Lecture #2
The Universe: from today back to the primordial nucleosynthesis having formulae at hand

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Outline

1. The FRW metric, reddening and the Hubble law
2. Gas of free particles in the expanding Universe
3. The dynamics of the expanding Universe
   - Einstein equations
   - Eamples of cosmological solutions
   - Particles horizon
   - Events horizon

Homogeneous and isotropic 3d manifolds

$$dl^2 = d\rho^2 + r^2(\rho)(d\theta^2 + \sin^2\theta d\phi^2)$$

$$r(\rho) = \begin{cases} R\sin(\rho/R)\,, & \text{3-sphere} \\ \rho\,, & \text{3-plane} \\ R\sinh(\rho/R)\,, & \text{3-hyperboloid} \end{cases}$$

ρ is a geodesic distance; $S = 4\pi r^2(\rho)$; $\Delta\theta = \frac{l}{r(\rho)}$

$$d\rho^2 = \frac{dr^2}{\cosh^2\frac{\rho}{R}} = \frac{dr^2}{1+\frac{r^2}{R^2}} \qquad d\rho^2 = \frac{dr^2}{\cos^2\frac{\rho}{R}} = \frac{dr^2}{1-\frac{r^2}{R^2}}$$

$$dl^2 = \frac{dr^2}{1-\varkappa\frac{r^2}{R^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Homogeneous and isotropic 3d manifolds

$$dl^2 = \gamma_{ji} dx^i dx^j = \frac{dr^2}{1 - \varkappa \frac{r^2}{R^2}} + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

$$\varkappa = \begin{cases} +1\,, & \text{3-sphere} \\ 0\,, & \text{3-plane} \\ -1\,, & \text{3-hyperboloid} \end{cases}$$

$$^{(3)}R_{ijkl} = \frac{\varkappa}{R^2}(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})$$

$$^{(3)}R_{ij} = 2\frac{\varkappa}{R^2}\gamma_{ij}\,, \qquad {}^{(3)}R = 6\varkappa R^{-2}$$

FRW(L) metric

$$ds^2 = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j \ ,$$

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

Special frame: different parts look similar
Also this is comoving frame: world lines of particles at rest are geodesics,

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = 0 \ .$$

$$\gamma_{ij} \approx \delta_{ij}$$

Photons in the expanding Universe

$$S = -\frac{1}{4}\int d^4x \sqrt{-g} g^{\mu\nu} g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho}$$

$dt = a d\eta$ conformally flat metric

$$ds^2 = dt^2 - a^2(t)\delta_{ij} dx^i dx^j \longrightarrow ds^2 = a^2(\eta)[d\eta^2 - \delta_{ij} dx^i dx^j]$$

$$S = -\frac{1}{4}\int d^4x \, \eta^{\mu\nu}\eta^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} \, , \qquad A_\mu^{(\alpha)} = e_\mu^{(\alpha)} e^{ik\eta - i\mathbf{k}\mathbf{x}} \, , \quad k = |\mathbf{k}|$$

$$\Delta x = 2\pi/k \, , \quad \Delta\eta = 2\pi/k$$

$$\lambda(t) = a(t)\Delta x = 2\pi\frac{a(t)}{k} \, , \quad T = a(t)\Delta\eta = 2\pi\frac{a(t)}{k}$$

Redshift and the Hubble law $\lambda_0 = \lambda_i \frac{a_0}{a(t_i)} \equiv \lambda_i(1+z(t_i))$

$$\mathbf{p}(t) = \frac{\mathbf{k}}{a(t)}\,, \quad \omega(t) = \frac{k}{a(t)}$$

for not very distant objects

$$a(t_i) = a_0 - \dot{a}(t_0)(t_0 - t_i) \quad \longrightarrow \quad a(t_i) = a_0[1 - H_0(t_0 - t_i)]$$

$$z(t_i) = H_0(t_0 - t_i) = H_0 r\,, \qquad z \ll 1$$

$$H_0 = h \cdot 100\,\frac{\mathrm{km}}{\mathrm{s}\cdot\mathrm{Mpc}}\,, \qquad h = 0.73^{+0.04}_{-0.03}$$

similar reddening for other relativistic particles (small H, $\dot{H}$, etc.)

$\mathbf{p} = \frac{\mathbf{k}}{a(t)}$ is true for massive particles as well

Gas of free particles in the expanding Universe

homogeneous gas $$dN = f(t,\mathbf{p})d^3\mathbf{X}d^3\mathbf{p}$$

$$d^3\mathbf{x} = \text{const}\,,\quad d^3\mathbf{k} = \text{const}\,,\quad f(k) = \text{const}$$

$$f(k)d^3\mathbf{x}d^3\mathbf{k} = \text{const}$$

comoving volume equals physical volume

$$d^3\mathbf{x}d^3\mathbf{k} = d^3(a\mathbf{x})d^3\left(\frac{\mathbf{k}}{a}\right) = d^3\mathbf{X}d^3\mathbf{p}$$

$$f(\mathbf{p},t) = f(\mathbf{k}) = f[a(t)\cdot\mathbf{p}]\,.$$

$$f_i(\mathbf{p}) \;\longrightarrow\; f(\mathbf{p},t) = f_i\left(\frac{a(t)}{a_i}\mathbf{p}\right)$$

Massless bosons (photons) fermions

$$f_i(\mathbf{p}) = f_{Pl}\left(\frac{|\mathbf{p}|}{T_i}\right) = \frac{1}{(2\pi)^3}\frac{1}{e^{|\mathbf{p}|/T_i}-1}$$

$$f(\mathbf{p},t) = f\left(\frac{a(t)|\mathbf{p}|}{a_i T_i}\right) = f\left(\frac{|\mathbf{p}|}{T_{eff}(t)}\right)$$

$$T_{eff}(t) = \frac{a_i}{a(t)}T_i$$

decoupling at $T \gg m$: neutrinos, hot dark matter

decoupling at $T \ll m$: $f(\mathbf{p}) = \frac{1}{(2\pi)^3}\exp\left(-\frac{m-\mu_i}{T_i}\right)\exp\left(-\frac{a^2(t)\mathbf{p}^2}{2ma_i^2 T_i}\right)$

$$f(\mathbf{p},t) = \frac{1}{2\pi^3}\exp\left(-\frac{m-\mu_{eff}}{T_{eff}}\right)\exp\left(-\frac{\mathbf{p}^2}{2mT_{eff}}\right)$$

$$T_{eff}(t) = \left(\frac{a_i}{a(t)}\right)^2 T_i\,, \qquad \frac{m-\mu_{eff}(t)}{T_{eff}} = \frac{m-\mu_i}{T_i}$$

Einstein equations

$$ds^2 = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j \ ,$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \ .$$

$$R_{00} = -3\frac{\ddot{a}}{a} \ , \quad R_{0i} = 0 \ , \quad R_{ij} = (\ddot{a}a + 2\dot{a}^2 + 2\varkappa)\gamma_{ij} \ ,$$
$$R = g^{\mu\nu}R_{\mu\nu} = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\varkappa}{a^2}\right)$$

$T_{\mu\nu}$: macroscopic description
$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - g_{\mu\nu}p$ liquid with $\rho(t)$ and $p(t)$

in the comoving frame $u^0 = 1$, $\mathbf{u} = 0$

$$T_{\mu\nu} = diag(\rho, \ a^2(t)\gamma_{ij}p)$$

Friedman equation $\quad (00): \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{\varkappa}{a^2}$

$$\nabla_\mu T^{\mu 0} = 0 \quad \longrightarrow \quad \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

the equation of state

$$p = p(\rho)$$

many-component liquid, in case of termal equilibrium

other equations

$$-3d(\ln a) = \frac{d\rho}{p+\rho} = d(\ln s)$$

entropy is conserved in a comoving frame

$$sa^3 = \text{const}$$

Examples of cosmological solutions

$$\varkappa = 0 \qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho$$

dust: $p = 0$ singular at $t = t_s$

$$\rho = \frac{\text{const}}{a^3}\,, \quad a(t) = const \cdot (t - t_s)^{2/3}\,, \quad \rho(t) = \frac{\text{const}}{(t - t_s)^2}$$

$$t_s = 0\,, \quad H(t) = \frac{\dot{a}}{a}(t) = \frac{2}{3t}\,, \quad \rho = \frac{3}{8\pi G} H^2 = \frac{1}{6\pi G}\frac{1}{t^2}$$

the Universe is too young

$$t_0 = \frac{2}{3H_0} = 0.93 \cdot 10^{10}\ \text{yr} \quad (h = 0.7)$$

Cosmological (particle) horizon $l_H(t)$

distnace covered by photons emitted at $t = 0$

the size of causally-connected region — the size of the visible part of the Universe

in conformal coordinates: $ds^2 = 0 \longrightarrow |d\mathbf{x}| = d\eta$
coordinate size of the horizon equals $\eta(t)$

$$l_H(t) = a(t)\eta(t) = a(t)\int_0^t \frac{dt'}{a(t')}$$

dust

$$l_H(t) = 3t = \frac{2}{H(t)}\,, \quad l_{H,0} = 2.7 \cdot 10^{28}\text{cm} \quad (h = 0.7)$$

Examples of cosmological solutions

radiation: $p = \frac{1}{3}\rho$ singular at $t = t_s$

$$\rho = \frac{\text{const}}{a^4}\,, \quad a(t) = const \cdot (t - t_s)^{1/2}\,, \quad \rho(t) = \frac{\text{const}}{(t - t_s)^2}$$

$$t_s = 0\,, \quad H(t) = \frac{\dot{a}}{a}(t) = \frac{1}{2t}\,, \quad \rho = \frac{3}{8\pi G}H^2 = \frac{3}{32\pi G}\frac{1}{t^2}$$

$$l_H(t) = a(t)\int_0^t \frac{dt'}{a(t')} = 2t = \frac{1}{H(t)}\,.$$

In case of termal equilibrium

$$\rho_b = \frac{\pi^2}{30}g_b T^4\,, \quad \rho_f = \frac{7}{8}\frac{\pi^2}{30}g_f T^4$$

$$\rho = \frac{\pi^2}{30}g_* T^4\,, \quad g_* = \sum_b g_b + \frac{7}{8}\sum_f g_f$$

$$\rho = \frac{\pi^2}{30} g_* T^4 \,, \qquad g_* = \sum_b g_b + \frac{7}{8} \sum_f g_f$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{\varkappa}{a^2} \quad \longrightarrow \quad H = \frac{T^2}{M_{Pl}^*}$$

$$M_{Pl}^* = \sqrt{\frac{90}{8\pi^3 g_*}} M_{Pl} = \frac{1}{1.66\sqrt{g_*}} M_{Pl} \,.$$

$g_* = g_*(T)$

$$T(t) \approx \frac{\text{const}}{a(t)}$$

Examples of cosmological solutions

vacuum: $T_{\mu\nu} = \rho_{vac}\eta_{\mu\nu}$ $\quad p = -\rho$

$$S_G = -\frac{1}{16\pi G}\int R\sqrt{-g}d^4x\ , \quad S_\Lambda = -\Lambda\int\sqrt{-g}d^4x\ .$$

$$a = \text{const}\cdot \mathrm{e}^{H_{\mathrm{dS}}t}\ , \quad H_{\mathrm{dS}} = \sqrt{\frac{8\pi}{3}G\rho_{vac}}$$

de Sitter space: space-time of constant curvature

$$ds^2 = dt^2 - \mathrm{e}^{2H_{dS}t}d\mathbf{x}^2$$

$\ddot{a} > 0$, no initial singularity

$ds^2 = dt^2 - e^{2H_{dS}t} d\mathbf{x}^2$

no cosmological horizon: $l_{\rm H}(t) = e^{H_{dS}t}\int_{-\infty}^{t} dt' e^{-H_{dS}t'} = \infty$

de Sitter (events) horizon ($\mathbf{x} = 0$, t):

from which distnace $l(t)$ one can detect light emitted at t?

in conformal coordinates: $ds^2 = 0 \longrightarrow |d\mathbf{x}| = d\eta$
coordinate size: $\eta(t \to \infty) - \eta(t) = \int_t^\infty \frac{dt'}{a(t')}$
physical size: $l_{dS} = a(t)\int_t^\infty \frac{dt'}{a(t')} = \frac{1}{H_{dS}}$

observer will never be informed what happens at distances larger $l_{dS} = H_{dS}^{-1}$

marginal matter: **gas of stright strings**

$p = w\rho$, $w = -\frac{1}{3}$, $p = -\frac{1}{3}\rho$