon$ –topological symmetry.
Quantize: pick gauge function $\mathcal{N} = 0$ and introduce Fadeev-Popov factor

$$Z = \int D\phi \, \det\left(\frac{\partial \mathcal{N}}{\partial \phi}\right) e^{-\frac{1}{2\alpha}\mathcal{N}^2(\phi)}$$

Interpret ψ,$\overline{\psi}$ as ghost fields ($\alpha = 1$) recover our model!

Moral of the story

- Fine tuning problems can be handled in $D < 4$ by perturbative lattice calcs.
- In some cases can do better – find combinations of the supersymmetries in (some) SUSY models which are nilpotent. No immediate conflict with SUSY algebra.
- (Twisted) reformulations are closely connected to construction of TQFT. Actions are Q-exact. Easy to translate to lattice.
- Help protect lattice theory from SUSY violating ops. Additional (broken) SUSYs now regained without fine tuning