AFFINE LIE ALGEBRAS ; KAC-MOODY ALGEBRA

Preamble:

The object of our study is a class of infinite dimensional Lie algebras called : Lie algebras with generalized Cartan matrix
or Kac. Moody algebras,
or Contragredient algebras

Let us consider a Lie algebra $g'(A)$ associated to the generalized Cartan matrix A, i.e. :

$A = (a_{ij})_{i,j=1..n}$ is a $n \times n$ matrix where the a_{ij} are (real) integers such that:

$a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$

and $a_{ij} = 0 \Rightarrow a_{ji} = 0$

The Kac. Moody algebra $g'(A)$ associated to A is a complex Lie algebra with $3n$ generators : e_i, f_i, h_i $(i=1,..n)$ satisfying the relations (Serre_ Chevalley basis):

$$[h_i, h_j] = 0 \qquad [e_i, f_i] = h_i \qquad [e_i, f_j] = 0 \quad i \neq j$$

$$[h_i, e_j] = a_{ji} e_j \qquad [h_i, f_j] = - a_{ji} f_j$$

$$(ad\, e_i)^{1-a_{ji}} e_j = 0 \qquad (ad\, f_i)^{1-a_{ji}} f_j = 0. \quad \text{for } i \neq j$$

Actually, the class of so-called Kac. Moody algebras decompose into 3 sub-classes.

One supposes A undecomposable, i.e. $\nexists$ partition of the set

$\{1, \dots, n\}$ into 2 non-empty parts such that $a_{ij} = 0$ for any $i \in 1^{st}$ subspace and $j \in 2^{nd}$ subspace (then one would get the direct sum of 2 or more K-M. algebras).

Then there are 3 (exclusive) possibilities:

a) $\exists$ a vector Θ made with integers > 0 such that the components of the vector $A\Theta$ are all > 0.

In this case, the principal minors of A are positive and $g'(A)$ is finite dimensional.

b.) $\exists$ a vector δ made with positive integers s.t: $A\delta = 0$

In this case, all the principal minors of A are non-negative and $\det A = 0$.

Then $g'(A)$ is infinite dimensional, but of polynomial growth.

Such Lie algebras are called affine Lie algebras.

c.) $\exists$ a vector α made with positive integers s.that all the components of $A\alpha$ are < 0.

In this case, $g'(A)$ is of exponential growth.

The main result of the Killing-Cartan theory can be formulated as follows: "A finite dimensional, complex, simple Lie algebra is isomorphic to an algebra of sub-class a:)".

Actually, most of the classical concepts of the Killing-Cartan-Weyl theory can be generalized to the entire class of Kac Moody algebras, i.e. Cartan subalgebra, root system, Weyl group, etc.

STRUCTURE of AFFINE LIE ALGEBRAS

Generators: Let g be complex, simple, finite dimensional Lie algebra, $\mathbb{C}(t, t^{-1})$ the set of Laurent polynomials in the t-variable. Let us consider the loop algebra $\tilde{g}$

$$\tilde{g} = g \otimes \mathbb{C}(t, t^{-1})$$

with $t = e^{i\alpha}$ α real, thus inducing an application

$$\gamma : t \longrightarrow \gamma(t) \in G \quad G \text{ group with } g \text{ as Lie algebra}$$

$$t \in S^1 = \text{circle of radius} = 1.$$

The group operation is defined on G in an obvious way:

$$\gamma_1, \gamma_2 : \quad S^1 \to G \qquad \text{then: } \gamma_1 . \gamma_2 (t) = \gamma_1(t) . \gamma_2(t)$$

providing the loop group of G: $\tilde{\tilde{g}}$.

The generators of $\tilde{g}$ can be taken as: $J^a \otimes t^n$

with $\{J^a\}_{a=1..\dim g}$ a basis of g s.t.: $[J^a, J^b] = i f^{ab}_c J^c$

then the algebra law naturally extends to $\tilde{g}$:

$$\left[J^a \otimes t^n, J^b \otimes t^m\right] = i f^{ab}_c J^c \otimes t^{m+n}$$

that is also, with: $J^a \otimes t^n = J^a_n$:

$$\left[J^a_n, J^b_m\right] = i f^{ab}_c J^c_{m+n}$$

We remark that the definition of the loop algebra $\tilde{g}$ and the one of the loop group ${}^{\text{eg}}\mathcal{G}$ are coherent.

An element of the loop group, which is supposed connected, is of the form:

$$\gamma = \exp\left(-i\, J^a \cdot \theta_a\right) \qquad a = 1, \dots \dim g$$

A typical element of ${}^{\text{eg}}\mathcal{G}$ – or rather the connected component which contains the identity, consisting of applications:

$$\gamma : S^1 \to G$$

topologically trivial, that is which can be continuously deformed toward the constant application: $\gamma(t) = 1$ – can then be described by dim-g functions $\theta_a(t)$ defined on the unit circle:

$$\gamma(t) = \exp\left[-i\, J^a \cdot \theta_a(t)\right]$$

For the elements close to identity: $\gamma \simeq 1 - i\, J^a \cdot \theta_a$

and: $\gamma(t) \simeq 1 - i\, J^a \cdot \theta_a(t)$

And, performing a Laurent development of $\theta_a(t)$:

$$\theta_a(t) = \sum_{n=-\infty}^{+\infty} \theta_a^{-n}\, t^n$$

one realizes, introducing the generators:

$$J^a_n = J^a \cdot t^n$$

that the θ_a^n, $1 \leq a \leq \dim g$, $n \in \mathbb{Z}$, provide an infinite set of parameters for ${}^{\text{eg}}\mathcal{G}$ with:

$$\gamma(t) \simeq 1 - i \sum_{n,a} J^a_{-n} \cdot \theta_a^n$$

Note that the generators $\{J^a_0\}$ generate a $\tilde{g}$ subalgebra isomorphic to g, corresponding to the ${}^{\text{eg}}\mathcal{G}$ subgroup constituted by the constant applications: $S^1 \to G$, obviously isomorphic to G.

Central Extension: A central extension is obtained by adding to $\tilde{g}$ a central element (i.e. an element commuting with all the elements of the algebra) $\hat{k}$ such that:

$$\left[J^a_n , J^b_m \right] = i\, f^{ab}_c\, J^c_{n+m} + \hat{k}\, n\, \delta_{n+m,0} \cdot \delta_{ab}$$

$$\left[J^a_n , \hat{k} \right] = 0$$

In fact, this c. extens. is unique. Let us pick it.

We consider, more generally:

$\xi,\eta \in \tilde{g}$: $[\xi,\eta] = \zeta + \omega(\xi,\eta)\, e$ $\qquad [e,\zeta] = 0 \quad \forall \zeta \in \tilde{g}$

ω is a bilinear form: $\tilde{g} \times \tilde{g} \to \mathbb{R}$

satisfying:

antisymmetry: $\omega(\xi,\eta) = -\,\omega(\eta,\xi)$

Jacobi condition: $\omega([\xi,\eta],\rho) + \omega([\eta,\rho],\xi) + \omega([\rho,\xi],\eta) = 0$

Let us remark that ω is invariant by conjugation of constant loops, i.e. $\forall g_0 \in G$ finite:

$$[g_0 \xi g_0^{-1}, g_0 \eta g_0^{-1}] = g_0 [\xi,\eta] g_0^{-1} = g_0 \zeta g_0^{-1} + \omega(\xi,\eta)\, e$$

but: $[g_0 \xi g_0^{-1}, g_0 \eta g_0^{-1}] = X + \omega(g_0 \xi g_0^{-1}, g_0 \eta g_0^{-1})\, e$

$X \in \tilde{g}$ "without" the c. extens.

Therefore: $X = g_0 \zeta g_0^{-1}$

and : $\omega(g_0 \xi g_0^{-1}, g_0 \eta g_0^{-1}) = \omega(\xi,\eta) \quad \forall \xi,\eta \in \tilde{g}$

Now, since $\xi \in \tilde{\mathcal{G}}$ can be decomposed in a Fourier series: $\sum_k \xi_k t^k$, $\xi_k \in \mathcal{G}$, let us consider:

$$\omega\left(\xi_p t^p, \eta_q t^q\right) = \omega_{p,q}\left(\xi_p, \eta_q\right) \qquad \xi, \eta \in \mathcal{G}_{\mathbb{C}}$$

$$\omega_{p,q} : \mathcal{G}_{\mathbb{C}} \times \mathcal{G}_{\mathbb{C}} \to \mathbb{C}$$

and $\omega_{p,q}$ G-invariant (see above ← and beside).

But, property " Any bilinear invariant form on $\mathcal{G}$ is symmetric, and proportional to the Killing form of $\mathcal{G}$"

It follows: • $\omega_{p,q} = -\omega_{q,p}$

since:

$$\omega\left(\xi t^p, \eta t^q\right) = \omega_{p,q}\left(\xi, \eta\right)$$

antisym: $$= -\omega\left(\eta t^q, \xi t^p\right) = -\omega_{q,p}\left(\eta, \xi\right) \underset{\text{sym.}}{=} -\omega_{q,p}(\xi, t) \blacksquare$$

The Jacobi identity reads: • $\omega_{p+q,r} + \omega_{q+r,p} + \omega_{r+p,q} = 0$

$\forall p,q,r \in \mathbb{Z}$

One obtains,

- with : $q=r=0 \longrightarrow \boxed{\omega_{p,0} = 0}$

- with : $r = -p-q \longrightarrow \omega_{p+q,-p-q} = -\omega_{-p,p} - \omega_{-q,q}$

$$= \omega_{p,-p} + \omega_{q,-q}$$

it follows:

$$\boxed{\omega_{p,-p} = p.\omega_{1,-1}}$$

- with : $r = n-p-q$: $\longrightarrow$ $\omega_{n-p-q,\, p+q} = \omega_{n-p,p} + \omega_{n-q,q}$

it follows: $\underline{\underline{\omega_{n-k,k} = k\, \omega_{n-1,1}}}$

From this last relation, we get: $\underset{\overset{\parallel}{0}}{\omega_{n-n,n}} = n\, \omega_{n-1,1}$

that is : $\omega_{n-1,1} = 0$

and then: $\boxed{\omega_{n-k,k} = 0 \ \forall n \neq 0}$

and $\omega_{p,q} = 0$ as soon as: $p+q \neq 0$.

Coming back to :

$$\omega\left(\sum_p \xi_p . t^p, \sum_q \eta_q . t^q\right) = \sum_p p\, \omega_{1,-1}\left(\xi_p, \eta_{-p}\right)$$

$t = e^{i\alpha}$ $\Rightarrow$ $$= \frac{1}{2\pi}\int_0^{2\pi} \omega_{1,-1}\left(\xi(\alpha), \eta'(\alpha)\right) d\alpha$$

and, since it exists only one bilinear form which is invariant, symmetric for g- simple, and this form is proportional to the Killing form :

$$\boxed{\omega\left(\xi, \eta\right) = \frac{1}{2\pi}\int_0^{2\pi} \left\langle \xi(\alpha), \frac{\partial}{\partial\alpha}\eta(\alpha)\right\rangle_{Kill} d\alpha}$$

We understand now the expression of ω given two pages before, with $\underline{\underline{Kill\left(J^a, J^b\right) = \delta^{ab}}}$ (supposing the generators orthonormed w.r. to Killing form).

Note: that the properties of antisym. and Jacobi are "encoded" in the derivative (appearing in the above formula)-

More on Central Extensions:

– generalisation to the torus $S^1 \times S^1$, or more generally to M, compact space, to $S^1 \times$ Grassmann alg. and more generally to $M \times$ Grassm.

see L. Frappat, et al Nucl. Phys. B334, 250 (1990)
E. Ragoucy & P.S. (a review) Int Journ. Mod. Phys. A7 2883 (1992).

Affine Cartan-Weyl basis:

Let us consider the Killing form on $\mathcal{g}$: the non-zero relations are:

$$K(H^i, H^j) = \delta^{ij}$$

$$K(E^\alpha, E^{-\alpha}) = \frac{2}{|\alpha|^2} \qquad (\alpha \text{ root}).$$

In the C-W. basis, the commutation relations read:

$$[H^i_n, H^j_m] = n\,\hat{k}\,\delta^{ij}\,\delta_{n+m,0}$$

$$[H^i_n, E^\alpha_m] = \alpha^i E^\alpha_{n+m}$$

$$[E^\alpha_n, E^\beta_m] = \frac{2}{\alpha^2}\left(\alpha . H_{n+m} + \hat{k}\,n.\,\delta_{n+m,0}\right) \text{ if } \alpha = -\beta$$

$$= N_{\alpha\beta}\, E^{\alpha+\beta}_{n+m} \qquad \text{if } (\alpha+\beta) \text{ racine}$$

$$= 0 \qquad \text{in other cases.}$$

Rk: With $n=0$, one recovers the case for finite $\mathcal{g}$.

Now, in the adjoint representation, let us look at the action of the Abelian subalgebra $\{H^1_0, .. H^r_0, \hat{k}\}$ on E^α_n

$$\mathrm{ad}(H_0^i)(E_n^\alpha) = [H_0^i, E_n^\alpha] = \alpha_i E_n^\alpha$$
$$\mathrm{ad}(\hat{k})(E_n^\alpha) = [\hat{k}, E_n^\alpha] = 0$$

The vector $(\alpha^1, \dots \alpha^r, 0)$ made with the eigenvalues is independent of n, and therefore infinitely degenerate, being the same for all E_m^α's whatever $m \in \mathbb{Z}$.

So the eigenvalues are not completely specified by the eigenvectors: let us introduce a <u>grading operator</u>:

$$\boxed{L_0 = -t \frac{d}{dt}}$$

acting as follows: $\mathrm{ad}(L_0)(J^a \otimes t^n) = [L_0, J_n^a] = -n J_n^a$

The "Cartan subalgebra" will then be: $\{H_0^1, \dots H_0^r, \hat{k}, L_0\}$

the E_n^α playing the role of ladder operators,

and, we define:

<u>the affine algebra</u>: $\boxed{\hat{g} = \tilde{g} \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}L_0}$

<u>A particular case</u>:

the Heisenberg algebra: $[a_n, a_m] = n\,\delta_{n+m,0}$

affine extension of $U(1)$ generated by a_0.

Note that the eigenvalue of $\hat{k}$, on the r.h.s. of the commutation relation, is 1, but can be changed by dilatation of the generators $\Rightarrow$ therefore, for $\hat{U}(1)$, the $\hat{k}$ eigenvalue has no meaning (which will not be the case for $\hat{g}$ with g simple).

Killing form:

One wishes to generalize the $\mathfrak{g}$- Killing form :

$$K(X,Y) \sim \mathrm{Tr}\,(ad(X)\;ad(Y))$$

One knows that K is $\mathfrak{g}$- invariant :

$$(*) \qquad K([Z,X],Y) + K(X,[Z,Y]) = 0 \qquad \forall\, X,Y,Z \in \mathfrak{g}$$

$\leftarrow$ (see proof aside)

and one wants the extension of K to $\hat{\mathfrak{g}}$ to be $\hat{\mathfrak{g}}$ invariant

- Let $X,Y \in \{J^a_n\}$ and $Z = L_0$: $(*) \Rightarrow (n+m)\, K(J^a_n, J^b_m) = 0$

that is : $K(J^a_n, J^b_m) = 0$ if $n+m \neq 0$.

When $n+m=0$, then the t-powers disappear, and we are back to the $\mathfrak{g}$- Killing form :

it follows: $\boxed{K(J^a_n, J^b_m) = \delta^{ab}\cdot \delta_{n+m,0}}$.

- Let $X,Z \in \{J^a_n\}$ and $Y = \hat{k}$

$$(*) \Rightarrow K([J^a_n, J^b_m], \hat{k}) + \cancel{K(J^a_n, 0)} = 0.$$

$$= f^{ab}_c\, K(J^c_{n+m}, \hat{k}) + n\,\delta^{ab}\delta_{n+m,0}\, K(\hat{k},\hat{k}) = 0$$

Considering separately $a=b$, $n=-m$, and $a \neq b$ $\Rightarrow$ $\boxed{K(J^a_n, \hat{k}) = 0}$ $\boxed{K(\hat{k},\hat{k}) = 0}$

- Let $X,Z \in \{J^a_n\}$ and $Y = L_0$:

$$(*) \Rightarrow K([J^a_n, J^b_m], L_0) + K(J^a_n, [J^b_m, L_0]) = 0$$

That is: $f_c^{ab} K(J^c_{n+m}, L_0) + \delta^{ab}\delta_{n+m}\, n K(\hat{k}, L_0) + K(J^a_n, m J^b_m)_{=0}$

ie: $f_c^{ab} K(J^c_{n+m}, L_0) + \delta^{ab}\delta_{n+m}\, n K(\hat{k}, L_0) + m\delta^{ab}\delta_{n+m,0} = 0$

if $a \neq b$: $\boxed{K(J^c_{n+m}, L_0) = 0.}$

and then, $n = -m$ $\Rightarrow$ $K(\hat{k}, L_0) = -1$ or $\boxed{K(\hat{k}, -L_0) = 1}$

The only non specified norm is $K(L_0, L_0)$: by convention, we will take:

$$\boxed{K(L_0, L_0) = 0.}$$

– let us remind that the Killing form in g establishes an isomorphism between the Cartan subalgebra $\mathcal{H}$ and its dual $\mathcal{H}^*$, via:

$\gamma \in \mathcal{H}^* \longrightarrow \exists\, H^\gamma$ s.t.: $\gamma := K(\cdot, H^\gamma)$

that is: $\gamma(H^i) = K(H^i, H^\gamma)$ $\forall H^i \in \mathcal{H}$

(in particular $\alpha \in \mathcal{H}^*$ $\quad H^\alpha = \alpha . H = \sum_i \alpha^i H^i$

$$\alpha(H^j) = K(H^j, \alpha . H) = \alpha^j \underbrace{K(H^i, H^j)}_{\text{chosen as } \delta^{ij}}$$

in accordance with: $[H^j, E^\alpha] = \alpha(H^j)\, E^\alpha = \alpha^j . E^\alpha$)

Such an isomorphism allows to define a (positive definite) scalar product on the dual space:

$$\boxed{(\beta, \gamma) = K(H^\beta, H^\gamma)}$$

and since roots are elements of $\mathcal{H}^*$, this defines a scalar product in the root space –

We will generalize this scalar product to $\hat{g}$.

Weights :

Consider the components of a vector $\hat{\lambda}$ constructed with the eigenvalues relative to a state, in a $\hat{g}$ representation space, which is eigenvector of all the Cartan subalgebra generators :

$$\hat{\lambda} = \left(\hat{\lambda}(H_0^1), \dots \hat{\lambda}(H_0^r) ; \hat{\lambda}(\hat{k}) ; \hat{\lambda}(-L_0) \right)$$

↳ beware the sign!

We note it as : $\hat{\lambda} = (\lambda ; k_\lambda ; n_\lambda)$ and $\hat{\lambda}$ is called an affine weight.

↳ finite part of the algebra.

The scalar product induced by the extended Killing form is:

$$\boxed{\left(\hat{\lambda}, \hat{\mu}\right) = (\lambda, \mu) + k_\lambda n_\mu + k_\mu . n_\lambda .}$$

_ But, let us refresh our mind with elementary notions about weights in the case of finite simple g :

<u>A reminder:</u>

In any repres. of g, one can always find a basis $\{ |\lambda\rangle \}$ such that:

$$H^i \; |\lambda\rangle = \lambda^i |\lambda\rangle = \lambda(H^i) \; |\lambda\rangle$$

$\lambda = (\lambda^1, \dots \lambda^r)$ is a weight

$\lambda \in \mathcal{H}^*$

the c. relations are:
$$[H^i, H^j] = 0 \qquad i,j = 1, \dots r = \text{rank}(g)$$
$$[H^i, E^\alpha] = \alpha^i E^\alpha$$
$$[E^\alpha, E^\beta] = N_{\alpha\beta} \, E^{\alpha+\beta} \qquad \text{if } (\alpha+\beta) \text{ root}$$
$$= \frac{2}{|\alpha|^2} \, \alpha . H \qquad \text{if } \alpha = -\beta$$
$$= 0 \qquad \text{otherwise.}$$

One remarks:

$$H^i (E^\alpha |\lambda\rangle) = [H^i, E^\alpha] |\lambda\rangle + E^\alpha H^i |\lambda\rangle = (\lambda^i + \alpha^i)(E^\alpha |\lambda\rangle)$$

that is $E^\alpha |\lambda\rangle$ ∝ to $|\lambda + \alpha\rangle$.

In the case of a finite dim. rep. $\Rightarrow \exists$ p and $q \in \mathbb{Z}_+$ such that:

$$(E^{\alpha})^{p+1}|\lambda\rangle \sim E^{\alpha}|\lambda+p\alpha\rangle = 0$$

$$(E^{-\alpha})^{q+1}|\lambda\rangle \sim E^{-\alpha}|\lambda-q\alpha\rangle = 0$$

(of course, we consider the smallest values of p & q).

<u>Example</u>: $SU(2)$ & rep. of dim $(2j+1)$

$$[J^3, J^{\pm}] = \pm J^{\pm}$$

$$[J^+, J^-] = 2J^3 \longrightarrow J^3 = \frac{\alpha.H}{|\alpha|^2}$$

From $|\lambda\rangle$, the state $|m\rangle = |j\rangle$ can be obtained by (repeated) applications of $J^+ \equiv E^{\alpha}$, and $|m\rangle = |-j\rangle$ by applic. of $J^- = E^{-\alpha}$

since: $J^3 = \frac{\alpha.H}{|\alpha|^2} \Longrightarrow$

$$j = \frac{(\alpha,\lambda)}{|\alpha|^2} + p$$

$$-j = \frac{(\alpha,\lambda)}{|\alpha|^2} - q \quad \Rightarrow -(p-q) = \frac{2(\alpha,\lambda)}{|\alpha|^2}$$

We also remind the notion of <u>fundamental weight</u>: ω:

defined as: $\underline{\underline{(\omega_i, \alpha_j^\vee) = \delta_{ij}}}$ with: $\underline{\underline{\alpha_j^\vee = \frac{2\alpha_j}{|\alpha_j|^2}}}$

$i = 1,..r.$

and the <u>weight</u> $\lambda = \sum_{i=1}^{r} \lambda_i \omega_i$ with $\lambda_i = (\lambda, \alpha_i^\vee)$

Dynkin labels
always ≥ 0 integers in an irred. repres.

In the Chevalley basis: $h_i|\lambda\rangle = \lambda(h^i)|\lambda\rangle = (\lambda, \alpha_i^\vee)|\lambda\rangle$
$= \lambda_i|\lambda\rangle$

In particular, in the adj^t represent., the weights are the roots, and the elements of the Cartan matrix are the Dynkin labels of the simple roots: $\alpha_i = \sum_{j=1}^{r} a_{ij}\,\omega_j$.

Still in the adjoint repres, but for $\hat{g}$, we will have <u>affine roots</u>:

$$\boxed{\hat{\beta} = (\beta; 0; n)}$$

and their scalar product will reduce to the one defined for g:

$$\underline{\underline{(\hat{\beta}, \hat{\alpha}) = (\beta, \alpha)}}$$

Note that the root relative to E^{α}_{n} reads:

$$\hat{\alpha} = (\alpha, 0, n) \qquad n \in \mathbb{Z} \quad , \alpha \in \Delta$$

The root $\delta = (0; 0; 1)$ is associated to H^{i}_{1} and $n\delta$ to H^{i}_{n}. Denoting: $\alpha = (\alpha; 0, 0)$

then: $\underline{\hat{\alpha} = \alpha + n\delta}$

and the <u>total set of roots</u> is:

$$\boxed{\hat{\Delta} = \{ \alpha + n\delta \mid n \in \mathbb{Z}, \underset{\substack{\downarrow \\ \text{root system of } g}}{\alpha \in \Delta} \} \cup \{ n\delta \mid n \in \mathbb{Z}, n \neq 0 \}}$$

Remark that $(\delta, \delta) = 0$ δ is often called "imaginary root".

in the same way $(n\delta, m\delta) = 0$ & <u>$n\delta$ is imaginary</u> and of <u>multiplicity r</u>

while the other roots are <u>real, of multiplicity one</u>.

Simple roots, Cartan Matrix & Dynkin Diagram:

Simple Roots: The next natural step is the identification of a basis of simple roots for $\hat{g}$. Such a basis must contain $(r+1)$ elements, r being the finite g - roots α_i.

Let us define:

$$\boxed{\alpha_0 = (-\theta\,;0\,;1) = -\theta + \delta}$$

with θ highest root of g.

$\Rightarrow \quad \{\alpha_0, \alpha_1 \dots, \alpha_r\}$ defines a basis of simple roots.

Remark that the set of positive roots is:

$$\boxed{\hat{\Delta}_+ = \{\alpha + n\delta \mid n>0\,,\ \alpha\in\Delta\} \cup \{\alpha \mid \alpha \in \Delta_+\}}$$

indeed, for $n>0$ & $\alpha \in \Delta$

$$\alpha + n\delta = \alpha + n\alpha_0 + n\theta$$
$$= n\alpha_0 + (n-1)\theta + (\theta + \alpha)$$

written as a sum with ≥ 0 integers on $(\alpha_1 \dots \alpha_r)$

even if α negative root of Δ $\theta + \alpha$ is a sum with ≥ 0 integers on $(\alpha_1 \dots \alpha_r)$, since θ highest weight –

– We note that there is **NO** highest root in $\hat{g}$, i.e. the adjoint repres. of $\hat{g}$ is not an highest weight represent.

Cartan Matrix: In the same way, we have defined co-root in g with: $\alpha_i^\vee = 2\frac{\alpha_i}{|\alpha_i|^2}$, we can define

affine co-roots:

$$\hat{\alpha}^\vee = \frac{2}{|\hat{\alpha}|^2}(\alpha; 0; n) = \frac{2}{|\alpha|^2}(\alpha; 0; n)$$

$$= \left(\alpha^\vee; 0; \frac{2}{|\alpha|^2} n\right).$$

As for simple roots, the hat will be omitted over the simple coroots:

$$\alpha_0^\vee = \alpha_0 \qquad \alpha_i^\vee \equiv (\alpha_i^\vee; 0; 0) \qquad i \neq 0.$$

And the extended Cartan Matrix reads:

$$\boxed{\hat{A}_{ij} = (\alpha_i, \alpha_j^\vee)} \qquad 0 \leq i,j \leq r$$

(with $\theta = \sum_{i=1}^{r} a_i \alpha_i$) with: $(\alpha_0, \alpha_j^\vee) = -(\theta, \alpha_j^\vee) = -\sum_{i=1}^{r} a_i(\alpha_i, \alpha_j^\vee).$

The Dynkin Diagram associated to a finite simple g is defined by associating to each simple root α_i a node ○ or ● ← or (following the relative length of the root: ○ = long; ● = short root) and by joining the nodes relative to α_i and α_j ~~with~~ with $A_{ij} A_{ji}$ lines.

We will do the same for $\hat{g}$, linking α_0 and α_i with $\hat{A}_{0i} \hat{A}_{i0}$ lines.

Note, in the finite g case: $\theta = \sum_{i=1}^{r} a_i \alpha_i = \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee$

with $a_i = a_i^\vee \frac{2}{|\alpha_i|^2}$

We define: $a_0 = 1$

Since θ is highest root in g; $|\alpha_0|^2 = 2$ (long root).

which implies: $a_0^\vee = a_0 \frac{|\alpha_0|^2}{2} = 1$

Then, by construction of $\hat{A}_{ij}$:

$$\sum_{i=0}^{2} a_i \hat{A}_{ij} = \Big(\underbrace{a_0\alpha_0}_{\sim -\theta} + \underbrace{\sum_{i=1}^{2} a_i\alpha_i}_{+\theta}, \alpha_j^\vee\Big) = 0$$

$$= \sum_{i=0}^{2} \hat{A}_{ij}\, a_j^\vee = 0$$

and this linear dependence between the rows of the Cartan matrix reveals that $\exists$ one zero eigenvalue: see def. of affine Lie algebras p.1 & 2!

Note finally that the imaginary root reads:

$$\delta = \sum_{i=0}^{2} a_i \alpha_i = \sum_{i=0}^{2} a_i^\vee \alpha_i^\vee .$$

It follows the table of D.D. for (non-twisted) simples affine Lie algebras.

Serre-Chevalley basis for $\hat{g}$:

In the case of finite g, the Serre-Chevalley basis can be seen on page 1. Note:

$$e^i = E_0^{\alpha_i} \qquad f^i = E_0^{-\alpha_i} \qquad h^i = \frac{2\alpha_i . H}{|\alpha_i|^2}$$

(Rk also: $K(h^i, h^j) = (\alpha_i^\vee, \alpha_j^\vee)$)

In the case of $\hat{g}$:

$$e^0 = E_1^{-\theta} \qquad f^0 = E_{-1}^{\theta} \qquad \underline{h^0 = \hat{k} - \vec{\theta}.\vec{H}_0}$$

from $\alpha_0 = (-\theta; 0; 1)$

and

$$e^i = E_0^{\alpha_i} \qquad f^i = E_0^{-\alpha_i}$$

($\vec{H}_0 \equiv$ finite Cartan part)

the C.R. read:

$$[h^i, h^j] = 0$$
$$[h^i, e^j] = \hat{A}_{ji}\, e^j$$
$$[h^i, f^j] = -\hat{A}_{ji}\, f^j$$
$$[e^i, f^j] = \delta_{ij}\, h^i$$

and

$$(\mathrm{ad}\, e^i)^{1-\hat{A}_{ji}} e^j = 0 \qquad i \neq j$$
$$(\mathrm{ad}\, f^i)^{1-\hat{A}_{ji}} f^j = 0$$

Note: for $\widehat{SU}(2)$

$$h^0 = \hat{k} - \vec{\alpha}.\vec{H} \qquad \text{and } \exists \text{ only one } H.$$
$$h^1 = 2\frac{\alpha . H}{|\alpha|^2} = \alpha . H.$$

It follows: $\underline{\underline{h^0 + h^1 = k}}$

Fundamental Weights:

Recall that for finite g: $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ $(i=1..2)$

Now: $\theta = \sum_{i=1}^{2} a_i \alpha_i$

$a_0 = 1$ by convention and $a_0^\vee = a_0 \frac{|\alpha_0|^2}{2} = 1$

Since: $\alpha_0 = (-\theta; 0; 1) = \alpha_0^\vee$ $\Rightarrow$ $\delta = \sum_{i=0}^{2} a_i \alpha_i = \sum_{i=0}^{2} a_i^\vee \alpha_i^\vee$

$\delta = (0; 0; 1)$

So, let us introduce $\hat{\omega}_i$ $i = 0, 1, .. 2$

such that: $\boxed{(\hat{\omega}_i, \alpha_j^\vee) = \delta_{ij}}$

It follows that: $(\hat{\omega}_i, \alpha_0^\vee) = 0$ for $i = 1..2$

and one deduces $\Rightarrow$ $\boxed{\begin{array}{l} \hat{\omega}_i = (\omega_i; a_i^\vee; 0) \\ \hline \hat{\omega}_0 = (0; 1; 0) \end{array}}$ $i = 1..2$

Since $\theta = \sum_{i=1}^{2} a_i \alpha_i$ and using the scalar product as defined before

One can rewrite ω_i $(i = 0, .., 2)$ with the help of basic fund. weights

$\omega_i = (\omega_i; 0; 0)$

and $\hat{\omega}_0 = (0; 1; 0)$

that is: $\begin{cases} \hat{\omega}_i = a_i^\vee \hat{\omega}_0 + \omega_i & (i = 1, .. 2) \\ \hat{\omega}_0 \end{cases}$

The affine weights can be written in terms of fundam. affine weight and δ: $\boxed{\hat{\lambda} = \sum_{i=0}^{2} \lambda_i \hat{\omega}_i + l\delta \quad l \in \mathbb{R}}$

Since each fundam. weight contributes to the $\hat{k}$ eigenvalue by a factor $a_i^\vee$, we get:

$$k = \hat{k}\text{-eigenvalue} \equiv \hat{\lambda}(\hat{k}) = \sum_{i=0}^{r} a_i^\vee \lambda_i .$$

(remember: $\hat{k} = h_0 + h_1$ in $\widehat{su}(2)$, with $a_0 = a_1 = 1$..).

k is called the <u>level</u>.

See differently (but it is exactly the same!):

$$(\hat{\lambda}, \delta) = \sum_{i=0}^{r} a_i^\vee (\hat{\lambda}, \alpha_i^\vee) \qquad \text{since: } \delta = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee$$

but $(\hat{\lambda}, \delta) = \hat{\lambda}(\hat{k})$ since: $\begin{cases} \hat{\lambda} = (\hat{\lambda}(H_0^1), \dots \hat{\lambda}(H_0^r); \hat{\lambda}(\hat{k}); \hat{\lambda}(-L_0)) \\ \delta = (\vec{0}; 0, 1) \end{cases}$

We use the decomposition: $\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i + l\delta$

which provides: $(\hat{\lambda}, \alpha_i^\vee) = \lambda_i$

Therefore: $(\hat{\lambda}, \delta) = \sum_{i=0}^{r} a_i^\vee \lambda_i = \hat{\lambda}(\hat{k})$ •

That leads to: $\lambda_0 = \hat{\lambda}(\hat{k}) - \sum_{i=1}^{r} a_i^\vee . \lambda_i$ with $a_0^\vee = 1$

i.e: $\lambda_0 = k - (\lambda, \theta)$.

The affine weights are generally given in terms of Dynkin labels: $\hat{\lambda} = [\lambda_0, \lambda_1, \dots \lambda_r]$

For simple roots, Dynkin labels are given by the rows of the affine Cartan matrix: $\alpha_i = [\hat{A}_{i0}, \hat{A}_{i1}, \dots \hat{A}_{ir}]$

Affine Weyl group:

For finite g, the Weyl group is defined as follows:

$\alpha, \beta \in \Delta$: $\quad S_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha = \beta - 2\frac{\alpha.\beta}{|\alpha|^2}\alpha$

(or α root and λ weight)

In the same way, we can define the Weyl transform:

$$\boxed{s_{\hat\alpha}\hat\lambda = \hat\lambda - (\hat\lambda, \hat\alpha^\vee)\hat\alpha} = \hat\lambda - \frac{2}{|\alpha|^2}(\hat\lambda, \hat\alpha)\hat\alpha$$

affine root

$\hat\alpha = (\alpha; 0; m)$

It is of some interest to perform the computation:

$$\boxed{s_{\hat\alpha}\hat\lambda} = \left(\lambda - \left[(\lambda,\alpha) + km\right]\alpha^\vee \;;\; k \;;\; n - \left[(\lambda,\alpha) + km\right]\frac{2m}{|\alpha|^2}\right)$$

$(\lambda; k; n)$

since: $(\hat\lambda, \hat\alpha^\vee)\,\hat\alpha = \left[(\lambda,\alpha) + km\right]\hat\alpha^\vee$

$= \quad (\alpha^\vee; 0; \frac{2m}{|\alpha|^2})$

that is also:

$$\boxed{s_{\hat\alpha}\hat\lambda = \left(s_\alpha(\lambda + km\alpha^\vee)\right.} \;;\; k \;;\; \left. n - \left[(\lambda,\alpha) + km\right]\frac{2m}{|\alpha|^2}\right)$$

since on finite g: $\begin{cases} S_\alpha(\lambda) = \lambda - (\lambda,\alpha)\alpha^\vee \\ S_\alpha(\alpha^\vee) = -\alpha^\vee \end{cases}$

One notes also:

$$s_{\hat\alpha}\hat\alpha = \left(s_\alpha(\alpha) \;;\; 0 \;;\; m - (\alpha,\alpha^\vee)m\right) = (-\alpha; 0; -m)$$

That is: $\boxed{s_{\hat\alpha}\hat\alpha = -\hat\alpha}$

Moreover, since: $(\delta, \hat\alpha) = 0 \implies \boxed{s_{\hat\alpha}\delta = 0}$

So, the imaginary roots are not affected by affine Weyl transforms

Let us analyse the structure of the affine Weyl group $\hat{W}$ of $\hat{\mathcal{G}}$ from the above computation. Let us define:

$$\boxed{t_{\alpha^\vee} = s_{-\alpha+\delta} \cdot s_\alpha = s_\alpha \cdot s_{\alpha+\delta}} \qquad \alpha \in \Delta$$

that is:

$$t_{\alpha^\vee}(\lambda; k; n) \quad : \quad \hat{\lambda} = (\lambda; k; n) \xrightarrow{s_\alpha} (\lambda; k; n) - (\hat{\lambda} \cdot \alpha^\vee)\alpha$$

$$\Big\downarrow s_{-\alpha+\delta}$$

$$(\lambda - (\lambda.\alpha^\vee).\alpha; k; n) - \underbrace{\left[\left(\lambda - (\lambda.\alpha^\vee).\alpha; k; n\right)\left(-\alpha^\vee; 0; \frac{2}{|\alpha|^2}\right)\right]}(-\alpha; 0; 1)$$

$$- \left[-\lambda.\alpha^\vee + (\lambda.\alpha^\vee)(\alpha.\alpha^\vee) + \frac{k.2}{|\alpha|^2}\right](-\alpha; 0; 1)$$

that is finally: $t_{\alpha^\vee}(\hat{\lambda}) = \left(\lambda + k\alpha^\vee; k; -\left((\lambda.\alpha^\vee) + \frac{2}{|\alpha|^2}k\right) + n\right)$

one can checks that $t_{\alpha^\vee}(\hat{\lambda}) = s_{+\alpha} \cdot s_{\alpha+\delta}(\hat{\lambda})$

$$t_{\alpha^\vee}(\hat{\lambda}) = \left(\lambda + k\alpha^\vee; k; n + [|\lambda|^2 - |\lambda + k\alpha^\vee|^2]/2k\right)$$

So, the action of $t_{\alpha^\vee}$ on the finite part λ corresponds to a <u>translation</u>: $\underline{\underline{\lambda \to \lambda + k\alpha^\vee}}$

Moreover: $\quad (t_{\alpha^\vee})(t_{\beta^\vee}) = t_{\alpha^\vee + \beta^\vee}$

and in particular: $\quad (t_{\alpha^\vee})^m = t_{m\alpha^\vee}$

and we verify that:

$$\boxed{s_{\hat{\alpha}}(\hat{\lambda}) = s_\alpha \cdot (t_{\alpha^\vee})^m (\hat{\lambda})}$$

with $\alpha = (\alpha; 0; 0)$.

This means that a reflection of the affine Weyl group appears as the product of a finite Weyl reflection by a translation of an appropriate coroot.

· The $t_{\alpha^\vee}$ generate the coroot lattice $Q^\vee$.

The affine Weyl group is infinite and has a semi-direct product structure:

$$\hat{W} = \underset{\substack{\downarrow \\ \text{finite Weyl group}}}{W} \rhd Q^\vee$$

Indeed:

$$\forall w \in \hat{W} : \quad w\, t_{\alpha^\vee}\, w^{-1} = t_{(w\alpha)^\vee} \qquad \text{(To be checked)}$$
$$\forall \alpha^\vee \in Q^\vee$$

. As a last remark, one notes that the generators of $\hat{W}$ are the reflections S_i $(i = 0, 1, \dots 2)$ w.r.t. simple roots.

_ For $i \neq 0 \longrightarrow$ same defin. as in finite case:

$$s_i(\hat{\lambda}) = (s_i(\lambda)\,;\, k\,;\, n) \quad \text{if } \hat{\lambda} = (\lambda; k; n)$$

since i associate to $\alpha_i = (\alpha_i, 0, 0)$

_ For $i = 0$:

$$s_0(\hat{\lambda}) = s_{(-\theta;0;1)}(\lambda\,;k\,;n) =$$
$$= \big(\lambda + k\theta - (\lambda.\theta)\theta\;;\;k\;;\;n - k + (\lambda.\theta)\big)$$

that is $\quad s_0(\hat{\lambda}) = s_\theta \cdot t_{-\theta}(\hat{\lambda})$

Highest Weight Representations:

Let us recall that the positive roots are:

$$E_0^{\alpha}\ ,\alpha>0 \quad ; \quad E_n^{\pm\alpha}\ , n>0 \quad ; \quad H_n^i\ , n>0$$

Therefore, the highest weight of a $\overset{h.w.}{\vee}$ rep. will be $|\hat{\lambda}\rangle$ s.t.:

$$E_0^{\alpha}|\hat{\lambda}\rangle = E_{n>0}^{\pm\alpha}|\hat{\lambda}\rangle = H_n^i|\hat{\lambda}\rangle = 0$$

One has the eigenvalues:

$$H_0^i\,|\hat{\lambda}\rangle = \lambda^i\,|\hat{\lambda}\rangle \qquad (i\neq 0)$$

$$\hat{k}\,|\hat{\lambda}\rangle = k\,|\hat{\lambda}\rangle$$

$$L_0\,|\hat{\lambda}\rangle = 0 \qquad \text{by convention (one can redefine } L_0)$$

Note that in the Chevalley basis:

$$h^i\,|\hat{\lambda}\rangle = \lambda^i\,|\hat{\lambda}\rangle \quad (i=0,1,\dots,r).$$

Integrable Representations:

<u>Def</u>: It is a $\hat{g}$-representation which decomposes into finite dim. irreducible $SU(2)$ rep., and moreover, can be written as a direct sum of finite-dim weight spaces [this must hold for any $SU(2)$ algebra associated with any real root].

Again considering the computation for $SU(2)$ weights in § WEIGHTS before, for each such an $SU(2)$, we will get, as <u>defined p. 13</u>

$$(\hat{\lambda}', \alpha_i^{\vee}) = -(p_i - q_i) \qquad (i=0,1\dots,r)$$

for any $\hat{\lambda}'$ in the weight lattice which implies : $\lambda_i' \in \mathbb{Z}$ $(i=0,1,..r)$

For the highest weight : $p_i = 0 \Rightarrow \lambda_i \in \mathbb{Z}_+$ $(i=0,1..r)$
(see p. 13)

and , since : $\lambda_0 = k - (\lambda, \theta)$ (see above).

with $(\lambda,\theta) \in \mathbb{Z}_+$

This shows that: $\boxed{k \in \mathbb{Z}_+ \quad \text{and} \quad k \geq (\lambda,\theta)}$

in an integrable h.w. represent.

As a consequence, for each k value, $k \in \mathbb{Z}_+$, it exists a finite number of repres. [by def, an affine weight for which all Dynkin labels are non-negative integers is said dominant].

Ex: $k=1$ for $\hat{SU}(2)$: h.w : $\lambda = \hat{\omega}_0$ $(0,1)$ basic rep.
$\hat{\omega}_1$ $(1,0)$

$k=1$ for $\hat{SU}(N)$: N possible dominant h.w. repres.
with h. weights $\hat{\omega}_i$, $i=0,1..r$.

$k=2$ for $\hat{SU}(3)$: $a_0^\vee = a_1 = a_2 = 1$ and we get
6 repres. : $(2,0,0)$; $(0,2,0)$
$(0,0,2)$; $(1,1,0)$
$(1,0,1)$; $(0,1,1)$

$k=2$ for $\hat{G}_2$ $a_0^\vee = a_2^\vee = 1$ and we get
$a_1^\vee = 2$ 4 rep.: $(2,0,0)$
$(0,0,2)$
$(0,1,0)$
$(1,0,1)$

Unitary Representations:

Choosing $\boxed{\left(J^a_n\right)^\dagger = J^a_{-n}}$ that is also: $\left(H^i_n\right)^\dagger = H^i_{-n}$
$\left(E^\alpha_n\right)^\dagger = E^{-\alpha}_{-n}$

then one can prove that the dominant h.w. repres. <u>are unitary</u>

Indeed, as an example:

$$\left(E^\alpha_{-n}\,|\hat\lambda\rangle\right)^2 = \langle\hat\lambda\,|\,E^{-\alpha}_n\,E^\alpha_{-n}\,|\hat\lambda\rangle$$

$|\hat\lambda\rangle$ = h.w. $\qquad (n>0)$

$$= \langle\hat\lambda|\,E^{-\alpha}_n E^\alpha_{-n} - \underbrace{E^\alpha_{-n}E^{-\alpha}_{+n}}_{=0 \text{ on h.w.}}|\,\hat\lambda\rangle$$

$$= \langle\hat\lambda|\,\frac{2}{|\alpha|^2}\left(-\alpha.H_0 + \hat{k}.n\right)|\hat\lambda\rangle$$

$H^i_0\,|\lambda\rangle = \lambda^i|\lambda\rangle$

$$\Rightarrow\ = \frac{2}{|\alpha|^2}\underbrace{\left(-(\alpha,\lambda) + nk\right)}\langle\hat\lambda|\hat\lambda\rangle$$

$$= \underbrace{(n-1)\,k}_{>0} + \underbrace{\left(k - (\theta,\lambda)\right)}_{\in \mathbb{Z}_+} + (\theta-\alpha,\lambda) > 0$$

(see previous page) (since θ h. root)

$$\Rightarrow \left(E^\alpha_{-n}\,|\hat\lambda\rangle\right)^2 > 0$$

An example of h.w. integrable Representation: $\hat{SU}(2)$ $k=1$

with $\hat{\lambda} = (1,0)$

Cartan Matrix: $\hat{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

The 2 "SU(2)" read:

$\{e_0, f_0, h_0\}$

$\{e_1, f_1, h_1\}$

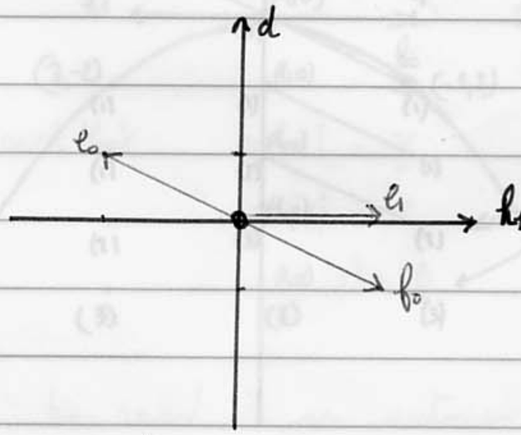
with: $e_0 \rightsquigarrow (-\alpha; 0; 1)$

$e_1 \rightsquigarrow (\alpha; 0; 0)$

$[h_i, e_j] = a_{ji}\, e_j$

$f_j] = -a_{ji}\, e_j$

$[e_i, f_j] = \delta_{ij}\, h_i$



With h.w: $\hat{\lambda} = (1,0) \Rightarrow h_0 |\hat{\lambda}\rangle = |\hat{\lambda}\rangle$

$h_1 |\hat{\lambda}\rangle = 0.$

Consider: $h_0 f_0 |\hat{\lambda}\rangle - f_0 h_0 |\hat{\lambda}\rangle = -2 f_0 |\hat{\lambda}\rangle \Rightarrow h_0 f_0 |\hat{\lambda}\rangle = (1-2) f_0 |\hat{\lambda}\rangle$
$= -f_0 |\hat{\lambda}\rangle$

Since we started with h.w $j=1$, we are already at $-j = -1$

It follows that we are in repres. $\frac{1}{2}$ of SU(2).

Now: $f_0 |\hat{\lambda}\rangle = |\hat{\lambda}'\rangle$

$h_1 f_0 |\hat{\lambda}\rangle = f_0 \underbrace{h_1 |\hat{\lambda}\rangle}_{=0} + 2 f_0 |\hat{\lambda}\rangle \Rightarrow$ eigenvalue: $+2$ for $f_0 |\hat{\lambda}\rangle$

$e_1 f_0 |\hat{\lambda}\rangle = f_0 e_1 |\hat{\lambda}\rangle \underset{h.w}{=} 0 \Rightarrow f_0 |\hat{\lambda}\rangle$ h.w. under $\{e_1, f_1, h_1\}$

$h_1 f_1 (f_0 |\hat{\lambda}\rangle) = f_1 h_1 f_0 |\hat{\lambda}\rangle - 2 f_1 f_0 |\hat{\lambda}\rangle = 0 \Rightarrow h_1 f_1 f_0 |\hat{\lambda}\rangle = 0$

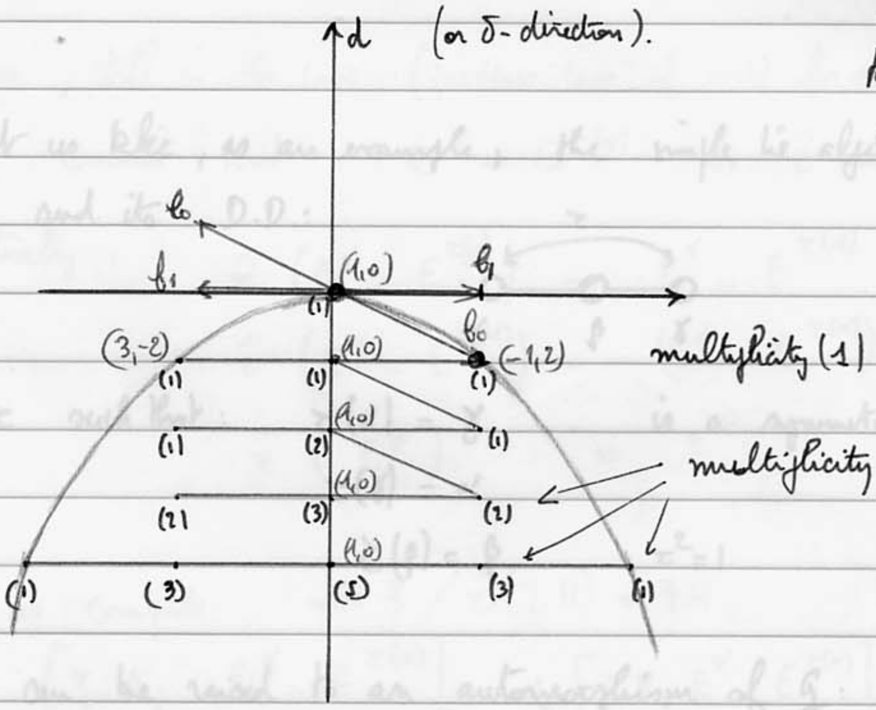
More generally: $(m_0, m_1) \xrightarrow{f_0} (m_0-2, m_1+2)$

$\xrightarrow{f_1} (m_0+2, m_1-2)$

In the same way:

$$h_0 f_1 f_0 |\hat\lambda\rangle = f_1 h_0 f_0 |\hat\lambda\rangle + 2 f_1 f_0 |\hat\lambda\rangle.$$
$$= - f_1 f_0 |\hat\lambda\rangle + 2 f_1 f_0 |\hat\lambda\rangle = f_1 f_0 |\hat\lambda\rangle$$

in accordance with the general law just above.



<u>Multiplicity</u>: let $\hat\mu$ a weight s.t. $\hat\mu+\delta$ is not a weight in the considered repres. associated to h.w $\hat\lambda$

Then the multiplicity of the $\neq$ weights in the "string" $(\hat\mu, \hat\mu-\delta, \hat\mu-2\delta, \dots, \hat\mu-n\delta, \dots)$ is given by the "string function" associated to the $\hat\lambda$-repres:

$$\sigma_{\hat\mu}^{(\hat\lambda)}(q) = \sum_{n=0}^{\infty} \text{mult}_{\hat\lambda}(\hat\mu - n\delta)\, q^n$$

For ex.,

$$\sigma_{[1,0]}^{[1,0]}(q) = \prod_{n=0}^{\infty} (1-q^n)^{-1} = \sum_{n=0}^{\infty} p(n).q^n$$

where $p(n)$ is the nber of inequivalent partitions of n into positive integers.

The first coeff^s are: 1, 1, 2, 3, 5, 7, 11, 15, ...

Twisted affine algebras:

The tableau of extended D.D p.17 is not complete: there is a second set of affine algebras, denoted "twisted" affine algebras. In order to introduce them, and also to show one can obtain them from usual (p.17) affine algebras, let us first discuss the notion of "Folding" in finite dim. simple $\mathcal{G}$.

$\mathcal{G}$ - Folding:

Let us take, as an example, the simple Lie algebra $SU(4)$ and its D.D.:

$$\overset{\alpha}{\circ} - \overset{\beta}{\circ} - \overset{\gamma}{\circ} \qquad \alpha \overset{\tau}{\longleftrightarrow} \gamma$$

τ such that: $\tau(\alpha) = \gamma$ is a symmetry of D.D.

$$\tau(\gamma) = \alpha$$

$$\tau(\beta) = \beta \qquad \tau^2 = 1$$

τ can be raised to an automorphism of $\mathcal{G}$: (note that τ does not change the D.D.)

$$\hat{\tau}(E^\alpha) = E^{\tau(\alpha)} \quad \text{if } \alpha \text{ simple root}$$

$$= \varepsilon_\alpha E^{\tau(\alpha)} \quad \varepsilon_\alpha = \pm 1 \text{ if } \alpha \text{ not simple root}$$

Since we have: $[H^i, H^j] = 0$

$$[H^i, E^\alpha] = \alpha^i E^\alpha \qquad \text{or: } [\beta.H, E^\alpha] = (\alpha.\beta) E^\alpha$$

$$[E^\alpha, E^\beta] = N_{\alpha\beta} E^{\alpha+\beta} \qquad (\alpha+\beta) \text{ root}$$

$$[E^\alpha, E^{-\alpha}] = \frac{2}{|\alpha|^2} \alpha.H$$

$$[E^\alpha, E^\beta] = 0 \qquad \text{otherwise}$$

We check: $[E^{\tau(\alpha)}, E^{\tau(-\alpha)}] = \frac{2}{|\tau(\alpha)|^2} \tau(\alpha).H = \frac{2}{|\alpha|^2} \hat{\tau}(\alpha.H)$

$$\Rightarrow \hat{\tau}(\alpha.H) = \tau(\alpha).H$$

and since (we want) $\hat{\tau}^2 = 1 \quad \rightarrow \quad \alpha.H = \tau(\alpha).\hat{\tau}(H)$

In the same way:

$$[\beta.H, E^\alpha] = (\alpha.\beta)\, E^\alpha \Rightarrow [\hat\tau(\beta H), E^{\tau(\alpha)}] = (\alpha.\beta)\, E^{\tau(\alpha)}$$

$$[\tau(\beta).H, E^{\tau(\alpha)}] = \tau(\beta).\tau(\alpha).\, E^{\tau(\alpha)}$$

$$\Rightarrow \quad (\alpha.\beta) = \tau(\alpha).\tau(\beta)$$

Then, still in the case (previous case of $SU(4)$ for ex.) $\tau^2 = 1$,

we remark that $E^\alpha + E^{\tau(\alpha)}$ is $\hat\tau$-invariant

Actually:

$$\hat\tau\,(E^\alpha + E^{\tau(\alpha)}) = E^\alpha + E^{\tau(\alpha)}$$

$$\hat\tau\,(E^\alpha - E^{\tau(\alpha)}) = -(E^\alpha - E^{\tau(\alpha)})$$

$$\hat\tau\,(E^\beta) = E^\beta \quad \text{in } SU(4)$$

Let us compute: with $\zeta \,/\, \hat\tau(\zeta.H) = \zeta.H$ (invariant part).

$$[\zeta.H, E^\alpha + E^{\tau(\alpha)}] = \tfrac12 [\zeta.H, E^\alpha + E^{\tau(\alpha)}] + \tfrac12 [\underbrace{\zeta}_{=\tau(\zeta).H}, E^\alpha + E^{\tau(\alpha)}]$$

$$= \tfrac12\left((\zeta.\alpha)E^\alpha + \zeta.\tau(\alpha)\, E^{\tau(\alpha)}\right) + \tfrac12\left(\tau(\zeta).\alpha.E^\alpha + \tau(\zeta)\tau(\alpha) E^{\tau(\alpha)}\right)$$

$$\downarrow$$

$$\tfrac12\left(\zeta.\tau(\alpha) E^\alpha + \zeta.\alpha E^{\tau(\alpha)}\right)$$

since: $\tau(\zeta).\alpha = \zeta.\tau(\alpha)$

$\tau(\zeta).\tau(\alpha) = \zeta.\alpha$

And finally: $[\zeta.H, E^\alpha + E^{\tau(\alpha)}] = \tfrac12\,(\alpha + \tau(\alpha))(E^\alpha + E^{\tau(\alpha)})$

Therefore on the $\hat\tau$-invariant part: the root relative to $E^\alpha + E^{\tau(\alpha)}$ is $\tfrac12(\alpha + \tau(\alpha))$

Note that: $\{\tfrac12(\alpha + \tau(\alpha))\}^2 = 1$ if $\alpha^2 = 2$ and $\alpha.\tau(\alpha) = 0$

But if $\alpha.\tau(\alpha) \neq 0$, ie $= -1 \Rightarrow \{\tfrac12(\alpha+\tau(\alpha))\}^2 = 1/2$ which will bring problem!

We will try to avoid this case.

$\hat{\mathcal{G}}$ - folding or twist:

- let τ be an outer automorphism of finite g, s.t. $\tau^N = 1$

$\mathcal{G}$ being considered on the complex field, one can diagonalize τ and the eigenvalues are: $e^{i2\pi m/N}$ $\quad m = 0, 1, \dots, N-1$.

Let us call $\mathcal{G}^{(m)}$ the $\mathcal{G}$-part the elements of which are τ-eigenvectors with eigenvalue $e^{i2\pi m/N}$.

One remarks that $\quad T \in \mathcal{G}_{(m)},\ T' \in \mathcal{G}_{(n)} \Rightarrow [T, T'] \in \mathcal{G}_{(n+m)}$ [mod N]

It follows that $\mathcal{G}$ can be decomposed as:

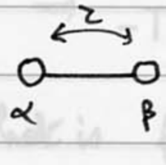
$$\mathcal{G} = \mathcal{G}_{(0)} \oplus \mathcal{G}_{(1)} \oplus \dots \oplus \mathcal{G}_{(N-1)}$$

satisfying the grading: $\quad [\mathcal{G}_{(m)}, \mathcal{G}_{(n)}] \subseteq \mathcal{G}_{(m+n)}$ [mod N]

Then, the "twisted" algebra $\hat{\mathcal{G}}^{\tau}$ will be defined from a basis constituted with elements:

$$\boxed{\begin{array}{l} T^a_r = T^a \otimes t^r \\ \text{with: } r \in \mathbb{Z} + m/N \\ \phantom{\text{with: }} T^a \in \mathcal{G}_{(m)} \end{array}}$$

Example: Consider $\hat{SU(3)}$

SU(3): 

$\tau(\alpha) = \beta$
$\tau(\beta) = \alpha$ $\quad ; \tau^2 = 1$

Define: $S^{\pm} = E^{\pm\alpha} + E^{\pm\beta}$

$H^1 = [S^+, S^-]$

and $\quad D^{\pm} = E^{\pm\alpha} - E^{\pm\beta} \quad ; \quad E^{\pm\gamma} = E^{\pm(\alpha+\beta)}$

(Note: $[E^{\alpha} + E^{\beta}, E^{\alpha} - E^{\beta}] = 2[E^{\alpha}, E^{\beta}] = 2E^{\alpha+\beta}$

it follows $\quad \tau(E^{\pm\gamma}) = -E^{\pm\gamma}$

as well as: $\quad \tau(D^{\pm}) = -D^{\pm}$

$\hat{SU}(3)^{\tau}$ also noted $S\hat{U}(3)^{(2)}$ (usually, the untwisted $\hat{SU}(3)$ is also written as $SU(3)^{(1)}$)

is therefore constituted with elements:

$$\{H^1_m \, ; \, S^{\pm}_m \} \text{ or } \{H^1_{2m} \, ; \, S^{\pm}_{2m}\} \equiv \mathcal{G}^{(0)}$$

$$\{D^{\pm}_{m+\frac{1}{2}} \, ; \, H^2_{m+1/2} \, ; \, E^{\pm\gamma}_{m+1/2}\} \text{ or } \{D^{\pm}_{2m+1} ; H^2_{2m+1} ; E^{\pm\gamma}_{2m+1}\} \equiv \mathcal{G}^{(1)}$$

More generally, from $\mathcal{G}^{(1)}$ with the outer automorphism τ s.t. $\tau^N = 1$, one constructs $\mathcal{G}^{(N)}$ as made with elements $T^a \otimes f(z)$ satisfying:

$$(*) \quad \boxed{\tau(T^a) \otimes f(z) = T^a \otimes f(z e^{2i\pi/N})}$$

As an example, consider again $\tau \,/\, \tau^2 = 1$:

- $T^a \in \mathcal{G}_{(0)}$, the $(*)$ condition reads $\Rightarrow T^a \otimes f(z) = T^a \otimes f(-z)$
  that is $f(z)$ made with even powers of z.
- $T^a \in \mathcal{G}_{(1)}$ $\Rightarrow -T^a \otimes f(z) = T^a \otimes f(-z)$

  that is $f(z)$ made with odd powers of z.

Easily checkable for τ s.t. $\tau^N = 1$ with $N = 3, 4 \ldots$

Actually, the DD associated to twisted affine algebras can be obtained by "folding" the extended DD of the $G^{(1)}$.

[Note: to avoid some difficulty, one restricts to τ such that

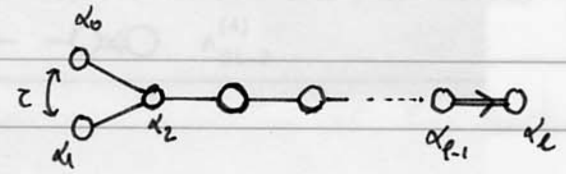
$$\underline{\alpha \neq \tau(\alpha) \Rightarrow \alpha . \tau(\alpha) = 0}$$

then $\frac{1}{2}(\alpha + \tau(\alpha))$ will be of norm ½ the norm of α or $\tau(\alpha)$.

Otherwise, if $\alpha . \tau(\alpha) = -1$ one would get, with $|\alpha|^2 = 2$, a norm for $\frac{1}{2}(\alpha + \tau(\alpha))$ of value $= 1/2$!].

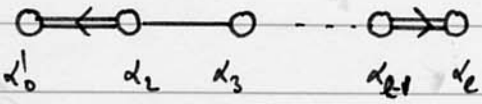
So, we look at the symmetries of the extended D.D which do not leave invariant the affine root : $\alpha_0 = -\theta + \delta$

example: from $B_\ell^{(1)}$



$$\alpha_0' = \frac{1}{2}(\alpha_0 + \alpha_1) = \frac{1}{2}(-\theta + \delta + \alpha_1) = -\frac{1}{2}(\theta - \alpha_1) + \frac{\delta}{2}$$

and we obtain :



The complete table showing the construction of twisted alg. from untwisted ones is shown next page.

For more details, cf the paper:

"Fermionic construction of vertex operators for twisted affine algebras" L. Frappat, A. Sciarrino & P. Sorba.

Nucl Phys. B305 (1988) [FS23] 164

VERTEX OPERATOR REPRESENTATION of an AFFINE ALGEBRA

(Frenkel-Kac-Segal construction)

Rk: One knows that the $SU(n)$ algebra can be realized in terms of creators and annihilators: let n creators $a_1^+, \dots a_n^+$ and n annihilators $a_1, \dots a_n$ satisfying:

$$[a_i, a_j^+] = \delta_{ij}$$

Considering the products: $a_i^+ a_j = \alpha_{ij}$: one can check that they satisfy the C.R. of the $U(n)$ algebra [one has to take off the $U(1)$-generator $\sum_{i=1}^n a_i^+ a_i$ to recover the $SU(n)$ algebra].
Of course such a realization provides only some representations of $SU(n)$ [antisym. ones for $[,]_-$ and symmetric ones for $[,]_+$).

It is to some extend a generalization of this construction to the case of affine algebras that we will consider now.

- We wish to represent the affine Lie algebra $\hat{g}$ defined by its C.R.:

$$[H_n^i, H_m^j] = \delta^{ij} m\, \delta_{n+m,0}$$

$$[H_n^i, E_m^\alpha] = \alpha^i E_{n+m}$$

$$[E_n^\alpha, E_m^\beta] = \varepsilon(\alpha,\beta)\, E_{n+m}^{\alpha+\beta} \quad \text{if } \alpha+\beta \in \Delta_{\text{root syst}}$$

$$= \alpha \cdot H_{m+n} + m\,\delta_{m+n,0}\,\hat{k} \quad \text{if } \alpha+\beta = 0.$$

$$= 0 \quad \text{otherwise}$$

the central extension.

Let us consider the affine algebra of l-dimensional oscillators:

$$\boxed{[\alpha^i_m, \alpha^j_n] = m\,\delta_{m+n,0}\,\delta^{ij}} \qquad i,j \in \{1,\dots l\} \quad m,n \in \mathbb{Z}$$

with generic functions: $\alpha^i(z) = \sum_{m\in\mathbb{Z}} \alpha^i_m z^{-m}$

One defines : $\underline{\alpha^i_0 = p^i}$ and q^i s.t. : $\underline{[q^i, p^j] = i\,\delta^{ij}}$

$q^i \downarrow$ which will soon appear as an integration constant

One also imposes: $\begin{cases} \alpha^{i\,+}_n = \alpha^i_{-n} \\ p^{i\,+} = p^i \end{cases}$

The space of considered states is a Fock space, constructed from the vacuum $|0\rangle$ and such that:

$$\begin{cases} \alpha^i_n |0\rangle = 0 & n>0 \quad \text{"annihilators"} \\ \alpha^i_n |0\rangle \neq 0 & n<0 \quad \text{"creators"} \\ p^i |0\rangle = 0 & \text{and } e^{iq.\lambda}|0\rangle = |\lambda\rangle \end{cases}$$

We also define a <u>normal ordering</u>:

$$:\alpha^i_n . \alpha^j_m: \;=\; \alpha^i_n . \alpha^j_m \quad \text{if } m>0$$

$$\alpha^j_m . \alpha^i_n \quad \text{if } m<0 \text{ and } n>0$$

$$:q^i . p^j: \;=\; q^i . p^j$$

One remarks that the H^i_m, H^i_{-m} $(m=1,\dots\infty)$ constitute independent harmonic oscillators (they correspond to the successive harmonics of a string vibrating in a space of dim. l = rank (g)).

The generating function is called a Fubini - Veneziano momentum field and writes :

$$\boxed{H^i(z) = P^i(z) = p^i + \sum_{n=1}^{\infty}\left(\alpha^i_n z^{-n} + \alpha^i_{-n} z^n\right)}$$

In string theory, there is also a F-V. coordinate field $Q^i(z)$ obtained from $P^i(z)$ by integrating the relation:

$$\boxed{P^i(z) = iz\frac{d}{dz}Q^i(z)}$$

that is: $$\boxed{Q^i(z) = q^i - ip^i \ln z + i\sum_{n\neq 0}\frac{\alpha^i_n}{n}z^{-n}.}$$

From the relation : $e^{iq\lambda}|0\rangle = |\lambda\rangle$ one can imagine the generalization of the "plane wave" $e^{iq\lambda}$ as $e^{i\alpha . Q(z)}$ and more precisely :

$$U^\alpha(z) = z^{\alpha^2/2} :\exp\left(i\alpha . Q(z)\right):$$

satisfying (see previous page)

$$U^\alpha(z)^+ = U^{-\alpha}(1/z^*).$$

- One can easily compute:

(see proof aside) (*) $$\boxed{P^i(z) . Q^j(\zeta) = \; : P^i(z) . Q^j(\zeta): - i\delta^{ij}\frac{z}{z-\zeta}} \quad \underline{\underline{|z|>|w|}}$$

which converges for $|z|>|w|$
It can be analytically continued for $|z|<w$ except at poles $z=0$, $\zeta=0$ and $z=\zeta$.

and by differentiating w.r.to ζ:

(**) $$\boxed{P^i(z) . P^j(\zeta) = \; : P^i(z) . P^j(\zeta): + \delta^{ij}\frac{z\zeta}{(z-\zeta)^2} \; ; \; |z|>|\zeta|}$$

From $(*)$, we can use Wick theorem to get:

$$(***)\quad \boxed{P^i(z)\cdot U^\alpha(\zeta) = :P^i(z)\cdot U^\alpha(\zeta): + \frac{z}{z-\zeta}\alpha^i\, U^\alpha(z) \qquad |z|>|\zeta|}$$

← — — We can also get this result by iteration from $(*)$.

see aside

It will be also necessary to know the formula:

$$(****)\quad \boxed{U(\alpha,z)\cdot U(\beta,w) = (z-w)^{\alpha\cdot\beta} : U(\alpha,z)\cdot U(\beta,w): \qquad |z|>w}$$

Note: these expression converges for $|z|>|w|$ and can be analytically continued for $|z|<|w|$ except for the poles at: $z=0$, $w=0$ and $z=w$.

To prove this last formula, we remark first that:

$$:e^{i\alpha(q-ip\ln z+Q_<+Q_>)}: = :e^{i\alpha(q-ip\ln z)}: e^{i\alpha Q_<}\cdot e^{i\alpha Q_>}$$

(with $Q_>(z) = i\sum_{n=1}^{\infty}\frac{1}{n}\alpha_n^i z^{-n}$
$Q_<(z) = i\sum_{n=-1}^{-\infty}\frac{1}{n}\alpha_n^i z^{-n}$)

↓ this last property can be checked directly.

$$:e^{i\alpha Q(z)}: = e^{i\alpha q}\cdot e^{\alpha\cdot p\ln z}\, e^{i\alpha\cdot Q_<}\cdot e^{i\alpha Q_>}$$
$$= e^{i\alpha\cdot q}\cdot z^{\alpha\cdot p}\cdot e^{i\alpha Q_<}\cdot e^{i\alpha Q_>}$$

but (Hausdorff-Campbell formula):

$$e^{i\alpha\cdot Q_>(z)}\cdot e^{i\beta\cdot Q_<(\zeta)} = e^{[i\alpha Q_>(z),\, i\beta\cdot Q_<(\zeta)]}\cdot e^{i\beta\cdot Q_<(\zeta)}\cdot e^{i\alpha Q_>(z)}$$

- since $e^A\cdot e^B = e^{A+B+\frac{1}{2}[A,B]}$ with $[A,B]=C^t$ (this is the case with $[Q_>,Q_<]=C^t$)
$e^B\cdot e^A = e^{A+B+\frac{1}{2}[B,A]}$

C^t = "central el." in the Lie algebra.

So: $(A+B)+\frac{1}{2}[A,B]$ commutes with $(A+B)+\frac{1}{2}[B,A]$

$$\Rightarrow e^A\cdot e^B = e^{[A,B]}\cdot e^B\cdot e^A \quad !$$

Therefore: $e^{i\alpha.Q_>(z)} . e^{i\beta.Q_<(\zeta)} = \left(1 - \frac{\zeta}{z}\right)^{\alpha\beta} . e^{i\beta Q_<(\zeta)} . e^{i\alpha Q_>(z)}$ ‖

in $e^{i\alpha Q_<(z)} . e^{i\alpha.Q_>(z)} . e^{i\beta.Q_<(\zeta)} . e^{i\beta.Q_>(\zeta)}$

We don't forget the (p,q) term: $z^{\alpha p} . e^{i\beta q} = e^{i\beta q} . z^{\alpha p} . z^{\alpha\beta}$ ‖

(coming back to $z^{\alpha p} = e^{\alpha p \ln z}$ and checking term by term, f.ex)

we are back to formula (****) !

Now that Eq. (***) and (****) can also provide:

$(***)'$: $\boxed{U^\alpha(\zeta) . P^i(z)} = h^{i\alpha}(z,\zeta)$ $\underline{\text{for } |\zeta| > |z|}$

if : $P^i(z) . U^\alpha(\zeta) = h^{i\alpha}(z,\zeta)$ $\underline{|z|>|\zeta|}$

and

$(****)'$: $\boxed{U(\beta,w) . U(\alpha,z)} = (-1)^{\alpha\beta}\, U(\alpha,z) . U(\beta,w)$ $\underline{\underline{|z|<|\zeta|}}$.

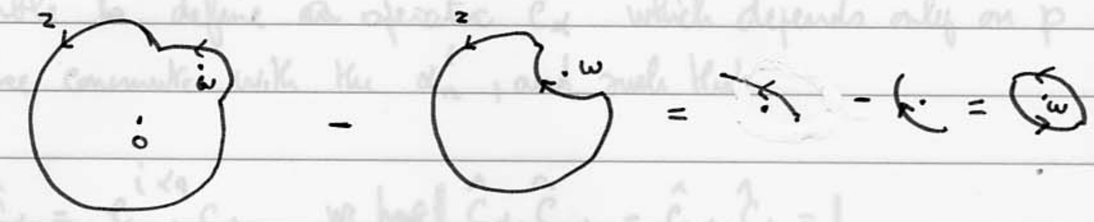
Therefore, let us consider: $\boxed{U(\alpha,z) = \sum_{m\in\mathbb{Z}} z^{-m}\, U^\alpha_m}$

$$\rightarrow U^\alpha_m . U^\beta_n - (-1)^{\alpha\beta} U^\beta_n . U^\alpha_m = \frac{1}{(2i\pi)^2} \oint_{C_0\ |z|>|w|} U(\alpha,z) . U(\beta,w)\ \frac{z^m}{z} dz\ \frac{w^n}{n} dw$$

$$- (-1)^{\alpha\beta} \frac{1}{(2i\pi)^2} \int_{C_0\ |w|>|z|} U(\beta,w) . U(\alpha.z)\ z^m w^n \frac{dz}{z}\frac{dw}{w}$$

$$= \frac{1}{(2i\pi)^2} \int_{C_0\ |z|>|w| - |w|>z} (z-w)^{\alpha\beta} \underbrace{: U(\alpha,z) . U(\beta,w) :}_{z^{\alpha^2/2} w^{\beta^2/2} : e^{i(\alpha.Q(z) + \beta.Q(w))} :}\ z^m w^n \frac{dz}{z}\frac{dw}{w}$$

That is:



this means $\displaystyle\int_{C} \left(|z|>|w| - |w|>|z|\right) \equiv \oint_0 \oint_w$

contour around 0 — contour around ω

If: $\alpha.\beta = -1 \Rightarrow \displaystyle\oint_0\oint_\omega :U(\alpha,z).U(\beta,w): \frac{1}{(z-w)} \frac{z^m dz}{z} \frac{w^n}{w} dw$

1 pole

$(-1)^{\alpha\beta} = -1$

$$\Rightarrow U^\alpha_m U^\beta_n + U^\beta_m U^\alpha_n = U^{\alpha+\beta}_{m+n} \quad (\alpha+\beta) \text{ root !}$$

If: $\alpha.\beta = -2 \Rightarrow \alpha = -\beta$.

then double pole $\displaystyle\oint_0\oint_\omega z^{\alpha^2/2}.w^{\beta^2/2} : e^{i(\alpha.Q(z)+\beta.Q(w))} : \frac{1}{(z-w)^2} z^m \frac{dz}{z} w^n \frac{dw}{w}$

$$\Rightarrow U^\alpha_m . U^{-\alpha}_n - U^{-\alpha}_n . U^\alpha_m = \alpha . P_{m+n} + m\,\delta_{m+n,0}$$

Then we get almost the C.R. between root generators, up to a sign (we have quasi-commutation relations). This is not surprising since we didn't yet speak about the kind of simple alg. we are dealing with.
- Actually, the $\varepsilon(\alpha,\beta)$ which is introduced p. 35 is a
2-cocycle satisfying:

$$\varepsilon(\alpha, \beta+\gamma).\varepsilon(\beta,\gamma) = \varepsilon(\alpha+\beta,\gamma).\varepsilon(\alpha,\beta)$$

← see aside

It is possible to define ~~an~~ cocycle operator c_α which depends only on p and therefore commute with the α^i_n, and such that:

with $\underline{\underline{\hat{c}_\alpha = e^{i\alpha q}\cdot c_\alpha}}$ we have
$$\hat{c}_\alpha\cdot\hat{c}_{-\alpha} = \hat{c}_{-\alpha}\cdot\hat{c}_\alpha = 1$$
$$\hat{c}_\alpha\cdot\hat{c}_\beta = (-1)^{\alpha\beta}\hat{c}_\beta\cdot\hat{c}_\alpha = \varepsilon(\alpha,\beta)\,\hat{c}_{\alpha+\beta}$$

Then setting the generating function:
$$\underline{\underline{E(\alpha,z) = U(\alpha,z)\cdot\hat{c}_\alpha}} = \sum_{n\in\mathbb{Z}} U^\alpha_n\cdot c_\alpha\cdot z^{-n} = \sum_{n\in\mathbb{Z}} E^\alpha_n\cdot z^{-n}$$

<u>we recover the correct C.R between E^α_m and E^β_n</u>

. Let us finally check that the other C.R ~~of~~ the K.M. alg. are obtained in the same manner:

\- Considering:
$$P^i(z)\cdot U^\alpha(w) = :P^i(z)\cdot U^\alpha(w): + \alpha^i\frac{z}{(z-w)}U^\alpha(w) \quad |z|>|w|$$
$$U^\alpha(w)\cdot P^i(z) = \quad \text{''} \quad |w|>|z|$$

Note that now, the regular part will not contribute to the integral, being regular, by Cauchy theorem, but the <u>second term will provide a pole at $z=w$</u>, and finally:

$$\bullet\ \left[H^i_m, U^\alpha_n\right] = \alpha^i U^\alpha_{m+n}.$$

In an even simpler way, we can compute:

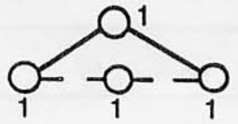
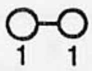
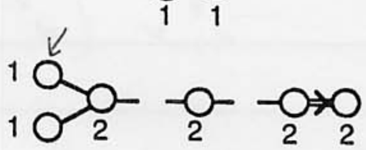
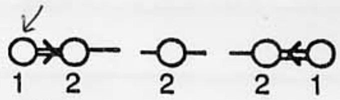
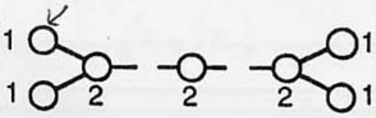
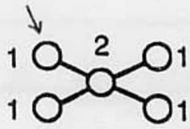
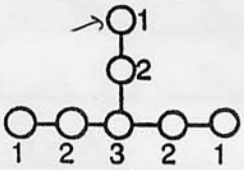
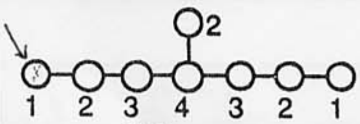
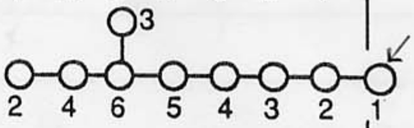
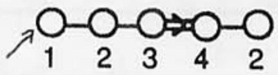
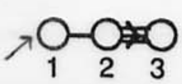
$$\bullet\ \left[H^i_m, H^j_n\right] = m\,\delta^{ij}\,\delta_{m+n,0}$$

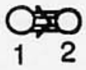
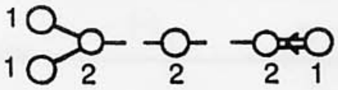
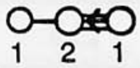
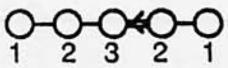
<u>Note</u>: the $\hat{k}$-eigenvalue is 1. <u>Only level one rep.</u> to obtained.

either from the C.R. of the α^i_n between themselves, or (equivalently) using:
$$P^i(z)\,P^j(w) = :P^i(z)\cdot P^j(w): + \delta^{ij}\frac{zw}{(z-w)^2} \quad |z|>|w|$$

and performing the integ. by parts, involving non trivially the <u>non regular second term</u> which provides <u>a second order pole</u>.

Table 1 : Symmetries of Extended Dynkin Diagrams

Affine algebra $\mathcal{G}^{(1)}$	*Dynkin diagram*	*Auto-morphism group* $\mathcal{F}(\mathcal{G}^{(1)})$	*Center* $Z(\overline{G})$	*Auto-morphism group* $\mathcal{F}(\mathcal{G})$
$A_l^{(1)}$ $l \geq 2$		D_{l+1}	Z_{l+1}	Z_2
$A_2^{(1)}$		Z_2	Z_2	1
$B_l^{(1)}$ $l \geq 2$		Z_2	Z_2	1
$C_l^{(1)}$ $l \geq 3$		Z_2	Z_2	1
$D_l^{(1)}$ $l > 4$		D_4	$Z_2 \times Z_2$ (l *even*)	Z_2
		D_4	Z_4 (l *odd*)	Z_2
$D_4^{(1)}$		S_4	$Z_2 \times Z_2$	S_3
$E_6^{(1)}$		S_3	Z_3	Z_2
$E_7^{(1)}$		Z_2	Z_2	1
$E_8^{(1)}$		1	1	1
$F_4^{(1)}$		1	1	1
$G_2^{(1)}$		1	1	1

Twisted algebra $\mathcal{G}^{(m)}$	*Dynkin diagram*	*Auto–morphism group* $\mathcal{F}(\mathcal{G}^{(m)})$	
$A_{2l}^{(2)}$ $l \geq 2$		1	$\Leftarrow A_{2l+1}^{(2)}$
$A_2^{(2)}$		1	$\Leftarrow D_4^{(1)}$
$A_{2l-1}^{(2)}$ $l \geq 3$		Z_2	$\Leftarrow D_{2l}^{(1)}$
$D_{l+1}^{(2)}$ $l \geq 2$		Z_2	$D_{l+2}^{(1)}$
$D_4^{(3)}$		1	$\Leftarrow E_6^{(1)}$
$E_6^{(2)}$		1	$\Leftarrow E_7^{(1)}$

We note that Z_n is the cyclic group of order n, S_n the permutation group of n objects and D_n the dihedral group with $2n$ elements [10].
Algebras labelled by the index l have DD with $l+1$ vertices.

Folding schemes for affine and twisted algebras