Introduction to $f(R)$-gravity

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- general $f(R)$
- R^2 example
- vacuum polarization and Starobinsky's inflation

$$S = \int d^4x\sqrt{-g}f(R) + S_m,$$

$$-\frac{1}{2}f g_{ik} + f_R R_{ik} - \nabla_i\nabla_k f_R + g_{ik}\Box f_R = T_{ik},$$

$$T_{ik} = 0,$$

$$3\Box f_R - 2f + f_R R = 0$$

$$f_R(R_0)R_0 - 2f(R_0) = 0,$$

$$m^2_{eff} = \frac{1}{3}\left(\frac{f_R}{f_{RR}} - R\right).$$

- $f_R > 0$ – graviton is not ghost
- $f_{RR} > 0$ – scalaron is not tachyon
- additional possible condition: $f(0) = 0$ – vanish cosmological constant

Cosmological constant Λ is a good candidate for dark energy (late time accelerating), but not for inflation one.
Cosmological constant can not explain possible fantom regime $w < -1$ for $p = w\rho$.

$$S = \int d^4x\sqrt{-\mathrm{g}}f(R)$$

$$\Downarrow$$

$$S = \int d^4x\sqrt{-\mathrm{g}}[\psi(\varphi)R - V(\varphi)]$$

where φ is defined by $\psi(\varphi) = f'(\varphi)$

and $V(\varphi) = \varphi f'(\varphi) - f(\varphi)$

$$S = \int d^4x\sqrt{-\mathrm{g}}\left[R - \frac{3}{2}\mathrm{g}^{ik}\nabla_i\sigma\nabla_k\sigma - V(\sigma)\right].$$

by conformal transformation $\mathrm{g}_{ik} \to e^{\sigma}\mathrm{g}_{ik}$ with $\sigma = -\ln f_R$

Several examples

- $f(R) = R + \frac{c_1\left(\frac{R}{\mu^2}\right)^n}{c_2\left(\frac{R}{\mu^2}\right)^n + 1}$

- $f(R) = R + \frac{(R - R_0)^n + R_0^n}{f_0 + f_1[(R - R_0)^n + R_0^n]}$

- $S = \int d^4x\sqrt{-g}\,[R + f_1(R) + f_2(R)L_d]$,

  with $L_d = \frac{1}{2}g^{ik}\nabla_i\varphi\nabla_k\varphi$

Another possibilities

$$f(R,G),\ f(R,C_{iklm}C^{iklm}),\ f(R,R_{ik}R^{ik}),\ f(R,\Box R)\ ...$$

$$S=\int d^Dx\sqrt{-\mathrm{g}}\left[\frac{R}{k}+c_1\alpha^{'}e^{-2\varphi}L_2+c_2\alpha^{'2}e^{-4\varphi}L_3+c_3\alpha^{'3}e^{-6\varphi}L_4+...\right]$$

$$L_2=R^2_{\mu\nu\alpha\beta}-4R^2_{\mu\nu}+R^2,$$

$$L_3=2\Omega_3+R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\rho}R^{\lambda\rho}{}_{\mu\nu}.$$

R^2 example

$$S = \int d^4x\sqrt{-\mathrm{g}}(R + \alpha R^2).$$

$$2\alpha\nabla_i\nabla_k R - (1 + 2\alpha R)R_{ik} + \mathrm{g}_{ik}[\frac{1}{2}\alpha R^2 + \frac{1}{2}R - 2\alpha\Box R] = 0.$$

$$6\alpha\Box R = R \quad \Rightarrow \quad m^2_{eff} = \frac{1}{6\alpha}.$$

$$\tilde{\mathrm{g}}_{ik} = (1 + 2\alpha\varphi)\mathrm{g}_{ik}.$$

$$\tilde{R}_{ik} - \frac{1}{2}\tilde{g}_{ik}\tilde{R} = \frac{6\alpha^2}{(1+2\alpha\varphi)^2}\left(\nabla_i\varphi\nabla_k\varphi - \frac{1}{2}\tilde{g}_{ik}\left[\nabla_i\varphi\nabla^i\varphi + \frac{\varphi^2}{6\alpha}\right]\right).$$

This correspond to the theory:

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} - \frac{6\alpha^2}{(1+2\alpha\varphi)^2}\left(\nabla_i\varphi\nabla^i\varphi + \frac{\varphi^2}{6\alpha}\right)\right].$$

and the field equation for φ:

$$6\alpha(1+2\alpha\varphi)\Box\varphi - 12\alpha^2\nabla_i\varphi\nabla^i\varphi = \varphi.$$

$$6\alpha(1+2\alpha\varphi)(-\ddot{\varphi}-3H\dot{\varphi})-12\alpha^2\dot{\varphi}^2=\varphi.$$

in the limit of large φ

$$\varphi \propto -t$$

and from Einstein equation we find

$$H^2=\frac{1}{24\alpha}$$

– that is quasi de Sitter solution.

$$R_{ik} - \frac{1}{2}g_{ik}R = \langle T_{ik} \rangle$$

$$\langle T_{ik} \rangle = \frac{m_2}{2880\pi^2}(R_i{}^l R_{kl} - \frac{2}{3}RR_{ik} - \frac{1}{2}g_{ik}R_{lm}R^{lm} + \frac{1}{4}g_{ik}R^2)$$
$$+\frac{m_3}{2880\pi^2}\frac{1}{6}(2R_{;i;k} - 2g_{ik}R{}^{;l}_{;l} - 2RR_{ik} + \frac{1}{2}g_{ik}R^2)$$

$$k_2 = \frac{m_2}{60(4\pi)^2} = \frac{N + 11N_{\frac{1}{2}} + 62N_1 + 1411N_2 - 28N_{HD}}{60(4\pi)^2}$$

$$k_3 = \frac{m_3}{60(4\pi)^2} = -\frac{N + 6N_{\frac{1}{2}} + 12N_1 + 611N_2 - 8N_{HD}}{60(4\pi)^2}$$

$$\rho_q = k_2 H^4 + k_3(2\ddot{H}H + 6\dot{H}H^2 - \dot{H}^2)$$

$$6H^2 = \rho_q$$

- Vacuum stability condition: $k_3 < 0$
- Exist de Sitter solution for $k_2 > 0$
- Singularity problem may be solved for $K = -1$