QUANTUM PHASE SPACE

Wigner, Weyl, Moyal 1927–49

Wigner transform

$$f(x,y) \longmapsto \overset{w}{f}(q,p) = \int dr\, f(q-\tfrac{r}{2}, q+\tfrac{r}{2})\, e^{\frac{i}{\hbar}pr}$$

map from operators A to phase-space functions $\overset{w}{A}$

$$A(\hat{q},\hat{p}) \longmapsto \overset{w}{A}(q,p) = \langle x|A|y\rangle^{w} = \int dr\, \langle q-\tfrac{r}{2}|A|q+\tfrac{r}{2}\rangle\, e^{\frac{i}{\hbar}pr}$$

special case: the density operator

$$\rho \longmapsto W(q,p) \qquad \text{"Wigner function"}$$

alternatively:

$$\overset{w}{A}(q,p) = \frac{1}{2\pi\hbar}\int d\sigma \int d\tau\, \operatorname{tr} A(\hat{q},\hat{p})\, e^{-i\tau(\hat{p}-p)-i\sigma(\hat{q}-q)}$$

$$= A_{*}(q,p) \qquad \text{"symbol map"}$$

where A_* is $A(\hat{q}\to q, \hat{p}\to p)$ but with $*$ product:

$$(f*g)(q,p) = f(q,p)\, e^{\frac{i}{2}\hbar\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q\right)} g(q,p)$$

Groenewold 1946
Moyal 1949

$$= (f\cdot g)(q,p) + \tfrac{i}{2}\hbar\,\{f,g\}(q,p) + \mathcal{O}(\hbar^2)$$

Poisson brackets

inverse map from functions to operators — Weyl 1927

$$A(\hat{q},\hat{p}) = \frac{1}{(2\pi\hbar)^2}\int d\sigma \int d\tau \int dq \int dp\, \overset{w}{A}(q,p)\, e^{i\tau(\hat{p}-p)+i\sigma(\hat{q}-q)}$$

$$= \overset{w}{A}(\hat{q},\hat{p})\Big|_{\text{Weyl-ordered}}$$

some properties

$$(A\cdot B)^W = \overset{W}{A} * \overset{W}{B} \quad \rightsquigarrow \quad [A,B]^W = i\hbar \{\overset{W}{A}, \overset{W}{B}\} + \mathcal{O}(\hbar^3)$$

$$\int dq \int \frac{dp}{2\pi\hbar} \overset{W}{A} = \mathrm{tr}\, A \quad \rightsquigarrow \quad \int dq \int \frac{dp}{2\pi\hbar} \overset{W}{A} * \overset{W}{B} = \int dq \int \frac{dp}{2\pi\hbar} \overset{W}{A} \cdot \overset{W}{B}$$

$$(A(q)+B(p))^W = A(q)+B(p)\,, \qquad \text{'}*\text{' is associative}$$

$$\langle A \rangle_\rho = \mathrm{tr}(A\rho) = \int dq \int \frac{dp}{2\pi\hbar} \overset{W}{A} \cdot W$$

time evolution — Moyal 1949

$$i\hbar\, \partial_t \rho = [H, \rho]$$

wigner transform $\downarrow$, $H = \frac{p^2}{2m} + V(q)$

$$i\hbar\, \partial_t W = H * W - W * H \qquad \text{q-Liouville}$$

$$= i\hbar \{H, W\} - \frac{i}{24}\hbar^3 V''' \partial_p^3 W + \dots$$

finite time:

$$\mathcal{U}_*(q,p,t) = e_*^{\frac{i}{\hbar} t H} \qquad \partial_t H = 0$$

$$\overset{W}{A}(t) = \mathcal{U}_*^{-1}(t) * \overset{W}{A}(0) * \mathcal{U}_*(t)$$

QM may be formulated in phase space (with $*$) rather than in Hilbert space (with $\cdot$) !

NONCOMMUTATIVE "GEOMETRY"

Connes, Rieffel > 1981

"commutative" "noncommutative"

algebra of continuous functions $C^\infty(\mathcal{M}), \cdot$ $\xrightarrow[\theta]{\text{deform}}$ noncommutative algebra of functions $C^\infty_\theta(\mathcal{M}), *$

$\updownarrow$ Gel'fand-Naimark $\qquad\qquad$ $\downarrow$ ideals

manifold $\mathcal{M}$ $\qquad\qquad$ NC space $\mathcal{M}_\theta$

→ notion of "point" no longer exists

Specialize to flat Euclidean space $\mathbb{R}^d_*$

want associative product ⟶ $*$ = Moyal product

$$(f*g)(x) = f(x)\, e^{\frac{i}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu}\, g(x)$$

$$= f\cdot g(x) + \frac{i}{4}\theta^{\mu\nu}\partial_{[\mu} f\, \partial_{\nu]} g(x) + \ldots$$

copy
(q,p) $\hbar\mathbb{I}$
$\downarrow$ $\downarrow$
(x^1,x^2) θ

coordinates as special (generating) functions:

$$[x^\mu, x^\nu]_* \equiv x^\mu * x^\nu - x^\nu * x^\mu = i\,\theta^{\mu\nu}$$ analog of phase-space

$$[\hat{x}^\mu, \hat{x}^\nu] = i\,\theta^{\mu\nu}\cdot\mathbb{1}$$ analog of Hilbert space

canonical basis: $\{x^i, y^i;\ i=1,\ldots,\frac{d}{2}\}$

$$[\hat{x}^i, \hat{y}^j] = i\,\delta^{ij}\,\theta^i \geq 0 \quad \leftrightarrow \quad (\theta^{\mu\nu}) = \begin{pmatrix} \begin{matrix} 0 & \theta^1 \\ -\theta^1 & 0 \end{matrix} & & 0 \\ & \begin{matrix} 0 & \theta^2 \\ -\theta^2 & 0 \end{matrix} & \\ 0 & & \ddots \end{pmatrix}$$ Darboux form

new combinations:

$$a_i = \frac{1}{\sqrt{2\theta^i}}(\hat{x}^i + i\hat{y}^i), \quad a_i^\dagger = \frac{1}{\sqrt{2\theta^i}}(\hat{x}^i - i\hat{y}^i) \quad \Rightarrow \quad [a_i, a_j^\dagger] = \delta_{ij}$$

symmetries of $\mathbb{R}^d_*$

translations $\partial_\mu f = [-i\,\theta^{-1}_{\mu\nu} x^\nu, f]_*$

internal symmetries $\delta f = [A, f]_*$

no more distinction

- $\theta^{\mu\nu}$ preserved by $Sp(d)$
- $g_{\mu\nu}$ preserved by $SO(d)$

$\Big\} \cap\ U(d/2)$ symmetry

$\rightsquigarrow$ cannot separately define integral and trace

dipole picture

Susskind 2000

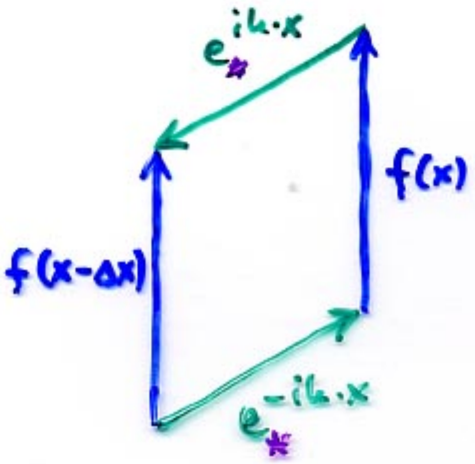
nc plane wave $\quad \partial_\mu e_*^{ik\cdot x} = ik_\mu e_*^{ik\cdot x}$

multiplication (BCH) $\quad e_*^{ik\cdot x} * e_*^{ik'\cdot x} = e^{-\frac{i}{2}k\wedge k'}\, e_*^{i(k+k')\cdot x}$

$$k \wedge k' \equiv k_\mu \theta^{\mu\nu} k'_\nu$$

translation $\quad e_*^{ik\cdot x} * f(x) * e_*^{-ik\cdot x} = f(x^\mu - \theta^{\mu\nu}k_\nu)$

nonlocality



plane wave ~ rigid oriented rod

size $\Delta x^\mu = \theta^{\mu\nu} k_\nu$

dipoles interact by joining at ends

NC INSTANTONS & SOLITONS

solitons in 2+1 dimensions $(\hat{x},\hat{y},t)_\theta$

noncommutativity overcomes Derrick's theorem

rescale $x^\mu \to \sqrt{\theta}\, x^\mu$, restrict to static fields

energy functional $E = \int d^2x \left[\frac{1}{2}(\vec{\nabla}\phi)^2 + \theta V_*(\phi) \right]$

θV large $\to$ neglect E_{kin} $\to$ E_{pot} minimal: $V'_*(\phi)=0$

<u>ex.:</u> $V = \frac{1}{2}\phi^2 + \frac{g}{3}\phi^3$ $\to$ solve $\phi + g\phi * \phi = 0$

[$\theta = 0$: $\phi = -1/g$ const. $(E=\infty)$ or $\phi = 0$]

$\theta \neq 0$: $\phi = -\frac{1}{g}\, \mathcal{U}^\dagger * P * \mathcal{U}$ with $\mathcal{U} * \mathcal{U}^\dagger = 1$

and $P * P = P$ projector

rank-one solution: $P = 2e^{-(x^2+y^2)/\theta}$ (no $*$!)

Gopakumar Minwalla Strominger 2000 $\Leftrightarrow$ $\hat{P} = |0\rangle\langle 0|$ in oscillator Fock space

- kinetic energy contribution is $\mathcal{O}(\theta^{-1})$ $\to$ perturbation
- $\theta \to 0$ limit is singular : $P \to \delta_{x,0}\,\delta_{y,0}$
- generalizations: multisolitons, sigma-model, higher-rank P...
- time dependence by adiabatic motion

vortices (Abelian Higgs model in $d=2+1$)

instantons (Yang-Mills in $d=4+0$) even for U(1)

NC ADHM construction — Nekrasov, Schwarz 1998

NC twistor approach — Lechtenfeld, Popov 2002

regulates zero-size instanton singularity in moduli space

monopoles (Yang-Mills-Higgs in $d=3+1$)

NC Nahm approach — Gross, Nekrasov 2000

NC twistor approach — Lechtenfeld, Popov 2003

NONCOMMUTATIVE FIELD THEORY

scalar fields

DFR 1994

deform classical action

$$S[\varphi] = \int d^dx \left[\tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \tfrac{1}{2}m^2\varphi^2 - V(\varphi)\right]$$

$$S_\theta[\varphi] = \int d^dx \left[\tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \tfrac{1}{2}m^2\varphi^2 - V_*(\varphi)\right]$$

this is nonlocal because

$$(f*g)(x) = \int d^dx_1 \int d^dx_2\, K(x|x_1,x_2)\, f(x_1)\, g(x_2)$$

with $K(x|x_1,x_2) = \pi^{-d}|\det\theta|^{-1} e^{2i(x-x_1)^\mu \theta^{-1}_{\mu\nu}(x-x_2)^\nu}$

similarity with matrix theories

$\Phi \in \mathrm{Mat}_N$

$$S_N[\Phi] = \underbrace{\int d^dx\, \mathrm{tr}}_{\mathrm{Tr}} \left[\tfrac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi - \tfrac{1}{2}m^2\Phi^2 - V(\Phi)\right]$$

deform

- $(\varphi, *) \leftrightarrow (\hat\varphi, \cdot) \leftrightarrow \varphi(a,a^\dagger) \leftrightarrow \Phi \in \mathrm{Mat}_\infty$ for $d=0$
- $(\varphi_{ij}, *) \leftrightarrow (\hat\varphi_{ij}, \cdot) \leftrightarrow \varphi_{ij}(a,a^\dagger) \leftrightarrow \Phi \in \mathrm{Mat}_{N\cdot\infty}$ $\quad ij=1\ldots N$

gauge fields

U(N) Yang-Mills

$$S[A] = -\frac{1}{4g^2}\int d^dx\, \mathrm{tr}\, F_*^2 \qquad F_*^{\mu\nu} = \partial^{[\mu}A^{\nu]} + i[A^\mu, A^\nu]_*$$

translations are gauge transfms: nontrivial for U(1)!

$$\delta A_\mu = \partial_\mu \varepsilon + i[A_\mu,\varepsilon]_* \xrightarrow[\varepsilon = v\cdot\theta^{-1}x]{} \delta A_\mu = (v\cdot\theta^{-1})_\mu + v\cdot\partial A_\mu$$

$$D_\mu \Phi = i[C_\mu, \Phi]_* \quad \text{with} \quad C_\mu := A_\mu - (\theta^{-1}\cdot x)_\mu$$

$$S[\hat C] = +\frac{1}{4g^2}\,\mathrm{Tr}\sum_{\mu\nu}\left(i[\hat C_\mu, \hat C_\nu] - \theta^{-1}_{\mu\nu}\right)^2$$

matrix model!

OLAF LECHTENFELD DUBNA

DIASTP 2003

Self-duality, integrability, and noncommutativity

D=4+0 cplx. coord.: $y = x^1 + \mathrm{i}x^2$ and $z = x^3 - \mathrm{i}x^4$

self-duality $F = *F \quad \in u(n) \qquad \Longleftrightarrow$

$$[D_y, D_z] = 0 = [D_{\bar{y}}, D_{\bar{z}}]$$
$$[D_y, D_{\bar{y}}] + [D_z, D_{\bar{z}}] = 0$$

these are the compatibility conditions of

$$(D_{\bar{y}} - \lambda D_z)\,\Psi(x,\lambda) \;=\; 0 \;=\; (D_{\bar{z}} + \lambda D_y)\,\Psi(x,\lambda)$$

$\Psi(x,\lambda) \in \mathrm{U}(n)$ matrix function, holomorphic in λ
λ: spectral parameter $\in \dot{\mathbb{C}} \simeq S^2$
twistor space: $\mathbb{R}^4 \times S^2 = \mathcal{U}_+ \cup \mathcal{U}_-$ (two patches)

gauge potential from auxiliary Ψ:

$$A_{\bar{y}} - \lambda A_z = \Psi(\partial_{\bar{y}} - \lambda\partial_z)\Psi^{-1}$$
$$A_{\bar{z}} + \lambda A_y = \Psi(\partial_{\bar{z}} + \lambda\partial_y)\Psi^{-1} \qquad (1)$$

reality condition $\longleftrightarrow$ normalization:

$$A_\mu^\dagger = -A_\mu \quad \Leftrightarrow \quad \Psi^\dagger(x, -1/\bar{\lambda})\,\Psi(x,\lambda) = 1 \qquad (2)$$

Atiyah, Ward 1977
Atiyah, Hitchin, Singer 1978

Splitting method

$\Psi(\lambda)$ has poles at $\lambda{=}0$ or $\lambda{=}\infty$ $\quad\rightarrow\quad$ patch up!

$$\Psi_+(\lambda) \quad \text{on } \mathcal{U}_+ \quad , \qquad \Psi_-(-1/\bar{\lambda}) \quad \text{on } \mathcal{U}_-$$

transition function $f_{+-} := \Psi_+^{-1}\Psi_-$ on $\mathcal{U}_+ \cap \mathcal{U}_-$

$$\Rightarrow \quad (\partial_{\bar{y}} - \lambda\partial_z) f_{+-} = 0 = (\partial_{\bar{z}} + \lambda\partial_y) f_{+-}$$

$$\Rightarrow \quad f_{+-} = f_{+-}(y{-}\lambda\bar{z}, z{+}\lambda\bar{y}, \lambda) \quad \text{holomorphic}$$

reality condition: $\quad f^\dagger_{+-}(-1/\bar{\lambda}) = f_{+-}(\lambda)$

$$\Psi_+(\lambda)\,\Psi^\dagger_-(-1/\bar{\lambda}) = g^2$$

with (nonunitary) gauge freedom $\quad \Psi_\pm \mapsto g^{-1}\Psi_\pm$

task:

solve parametric Riemann-Hilbert problem on S^2
$\Leftrightarrow$ split given f_{+-} in $\Psi_+^{-1}\Psi_-$ s.t. $\Psi_\pm$ extends to $\mathcal{U}_\pm$
$\Leftrightarrow$ find trivialization of hol. bundle over twistor space

then reconstruct A_μ via (1), gauge-trsfm. to real A_μ

Zakharov, Mikhailov 1978
Zakharov, Shabat 1979
Forgács, Horváth, Palla 1983

Dressing method

$\Psi(\lambda)$ has poles at $\lambda{=}\mu_k,\ k{=}1,\ldots,m \quad \rightarrow \quad$ ansatz!

simplify: complex gauge $A_{\bar{y}} = 0$ and put $A_{\bar{z}} = 0$

"solution" $A_y = \mathrm{e}^{-\Phi}\partial_y \mathrm{e}^{\Phi}\,, \quad A_z = \mathrm{e}^{-\Phi}\partial_z \mathrm{e}^{\Phi}$

$$\Rightarrow \quad \partial_{\bar{y}}\Big(\mathrm{e}^{-\Phi}\partial_y \mathrm{e}^{\Phi}\Big) + \partial_{\bar{z}}\Big(\mathrm{e}^{-\Phi}\partial_z \mathrm{e}^{\Phi}\Big) = 0 \qquad \text{Yang eq.}$$

linear system:

$$(\partial_{\bar{y}} - \lambda\partial_z)\Psi = \lambda A_z \Psi \qquad\qquad (\partial_{\bar{z}} + \lambda\partial_y)\Psi = -\lambda A_y \Psi$$

asymptotics: may choose
$\Psi(\lambda{\to}0) = 1$ and $\Psi(\lambda{\to}\infty) = \mathrm{e}^{-\Phi}$

reality property: $\mathrm{e}^{-\Phi} = \Psi(\lambda)\,\Psi^{\dagger}(-1/\bar{\lambda}) \qquad (3)$

reconstruct gauge potential:

$$A_z = \Psi(\lambda)\,(\partial_z - \tfrac{1}{\lambda}\partial_{\bar{y}})\,\Psi(\lambda)^{-1}$$
$$A_y = \Psi(\lambda)\,(\partial_y + \tfrac{1}{\lambda}\partial_{\bar{z}})\,\Psi(\lambda)^{-1} \qquad (4)$$

generate new solutions from old ones: $\Psi_{\text{new}} = \chi\,\Psi_{\text{old}}$
with ansatz

$$\chi = 1 - \frac{\lambda(1{+}\mu\bar{\mu})}{\lambda - \mu}\,P$$

task:
insert ansatz into (3), (4) and exploit pole structure!

Single-pole ansatz

simplest case (single pole, moduli μ):

$$\Psi_{\text{old}} = 1 \quad \Rightarrow \quad \Psi(\lambda) \;=\; \mathbf{1} - \frac{\lambda(1+\mu\bar{\mu})}{\lambda-\mu}\, P \qquad (5)$$

P to be determined, group-valued but λ-independent

eqs. (3) and (4): LHS are λ-independent $\longrightarrow$
RHS have zero residues at poles $\lambda{=}\mu$ and $\lambda{=}-1/\bar{\mu}$

$$(3) \;\Rightarrow\quad P^2 \;=\; P \;=\; \overline{P} \qquad \text{hermitian projector}$$

$$\Leftrightarrow\quad P \;=\; T\,\frac{1}{\overline{T}T}\,\overline{T} \qquad \text{with “column vector” } T$$

$$(4) \;\Rightarrow\quad P\,(\partial_{\bar{y}}{-}\mu\partial_z)\,P \;=\; 0 \;=\; (\mathbf{1}{-}P)\,(\partial_z{+}\bar{\mu}\partial_{\bar{y}})\,P$$

$$P\,(\partial_{\bar{z}}{+}\mu\partial_y)\,P \;=\; 0 \;=\; (\mathbf{1}{-}P)\,(\partial_y{-}\bar{\mu}\partial_{\bar{z}})\,P$$

$$\Leftrightarrow\quad (\mathbf{1}{-}P)\,L\,T \;=\; 0 \quad \text{with} \quad L = \begin{cases} \partial_z{+}\bar{\mu}\partial_{\bar{y}} \\ \partial_y{-}\bar{\mu}\partial_{\bar{z}} \end{cases}$$

$$\Leftrightarrow\quad L\,T \;=\; T\,\alpha \qquad \text{eigenvalue equation} \qquad (6)$$

preprotentials and gauge connection:

$$\mathrm{e}^{-\Phi} \;=\; \mathbf{1} - (1{+}\mu\bar{\mu})\,P \quad ,$$

$$A_z \;=\; -\frac{1{+}\mu\bar{\mu}}{\mu}\,\partial_{\bar{y}}P \quad \text{and} \quad A_y \;=\; \frac{1{+}\mu\bar{\mu}}{\mu}\,\partial_{\bar{z}}P$$

to be constructed from solution of eigenvalue eqn.

Noncommutativity

$$(f \cdot g)(x) = f(x)\, g(x) \quad \text{deformed to}$$

$$(f \star g)(x) = f(x)\ \exp\left\{\tfrac{\mathrm{i}}{2} \overleftarrow{\partial}_\mu\, \theta^{\mu\nu}\, \overrightarrow{\partial}_\nu\right\} g(x)$$

with $\quad \theta^{\mu\nu} = -\theta^{\nu\mu} = \text{constant}$

coordinate functions: $\quad x^\mu \star x^\nu - x^\nu \star x^\mu \;=\; \mathrm{i}\theta^{\mu\nu}$

standard form: $\quad \left(\theta^{\mu\nu}\right) = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta' \\ 0 & 0 & -\theta' & 0 \end{pmatrix}$

specialize to $\quad \theta' = -\theta \quad$ (anti-self-dual)

$\Rightarrow \quad y \star \bar{y} - \bar{y} \star y \;=\; 2\theta \;=\; z \star \bar{z} - \bar{z} \star z$

Moyal-Weyl map $\quad \left(f(y,\bar{y},z,\bar{z}), \star\right) \leftrightarrow \left(F(a,a^\dagger,b,b^\dagger), \cdot\right)$

with oscillators $\quad [a,a^\dagger] = 1 = [b,b^\dagger] \qquad$ put $2\theta{=}1$

$\rightarrow \quad F = \text{Weyl-order}\left[f(a,a^\dagger,b,b^\dagger)\right]$

$\leftarrow \quad f = F_\star(y,\bar{y},z,\bar{z}) \qquad$ "symbol of F"

$\partial_y f \;\leftrightarrow\; -[a^\dagger, F] \qquad \partial_{\bar{y}} f \;\leftrightarrow\; [a, F] \qquad$ etc.

$$\int \mathrm{d}^4x\ f(x) \;=\; (2\pi\theta)^2\, \mathrm{tr}_{\mathcal{H}} F$$

two-oscillator Fock space $\mathcal{H}$ spanned by

$|n_1,n_2\rangle = \frac{1}{\sqrt{n_1!\,n_2!}}\,(a^\dagger)^{n_1}\,(b^\dagger)^{n_2}\,|0,0\rangle \qquad n_1,n_2 \in \mathbb{N}_0$

Nekrasov, Schwarz 1998
Furuuchi 1999
Kraus, Shigemori 2001

D=4+0: Instantons in nc Yang-Mills $u(2)$

splitting method generalizes to noncommutative case!

deform simplest Atiyah-Ward ansatz:

$$f_{+-} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ \rho & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \rho & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix}$$

reality: $\rho^\dagger(x, -1/\bar{\lambda}) = \rho(x, \lambda)$
holomorphicity: $(\partial_{\bar{y}} - \lambda\partial_z)\rho = 0 = (\partial_{\bar{z}} + \lambda\partial_y)\rho$
Laurent decomposition $\rho(x,\lambda) = \rho_- + \rho_0 + \rho_+$

splitting: $f_{+-} = \Psi_+^{-1}\Psi_-$ with

$$\Psi_+ = \frac{1}{\sqrt{\rho_0}} \begin{pmatrix} 1 & \lambda^{-1}\rho_+ \\ \lambda & \rho_0+\rho_+ \end{pmatrix} \qquad \Psi_- = \frac{1}{\sqrt{\rho_0}} \begin{pmatrix} \rho_0+\rho_- & \lambda^{-1} \\ \lambda\rho_- & 1 \end{pmatrix}$$

this yields nc generalization of CFtHW ansatz (7)

$$A_\mu = \bar{\eta}^i_{\mu\nu}\frac{\sigma_i}{2\mathrm{i}}\left(\phi\,\partial_\nu\phi^{-1} - \phi^{-1}\partial_\nu\phi\right) + \frac{\mathbf{1}_2}{2}\left(\phi\,\partial_\mu\phi^{-1} + \phi^{-1}\partial_\mu\phi\right)$$

where $\phi = \sqrt{\rho_0}$ and $\bar{\eta}^i_{\mu\nu}$ is the asd 't Hooft tensor

but $F - *F \,\propto\, \phi^{-1}(\partial_y\partial_{\bar{y}}\phi^2 + \partial_z\partial_{\bar{z}}\phi^2)\,\phi^{-1} \stackrel{!}{=} 0$

dressing method also works in noncommutative case
but is easy only for self-dual θ
yields the nc BPST instanton:

$$A_y = \begin{pmatrix} -\frac{\bar{y}}{2\theta}\left(\sqrt{\frac{r^2+\Lambda^2-2\theta}{r^2+\Lambda^2}}-1\right) & 0 \\ -\bar{z}\frac{1}{\sqrt{r^2+\Lambda^2}\sqrt{r^2+\Lambda^2-2\theta}} & -\frac{\bar{y}}{2\theta}\left(\sqrt{\frac{r^2+\Lambda^2+4\theta}{r^2+\Lambda^2+2\theta}}-1\right) \end{pmatrix}$$

$$A_z = \begin{pmatrix} \left(\sqrt{\frac{r^2+\Lambda^2-2\theta}{r^2+\Lambda^2}}-1\right)\frac{\bar{z}}{2\theta} & -\frac{1}{\sqrt{r^2+\Lambda^2}\sqrt{r^2+\Lambda^2-2\theta}}\bar{y} \\ 0 & \left(\sqrt{\frac{r^2+\Lambda^2+4\theta}{r^2+\Lambda^2+2\theta}}-1\right)\frac{\bar{z}}{2\theta} \end{pmatrix}$$

Horváth, O.L., Wolf 2002

(by dressing & splitting)

Gross, Nekrasov 2000

D=3+0: Monopoles in nc Yang-Mills-Higgs

dimensional reduction: $\partial_4 = 0$ and $A_4 = \varphi$, $\theta' = 0$

$\Leftrightarrow$ $\partial_z - \partial_{\bar{z}} = 0$ and $D\varphi = *F$ Bogomolny

splitting method: transition function simplifies

$f_{+-} = f_{+-}(w, \lambda)$ with $w(\lambda) = 2x^3 + \lambda\bar{y} - \lambda^{-1}y$

deform simplest $su(2)$ BPS monopole transition function:

$$f_{\text{BPS}}(\lambda) = \begin{pmatrix} (e^w - e^{-w})w^{-1} & -\lambda e^{-w} \\ \lambda^{-1}e^{-w} & w\, e^{-w} \end{pmatrix} = f^\dagger_{\text{BPS}}(-1/\bar{\lambda})$$

noncommutativity:

$$w = 2x^3 + \lambda a^\dagger - \lambda^{-1}a = \tfrac{1}{\lambda}(a\xi^{-1} + \lambda\xi)(\lambda\xi^{-1}a^\dagger - \xi)$$

one can split $\mathrm{e}^{-\theta} f_{\text{BPS}} = \tilde{\Psi}_+^{-1}\tilde{\Psi}_-$, $w = u - v$:

$$\tilde{\Psi}_+ = \Psi_+ \cdot \begin{pmatrix} \mathrm{e}^u & 0 \\ -\lambda \mathrm{e}^{-u}w & \mathrm{e}^{-u} \end{pmatrix} \qquad \tilde{\Psi}_- = \Psi_- \cdot \begin{pmatrix} \mathrm{e}^{-v} & 0 \\ 0 & \mathrm{e}^{v} \end{pmatrix}$$

w/ $u = z + \lambda a^\dagger$, $v = \lambda^{-1}a - \bar{z}$, $\Psi_\pm(\rho)$ from (7)

$$\begin{aligned} \rho &= \mathrm{e}^u\, w^{-1}\, \mathrm{e}^u - \mathrm{e}^v\, w^{-1}\, \mathrm{e}^v \qquad \text{Weyl ordered} \\ &= \mathrm{e}^{-2\mathrm{i}x^4} \int_{-1}^{+1} \mathrm{d}t\; \mathrm{e}^{2tx^3}\, \mathrm{e}^{\lambda(1+t)a^\dagger + \lambda^{-1}(1-t)a} \\ &= \rho_- + \rho_0 + \rho_+ \qquad \text{Laurent decomposition} \end{aligned}$$

$$\rho_0 = \sinh(2R)/R \qquad \text{with} \qquad R = x^3 + \xi\xi$$

O.L. & A.D. Popov 2003

also: Dirac U(1) & Wu-Yang U(2) NC monopoles ✓

solution:

$$A_i \;=\; \varepsilon_{ijk}\,\frac{\sigma_k}{2\mathrm{i}}\Big(\rho_0^{+\frac12}\partial_j\,\rho_0^{-\frac12}-\rho_0^{-\frac12}\partial_j\,\rho_0^{+\frac12}\Big) + \frac{\mathbf{1}_2}{2}\Big(\rho_0^{-\frac12}\partial_i\,\rho_0^{+\frac12}+\rho_0^{+\frac12}\partial_i\,\rho_0^{-\frac12}\Big) + \;\sigma_i$$

$$\varphi \;\equiv\; A_4 \;=\; \frac{\sigma_i}{2\mathrm{i}}\Big(\rho_0^{+\frac12}\partial_i\,\rho_0^{-\frac12}-\rho_0^{-\frac12}\partial_i\,\rho_0^{+\frac12}\Big)$$

is not real (except for φ):
need to gauge transform to a *real* solution

$$A_i^g = g^{-1}(A_i+\partial_i)\,g \qquad \text{and} \qquad \varphi^g = g^{-1}\varphi\,g$$

via a *nonunitary* gauge transformation from

$$g^2 = \tilde\Psi_+(\lambda)\,\tilde\Psi_-^\dagger(-1/\bar\lambda)\,\Big|_{\lambda=0}$$

matrix g^2 is complicated and involves $\rho_{\pm1}$ as well

commutative limit: $\quad g^2 = \mathrm{e}^{2x^i\sigma_i} \quad\rightarrow\quad g=\mathrm{e}^{x^i\sigma_i}$

$$A_i \;=\; \varepsilon_{ijk}\,\frac{\sigma_k}{2\mathrm{i}}\,\frac{x_j}{r}\Big(\frac1r - 2\coth(2r)\Big) + \;\sigma_i$$

$$\Rightarrow \quad A_i^g \;=\; \varepsilon_{ijk}\,\frac{\sigma_k}{2\mathrm{i}}\,\frac{x_j}{r}\Big(\frac1r - \frac{2}{\sinh(2r)}\Big)$$

$$\varphi^g \;=\; \frac{\sigma_i}{2\mathrm{i}}\,\frac{x_i}{r}\Big(\frac1r - 2\coth(2r)\Big)$$

BPS ✓

D=2+1: Solitons in nc Yang-Mills-Higgs

reduction $\partial_4 = 0$ and Wick rotation $x^1 = \mathrm{i}t$
combine $z = x^3 + \mathrm{i}x^2 \to a$ s.t. $[a, a^\dagger] = 1$
obtain WZW-modified integrable sigma model Ward 1988

the dressing method goes noncommutative:
start from $\Psi_{\text{old}} = 1$ and obtain [see (5)]

$$\Psi(a, a^\dagger, t, \lambda) = 1 - \frac{\lambda(1+\mu\bar{\mu})}{\lambda - \mu} P(a, a^\dagger, t)$$

absence of λ-poles in (3) yields $P^2 = P = P^\dagger$
parametrize projector $P = T(T^\dagger T)^{-1} T^\dagger$
$u(n)$, rank r: T is $n\times r$ and P is $n\times n$ matrix

absence of λ-poles in (4) yields $(1-P)\, c\, P = 0$
$c = (\cosh\tau)\, a - (\mathrm{e}^{\mathrm{i}\vartheta} \sinh\tau)\, a^\dagger - \beta\, t = U(t)\, a\, U^\dagger(t)$
moving-frame coordinate via $ISU(1,1)$ transform
$U(t) = \mathrm{e}^{\alpha\, a^{\dagger 2} - \bar{\alpha}\, a^2}\, \mathrm{e}^{(\beta\, a^\dagger - \bar{\beta}\, a)t}$
produces "squeezed states" $|n\rangle_t = U(t)|n\rangle$

solution $\mathrm{e}^{-\Phi} = \Psi|_\infty = 1 - (1+\bar{\mu}\mu) P(t)$

describes soliton moving with constant velocity $\vec{v}(\mu)$
and energy $E(\vec{v}) = \frac{\pi}{2} \mathrm{tr}[\nabla\Phi^\dagger \cdot \nabla\Phi] = f(\vec{v}) E(\vec{0})$

static solutions: $c = a \quad \longleftrightarrow \quad \mu = \mathrm{i}$

equation of motion: $(1-P)\,a\,T \stackrel{!}{=} 0$

since $(1-P)T = 0$ it suffices to have

$$a\,T \;=\; T\,\alpha \qquad \text{for any } r\times r \text{ matrix } \alpha$$

nonabelian soliton: $\alpha = \mathbf{1}\,a \quad \Rightarrow \quad [a,T] \stackrel{!}{=} 0$

solution: $T =$ matrix of rational functions of a

example (n=2, r=1): $E = 8\pi \qquad N := a^\dagger a$

$$T = \begin{pmatrix} \gamma \\ a \end{pmatrix} \quad \Rightarrow \quad P = \begin{pmatrix} \frac{\gamma\bar\gamma}{N+\gamma\bar\gamma} & \frac{\gamma}{N+\gamma\bar\gamma}\,a^\dagger \\ a\,\frac{\bar\gamma}{N+\gamma\bar\gamma} & a\,\frac{1}{N+\gamma\bar\gamma}\,a^\dagger \end{pmatrix}$$

abelian soliton: $T = \big(|z^1\rangle, |z^2\rangle, \ldots, |z^r\rangle\big)$ states

take $\alpha = \mathrm{diag}(z^1, z^2, \ldots, z^r)$ with $z^i \in \mathbb{C}$

then $a\,|z^i\rangle = z^i\,|z^i\rangle \quad \Rightarrow \quad |z^i\rangle = \mathrm{e}^{z^i a^\dagger - \bar z^i a}\,|0\rangle$

rank-r projector $P \;=\; \sum_{i,j=1}^r |z^i\rangle(\langle z^\cdot|z^\cdot\rangle)^{-1}\langle z^j|$

$E = 8\pi r \quad \to \quad r$ lumps of energy in positions z^i

although $Q_{\mathrm{top}} = r$ these are no true multi-solitons since overall “boost” $a \to c$ gives *no relative velocity* multi-solitons by iterated dressing (5):

$$\Psi = \prod_{j=1}^{m}\Big(1+\frac{\lambda\,\alpha_j}{\lambda-\mu_j}P_j\Big) = 1+\lambda\sum_{q=1}^{m}\frac{R_q}{\lambda-\mu_q}$$

if all μ_j are mutually different

take $R_q = -\mu_q\sum_{p=1}^{m} T_p\,\Gamma^{pq}\,T_q$

no poles in (3) $\rightarrow$ $\Gamma^{pq} = \Gamma^{pq}(T_k,\mu_j)$, $e^{-\Phi}$

no poles in (4) $\rightarrow$ $c_k\,T_k \stackrel{!}{=} T_k\,\alpha_k$
where $c_k = U_k(t)\,a\,U_k^\dagger(t)$ given by μ_k

choice of matrix $T_k(c_k)$ creates explicit solutions

m lumps of energy moving with velocities $\vec{v}_k(\mu_k)$
and carrying energies of $E_k = 8\pi\,f(\vec{v}_k)\,r_k$ each

O.L. & A.D. Popov 2001

Scattering?

analysis of asymptotic behavior $\rightarrow$ no scattering!

allow for *coincident* poles μ_k $\rightarrow$ scattering:

nonabelian multi-solitons may scatter at angles $\vartheta = \pi/\ell$
abelian configurations form bound states

O.L. & A.D. Popov 2001

other possibilities:
wave fronts, soliton-antisoliton configurations

Bieling 2001
Wolf 2002
Ihl, Uhlmann 2002

D=2+2 and D=9+1: String field theory

Berkovits' WZW-like (super)string field action:

Berkovits 1995

$$S = \frac{1}{2}\int \Big[(\mathrm{e}^{-\Phi}G^{+}\mathrm{e}^{\Phi})(\mathrm{e}^{-\Phi}\tilde{G}^{+}\mathrm{e}^{\Phi}) - \int_0^1 \mathrm{d}t\,(\mathrm{e}^{-t\Phi}\partial_t\mathrm{e}^{t\Phi})\Big\{\mathrm{e}^{-t\Phi}G^{+}\mathrm{e}^{t\Phi}\,,\mathrm{e}^{-t\Phi}\tilde{G}^{+}\mathrm{e}^{t\Phi}\Big\}\Big]$$

where $\Phi = \Phi[x^\mu(\sigma),\psi^\mu(\sigma)]$ is a *string field* carrying $u(n)$ Chan-Paton labels; $D = 4$ or 10

nilpotent currents $G^{+}{=}Q$ and $\tilde{G}^{+}{=}\eta_0$ are part of twisted $\mathcal{N}{=}4$ superconformal constraint algebra $\{T, G^{+}, \tilde{G}^{+}, G^{-}, \tilde{G}^{-}, J,\, J^{++}, J^{--}\}$

string fields are multiplied via Witten's star product

eq. of motion: $\eta_0(\mathrm{e}^{-\Phi}Q\mathrm{e}^{\Phi}) = 0$ "Yang"

$A := \mathrm{e}^{-\Phi}Q\mathrm{e}^{\Phi} \quad\Rightarrow\quad \eta_0 A = 0 = QA + A^2$

$\Rightarrow\quad (\eta_0 + \lambda Q + \lambda A)^2 = 0$ "zero curvature"

linear problem $(\eta_0 + \lambda Q + \lambda A)\Psi[x,\psi,\lambda] = 0$

impose asymptotics and reality condition on Ψ

string field solution $\mathrm{e}^{-\Phi}[x,\psi] = \Psi[x,\psi,\lambda{=}\infty]$

apply splitting or dressing method to string fields

O.L. & A.D. Popov 2000