Inflation and the theory of cosmological perturbations

(3000 Mpc << L << 3 x 10^{26} Gpc from us)

Will be quite technical this time

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1. Some problems of the Big Bang cosmology
2. Standard inflationary paradigm
3. Inflationary zoology: large-field, small-field, hybrid
4. Theory of cosmological perturbations; primordial inhomogeneities
5. Inflation and observations
6. Alternatives to inflation (if I have enough time)

What is the origin of the Cosmic Web?
Why the initial state is highly symmetric?
What happens at superhorizon scales?

Dubna, 24 Jul 2007

Some problems of the Big Bang cosmology 1

Trying to answer those questions we also have to answer these (it turns out that the answers are closely related)

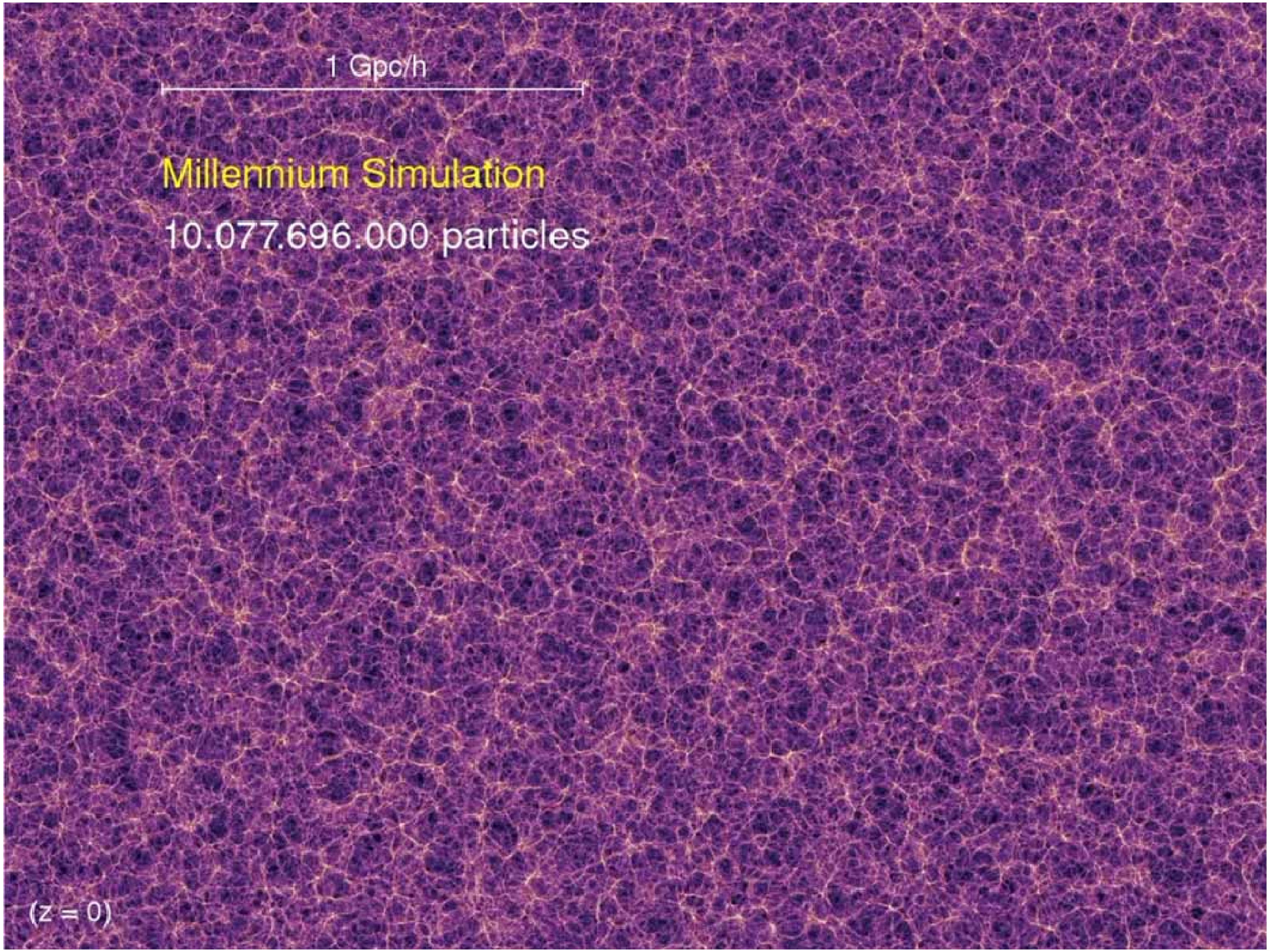
1. **Homogeneity problem.** The Universe is nearly homogeneous on the scales > 500 Mpc. Why?
2. **Isotropy problem**. Why all directions in the Universe are equivalent? Why for example there is no overall rotation of the Universe?
3. **Horizon problem.** If different parts of the Universe have not been in a causal contact when the universe was born, why do they look so similar? (CMB which came from the “opposite” sides of the Universe is correlated at the level 1 to 100000)
4. **Spatial flatness problem**. Why the total energy density in the Universe is so close to the critical one?

Some problems of the Big Bang cosmology 2

5. **Total entropy problem.** The total entropy of the observable part of the Universe is greater than 10^{87}. Why it is so huge? Note that the lifetime of a closed universe filled with hot gas with total entropy S is $S^{2/3} \times 10^{-43}$ seconds, so S **must** be huge. But why in our world?
6. **Total mass problem.** The total mass of the observable part of the Universe is 10^{60} M_P . The lifetime of a closed universe filled with non-relativistic particles of total mass M is $M/M_P \times 10^{-43}$ seconds. Thus M must be huge. Why?
7. **Heavy relics problem.** According to the BB paradigm, the Universe was hot in the beginning of its evolution, and it was possible for heavy relics to be spontaneously created in the primordial plasma. Why we don’t see them? Monopoles, gravitino, etc.
8. **Structure formation problem**. Although the Universe is homogeneous on the scale of horizon (400 Mpc < L < 3000 Mpc), there should be primordial inhomogeneities for the large scale structure formation. Why at the level of only 1:100000? Reminder of the LSS – Millennium simulation

(The ultimate answer is inflation)

Reminder: Large scale structure of the Universe



What is the main idea of inflation 1

Metric of the background spacetime $ds^2 = dt^2 - a(t)^2 d\mathbf{x}^2$

Its rate of expansion is $H(t) = \frac{\dot{a}}{a}(t)$

Key scale in cosmology is the **curvature scale** $l_H(t) = H^{-1}(t)$ When SEC is satisfied for a dominant matter component, this scale also defines the size of causal patch (**horizon** size).

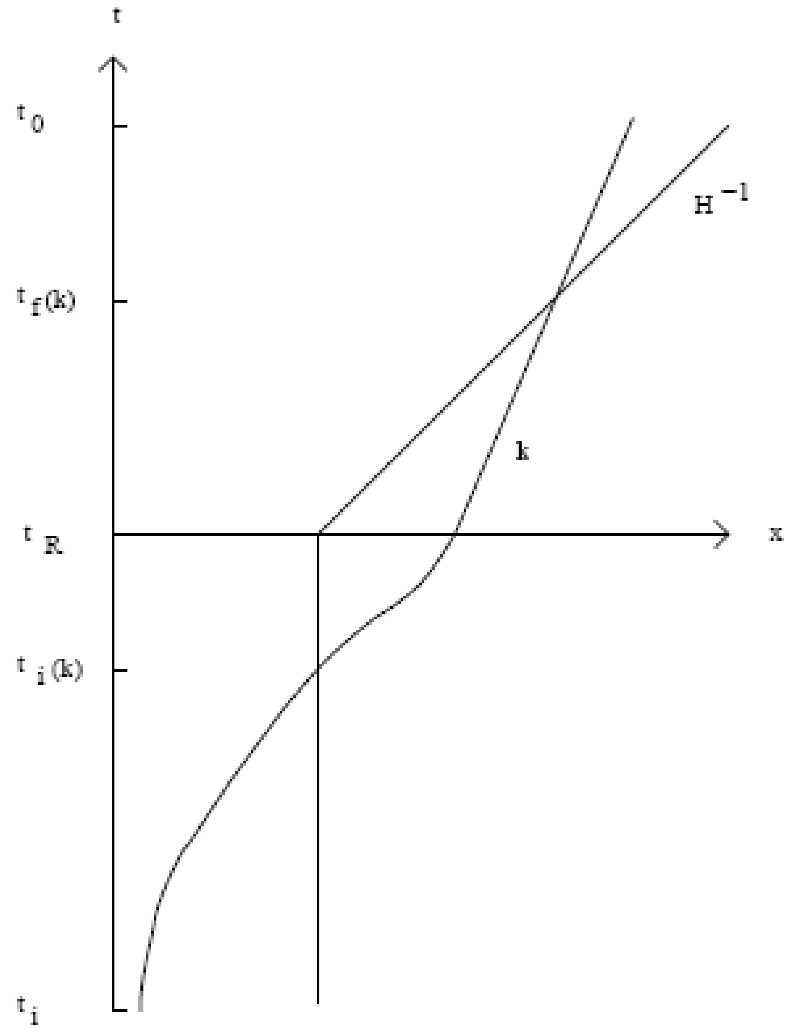
Friedmann equations (Einstein equations):

$$\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho$$

$$\frac{\ddot{a}}{a} = -4\pi G(\rho + 3p)$$

If SEC is violated, i.e. $p < -\frac{1}{3}\rho$, we have **accelerated expansion**. In the degenerate case the Hubble rate is constant but the physical wavelengths are exponentially growing and cross the Hubble scale at some point.

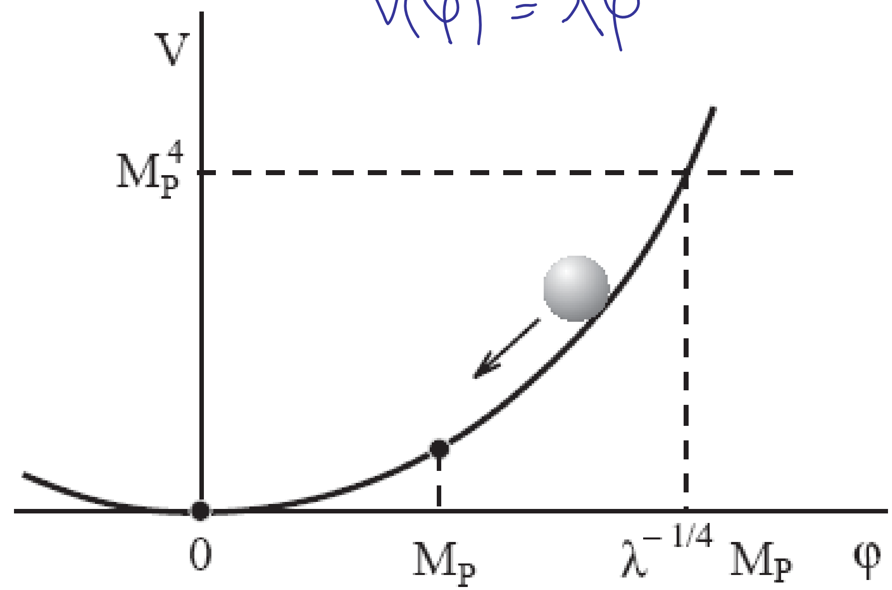
After the end of accelerated expansion stage they reenter the horizon.



$\left(\frac{K}{a} \sim \frac{K}{t^{1/2}};\ H \sim 1/t, \text{ so } \frac{K}{a} > H \text{ eventually}\right)$

What is the main idea of inflation 2

$V(\varphi) = \lambda\varphi^4$



How to realize this scenario at the fundamental level?

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right]$$

Energy density and pressure are

$$\rho = \frac{1}{2}(\dot{\varphi})^2 + \frac{1}{2}a^{-2}(\nabla\varphi)^2 + V(\varphi)$$
$$p = \frac{1}{2}(\dot{\varphi})^2 - \frac{1}{6}a^{-2}(\nabla\varphi)^2 - V(\varphi)$$

If the conditions

$$\frac{1}{2}(\nabla_p\varphi)^2 \ll V(\varphi)$$
$$\frac{1}{2}(\dot{\varphi})^2 \ll V(\varphi)$$

and $\ddot{\varphi} \ll 3H\dot{\varphi}$ are satisfied (or $\left(\frac{V'}{V}\right)^2 \ll 48\pi G$ and $\frac{V''}{V} \ll 24\pi G$)

then the scalar field is in the "**slow roll**" regime; effective equation of state is p=-ρ, so the expansion of the Universe is accelerated.

Slow roll parameters: $\varepsilon \sim M_P^2\left(\frac{V'}{V}\right)^2$, $\eta \sim M_P^2\left(\frac{V''}{V}\right)$,
derivatives are w.r.t. φ.

How inflation solves the problems of the Big Bang cosmology

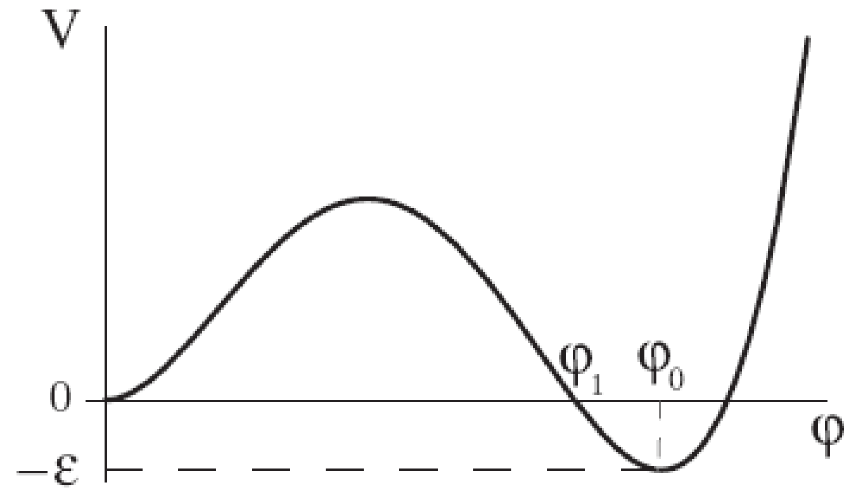
1. **Homogeneity and horizon problems:** during inflation the Universe expanded so much that now it contains large number of Hubble patches in the end of inflation. Primordial ihomogeneities were stretched to horizon (or much larger) scales during inflation (physical λ is proportional to a)
2. **Isotropy problem:** anisotropies are stretched as well. The contribution of the anisotropy into the right-hand side of the Friedmann equation is proportional to a^{-6} and a is growing (quasi-) exponentially
3. **Spatial flatness problem:** $|\Omega-1|=k^2/H^2a^2$ where $k=\{-1,0,1\}$
4. **Heavy relics:** they are redshifted. Energy density of relativistic matter behaves as a^{-4}, and energy density of non-relativistic matter – as a^{-3}
5. **Entropy and total mass problems:** strictly speaking, both are solved by ”greaceful exit” from inflation – preheating; huge (initially Planckian) energy density of the inflaton condensate is transferred into particles of the standard model.
6. **Structure formation problem:** inflation predicts small inhomogeneities (at the level of 1 to 100000); sections of my talk devoted to the cosmological perturbations

Inflationary zoology 1: small-field models

Old inflation (Guth, 1981)
(first order phase transition)

$$V(\varphi) = \frac{M}{2}\varphi^2 - \frac{\delta}{3}\varphi^3 + \frac{\lambda}{4}\varphi^4$$

Inflation ends by tunneling; nucleation of bubbles; no ”graceful exit” (collision of bubbles and inhomogeneities instead of very homogeneous initial state of the Hot Universe)



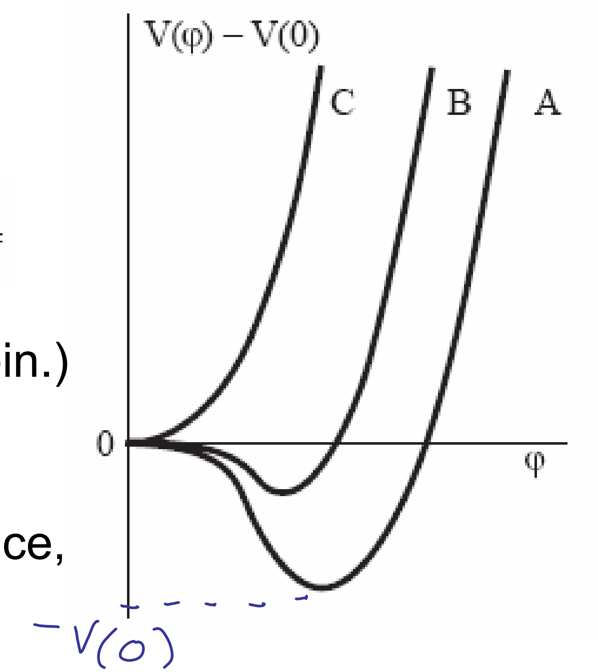
New inflation (Linde, 1982)
(SSB: second order phase transition)

$V(\phi) = \frac{\lambda}{4}(\phi^2 - \mu^2)^2 + \frac{\lambda}{8}T^2\phi^2$, critical temperature $T_c = \frac{2\mu}{\sqrt{\lambda}}$

No slow roll for $\mu \ll M_P$, situation improves with radiative corrections taken into account (Coleman-Wein.)

$$V(\phi) = \frac{\lambda}{4}\phi^4\left(\log\frac{\phi}{\mu} - \frac{1}{4}\right) + \frac{\lambda}{16}\mu^4$$

Slow roll trajectory is not an attractor in the phase space, so the fine tuning of initial conditions is necessary



Inflationary zoology 2: large-field models

Chaotic inflation (Linde, 1983)

Friedmann eq. and EOM for φ (M_P=1):

$$\ddot{\phi} + 3H\dot{\phi} = -m^2\phi$$

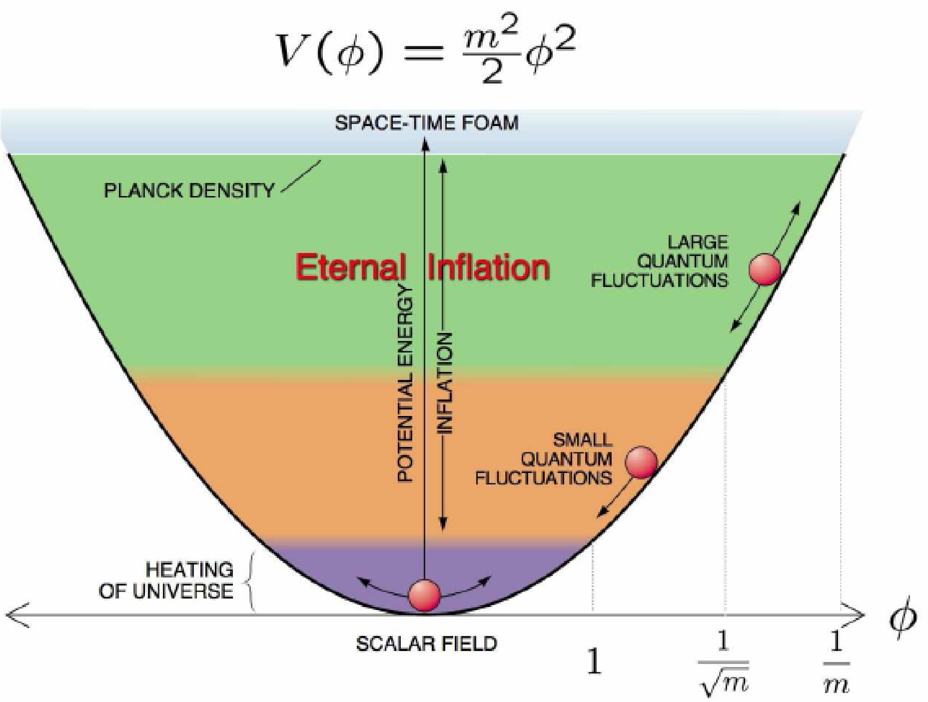
$$H^2 + \frac{k}{a^2} = \frac{1}{6}\left(\dot{\phi}^2 + m^2\phi^2\right)$$

In the slow roll regime, when

$$\ddot{\phi} \ll 3H\dot{\phi},\ H^2 \gg \frac{k}{a^2},\ \dot{\phi}^2 \ll m^2\phi^2$$

one has

$$H = \frac{\dot{a}}{a} = \frac{m\phi}{\sqrt{6}}\ , \qquad \dot{\phi} = -m\sqrt{\frac{2}{3}}$$



so the Universe expands quasiexponentially: $a = a_0\ e^{\phi^2/4}$

The problem of initial conditions: hard to embed into supergravity or superstring theory models, corrections from supergravity spoil the flatness of the potential at large values of the inflaton.

Inflationary zoology 3: hybrid inflation

Suppose we have two scalars: $V(\phi,\psi) = \frac{\lambda}{4}(\psi - M)^4 + V(\phi) + \frac{1}{2}g^2\psi^2\phi^2$

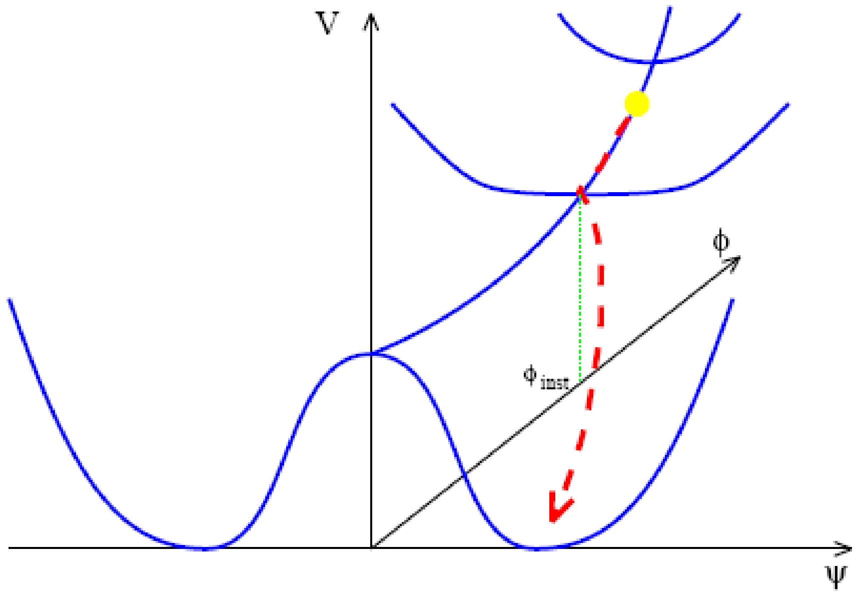
Effective mass of ψ is $\boxed{m^2_{\text{eff},\psi} = -\lambda M^2 + g^2\phi^2}$, so it can be quite massive at large values of φ, in particular, at

$$\phi > \phi_c = \sqrt{\lambda}\frac{M}{g}$$

and is not excited. The inflaton φ sees the potential

$$V_{\text{eff},\phi} = V(\phi) + \frac{\lambda}{4}M^4$$

(very easy for slow roll conditions to be satisfied)



Below the critical value, phase transition happens, the system quickly relaxes to the equilibrium state ψ=±M, φ=0. The slow roll conditions break down, and inflation is followed by the tachyonic preheating stage.

The major prediction of inflation is flat spectrum of primordial perturbations. Let us see how it works, but first we need some background.

Cosmological perturbations: Minkowski spacetime, Newtonian approximation

Hydrodynamics of matter in the flat spacetime (Newtonian approximation):

$$\dot{\rho} + \nabla_p \cdot (\rho \mathbf{v}) = 0$$

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla_p)\mathbf{v} + \frac{1}{\rho}\nabla_p p + \nabla_p \varphi = 0$$

$$\nabla_p^2 \varphi = 4\pi G \rho$$

$$\dot{S} + (\mathbf{v} \cdot \nabla_p) S = 0$$

$$p = p(\rho, S)$$

$\rho = \rho_0 + \delta\rho$ - matter density

$\mathbf{v} = \delta\mathbf{v}$ - 3-velocity

$p = p_0 + \delta p$ - pressure

$\varphi = \varphi_0 + \delta\varphi$ - gravitational potential

$S = S_0 + \delta S$ - entropy

$$\ddot{\delta\rho} - c_s^2 \nabla_p^2 \delta\rho - 4\pi G \rho_0 \delta\rho = \sigma \nabla_p^2 \delta S$$

$$\dot{\delta S} = 0,$$

EOS: $\delta p = c_s^2 \delta\rho + \sigma \delta S$

Speed of sound: $c_s^2 = \left(\frac{\delta p}{\delta \rho}\right)_{|S}$

Translational invariance in 3D space, so using Fourier $\delta\rho(\mathbf{x}, t) = \int e^{i\mathbf{k}\cdot\mathbf{x}} \delta\rho_k(t)$

Adiabatic perturbations: entropy is fixed, density is changing; time-dependent

Entropy perturbations: density is fixed, entropy perturbation is non-zero; they don't grow, they seed adiabatic perturbations

Jeans length: $k_J = \left(\frac{4\pi G \rho_0}{c_s^2}\right)^{1/2}$, at $k \ll k_J$ adiabatic perturbations exponentially grow:

$\delta\rho_k(t) \sim e^{\omega_k t}$ with $\omega_k \sim 4(\pi G \rho_0)^{1/2}$, at smaller scales – oscillate with $\omega_k \sim c_s k$

Cosmological perturbations: expanding spacetime, Newtonian approximation

The next step: let's take cosmological expansion into account (still Newtonian approximation) by modelling it with background velocity $\mathbf{v}_0 = H(t)\mathbf{x}_p$

$$\rho(t,\mathbf{x}) = \rho_0(t)\big(1+\delta_\varepsilon(t,\mathbf{x})\big)$$
$$\mathbf{v}(t,\mathbf{x}) = \mathbf{v}_0(t,\mathbf{x}) + \delta\mathbf{v}(t,\mathbf{x})$$
$$p(t,\mathbf{x}) = p_0(t) + \delta p(t,\mathbf{x}),$$

Fractional density perturbation δ satisfies

$$\ddot{\delta}_\varepsilon + 2H\dot{\delta}_\varepsilon - \frac{c_s^2}{a^2}\nabla^2\delta_\varepsilon - 4\pi G\rho_0\delta_\varepsilon = \frac{\sigma}{\rho_0 a^2}\delta S$$

The major difference is the Hubble damping: UV modes are exponentially damped, and the Jeans instability is smoother – $\delta_k(t) = c_1 t^{2/3} + c_2 t^{-1}$

Note: due to this growth the perturbation theory will break down for these modes and they will leave the "Hubble flow" – the origin of the Cosmic Web.

Spectra: the root mean square mass fluctuation $\delta M/M(k,t)$ in the sphere with radius $l = 2\pi/k$ at time t:

$$\left(\frac{\delta M}{M}\right)^2(k, t_H(k)) \sim k^{n-1}$$

n is the spectral index, n=1 corresponds to the Harrison-Zeldovich scale invariant spectrum

In the momentum space $\delta_\varepsilon(\mathbf{x},t) = \int d^3k\,\tilde{\delta}_\varepsilon(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}$ the power spectrum of density fluctuations is defined as

$P_\delta(k) = \frac{1}{2\pi^2}k^3|\tilde{\delta}_\varepsilon(k)|^2$, the power spectrum of the gr. potential $P_\varphi(k) = \frac{1}{2\pi^2}k^3|\delta\tilde{\varphi}(k)|^2$

The two are related by the Poisson equation: $P_\varphi(k) \sim k^{-4}P_\delta(k)$

Cosmological perturbations: relativistic theory 1

Using conformal time $dt = a(t)d\eta$; the metric is now $ds^2 = a(\eta)^2(d\eta^2 - d\mathbf{x}^2)$

Possible perturbations $g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}$ are:

scalar perturbations
(4 modes)

$$\delta g_{\mu\nu} = a^2 \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2(\psi\delta_{ij} - E_{,ij}) \end{pmatrix}$$

vector perturbations
(4 modes,
divergence-free vectors)

$$\delta g_{\mu\nu} = a^2 \begin{pmatrix} 0 & -S_i \\ -S_i & F_{i,j} + F_{j,i} \end{pmatrix}$$

tensor modes
(2 modes,
transverse traceless)

$$\delta g_{\mu\nu} = -a^2 \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}$$

with $h_i^i = h_{ij}^j = 0$.

There are pure gauge modes among these: $x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$; $\xi^i = \xi^i_{tr} + \gamma^{ij}\xi_{,j}$

$$\begin{aligned}
\tilde{\phi} &= \phi - \frac{a'}{a}\xi^0 - (\xi^0)' \\
\tilde{B} &= B + \xi^0 - \xi' \\
\tilde{E} &= E - \xi \\
\tilde{\psi} &= \psi + \frac{a'}{a}\xi^0,
\end{aligned}$$

Longitudinal gauge is $B = E = 0$

Cosmological perturbations: relativistic theory 2

Let's focus more on inflation. At the linear level the inflaton does not excite vector and tensor degrees of freedom; so we will write

$$ds^2 = a^2\left[(1+2\phi)d\eta^2 - (1-2\psi)\gamma_{ij}dx^i dx^j\right]$$

There is also no anisotropic stress as well, so $\psi = \phi$. Then, EOM for gravitational potential and inflaton perturbation are

$$\begin{aligned}
\nabla^2\phi - 3\mathscr{H}\phi' - (\mathscr{H}' + 2\mathscr{H}^2)\phi &= 4\pi G(\varphi_0'\delta\varphi' + V' a^2\delta\varphi) \\
\phi' + \mathscr{H}\phi &= 4\pi G\varphi_0'\delta\varphi \\
\phi'' + 3\mathscr{H}\phi' + (\mathscr{H}' + 2\mathscr{H}^2)\phi &= 4\pi G(\varphi_0'\delta\varphi' - V' a^2\delta\varphi)
\end{aligned}$$

which can be recombined into

$$\phi'' + 2\left(\mathscr{H} - \frac{\varphi_0''}{\varphi_0'}\right)\phi' - \nabla^2\phi + 2\left(\mathscr{H}' - \mathscr{H}\frac{\varphi_0''}{\varphi_0'}\right)\phi = 0$$

Here $\mathscr{H} = a'/a$

This is similar to what we found for the Neutonian approximation case (second term is the Hubble friction, the last term is due to the gravitational instability)

Cosmological perturbations: relativistic theory 3

It is often instructive to use the variable $\zeta \equiv \phi + \frac{2}{3}\frac{\left(H^{-1}\dot{\phi} + \phi\right)}{1+w}$ where $w = \frac{p}{\rho}$
(it is actually curvature perturbation in the comoving gauge)

The EOM is $\frac{3}{2}\dot{\zeta}H(1+w) = \mathscr{O}(\nabla^2\phi)$ and the nice thing is that $\dot{\zeta}(1+w) = 0$
at super-Hubble scales. For a single scalar field it does not change during cosm. evolution. Therefore,

$\phi(t_f(k)) \simeq \frac{(1+w)(t_f(k))}{(1+w)(t_i(k))}\phi(t_i(k))$. Initially, $\phi(t_i(k)) \sim H\frac{V'}{V}(t_i(k))$

Also, during slow roll $1+w(t_i(k)) \simeq \frac{\dot{\varphi}_0^2}{V}(t_i(k))$ while w=1/3 during RD stage.

Using the equations of motion

$$3H\dot{\varphi}_0 \simeq -V'$$
$$H^2 \simeq \frac{8\pi G}{3}V$$

one finally finds $\phi(t_f(k)) \sim \frac{V^{3/2}}{V' m_{pl}^3}(t_i(k))$ Since the slow roll parameter is very

small initially, r.h.s. depends on k only very weakly – **almost flat spectrum**.

Cosmological perturbations: quantum theory 1

No particles in the inflationary universe, just the inflaton condensate.

But where did the **initial** perturbations come from?

We have $S = \int d^4x \sqrt{-g}[-\frac{1}{16\pi G}R + \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - V(\varphi)]$

and work in the longitudinal gauge (where also $\psi = \phi$). The goal is to find some variable such that the corresponding effective action is quadratic – looking for oscillators to quantize (want to understand the vacuum of the theory).

The answer is the **Mukhanov-Sasaki variable**

$S^{(2)} = \frac{1}{2}\int d^4x [v'^2 - v_{,i}v_{,i} + \frac{z''}{z}v^2]$, where $v = a[\delta\varphi + \frac{\varphi_0'}{\mathscr{H}}\phi]$ and $z = \frac{a\varphi_0'}{\mathscr{H}}$

When effective EOS does not change, $z(\eta) \sim a(\eta)$

Physical meaning of Mukhanov-Sasaki variable is **curvature perturbation**: $v = z\mathscr{R}$

The EOM is $v_k'' + k^2 v_k - \frac{z''}{z}v_k = 0$, so at $k > k_H$ where $k_H^2 \equiv \frac{z''}{z} \simeq H^2$

curvature perturbation oscillates, while $v_k \sim z$, $k \ll k_H$ - old story of the Jeans instability and freezing out at super-Hubble scales.

Cosmological perturbations: quantum theory 2

Initial conditions are as in the **Bunch-Davies ”adiabatic” vacuum** (de Sitter invariant quantum state with zero number of particles defined as WKB adiabatic invariant):

$$v_k(\eta_i) = \frac{1}{\sqrt{2\omega_k}}$$

$$v'_k(\eta_i) = \frac{\sqrt{\omega_k}}{\sqrt{2}}$$

of growth rate superhorizon modes

For the spectrum one has

used $z(\eta) \sim a(\eta)$;
V_k amplitude is constant at subhorizon scales

$$\begin{aligned} 2\pi^2 \mathscr{P}_{\mathscr{R}}(k,t) \equiv k^3 |\mathscr{R}_k|^2(t) &= k^3 z^{-2}(t) |v_k(t)|^2 \\ &= k^3 z^{-2}(t) \underline{\left(\frac{z(t)}{z(t_H(k))}\right)^2 |v_k(t_H(k))|^2} \\ &= k^3 z^{-2}(t_H(k)) |v_k(t_H(k))|^2 \\ &\sim k^3 a^{-2}(t_H(k)) |v_k(t_i)|^2 , \end{aligned}$$

Therefore, the spectrum is scale invariant $\mathscr{P}_{\mathscr{R}}(k,t) \sim k^3 k^{-2} k^{-1} H^2$

(Used $a^{-1}(t_H(k))k = H$ and BD initial conditions)

Slow roll inflation and eternal inflation

$N = -\ln \frac{a}{a_f}$

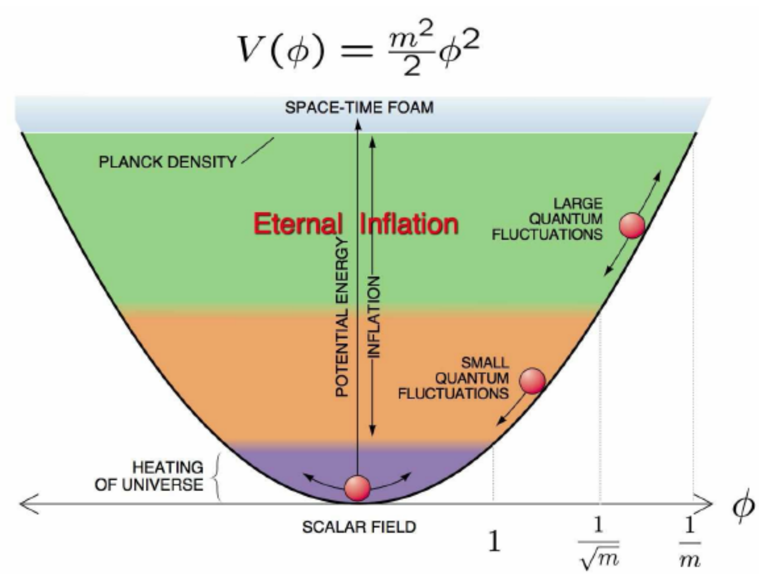
Please note that during one Hubble time (one e-folding) the EV of inflaton changes due to quantum fluctuations according to

$$\langle \delta_q \varphi^2 \rangle = \int_{k=aH}^{k=(e\times a)H} \frac{dk}{k} \frac{k^3}{2\pi^2} \delta_q\varphi_k \delta_q\varphi_{-k} \approx \frac{H^2}{4\pi^2} \text{ so that } |\delta\phi| \sim \frac{H}{2\pi} \sim \sqrt{\frac{2V}{3\pi M_P^2}}$$

On the other hand, during the same time the value of the inflaton decreases due to the classical force as

$$\Delta\phi \sim \dot{\phi}\Delta t \sim \frac{1}{3H^2}\frac{\partial V}{\partial \phi} \sim \frac{M_P^2}{8\pi V}\frac{\partial V}{\partial \phi}$$

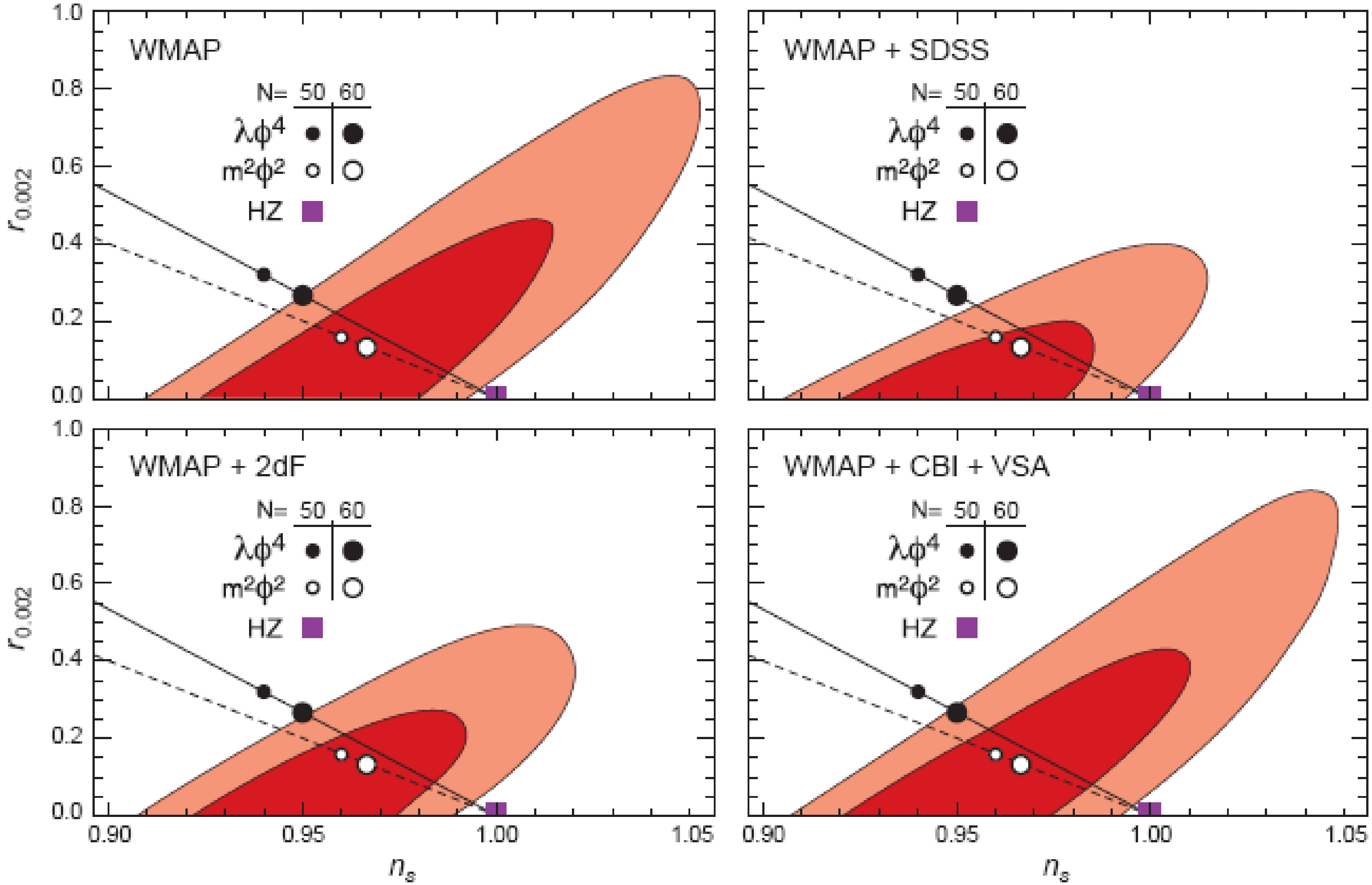
One can see that there could be a moment of time when the first effect will be stronger than the second one. For chaotic inflation this happens if the value of φ is large enough. This is dangerous regime of **eternal inflation** when perturbations are of the same order as the background. If the inflaton value drops, it can enter the slow roll regime.



$\| \quad \left(\frac{M_p}{m}\right)^{1/2} M_p$

More: the next lecture

Inflation and observations



r is tensor-to-scalar ratio, for $V \sim \phi^n$ one has $r = 4n/N$ where N is the number of efoldings (N≈60)

By definition, $N = -\ln \frac{a}{a_f}$.

From WMAP3

Some alternatives to inflation

1. **Pre-Big-Bang (Veneziano, 1991).** Using a certain duality in LEEA of string theory, introduce a stage of accelerated expansion without inflaton potential (dilaton plays a role of ”inflaton”). Necessary to pass through singularity in order for expansion regime to change from acceleration to deceleration. The problem of passing through the singularity (BKL anisotropic type) remains unsolved.
2. **Ekpyrotic/cyclic scenario (Khoury et al., 2001; Steinhardt, Turok, 2002)** Brane world; tachyonic potential → ”superluminal” expansion → effects of inflation without the inflaton. The spectrum of perturbations (without extreme fine tuning) is very far from being flat.
3. **String gas cosmology (Brandenberger, Vafa, 1989).** Using T-duality of string theory; Hagedorn phase (winding modes and momentum modes; infinite horizon) and RD phase (winding modes annihilated and the expansion started; finite horizon) – perturbations are correlated at scale much larger than the horizon in the RD phase. Actually leads to the strongly blue spectrum of perturbations with n=5.