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Lectures on Exact Solutions of Landau 1+1 Hydrodynamics

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1. Introduction
2. Exact analytical solution displayed
   Analogous behavior in higher dimensions
3. Riemann solution
4. Khalatnikov solution
5. Comparison of Landau hydrodynamics with boost-invariant Hwa-Bjorken hydrodynamics

Exact 1+1 dimensional hydrodynamical solutions

1. Help guide our intuition
2. Summarize important features
3. Point out essential elements
4. Can be used to investigate whether Hwa-Bjorken hydrodynamics is a limiting case of Landau hydrodynamics

Contents of Landau Hydrodynamics in z, t

1. A uniform energy density of length $z = 2l$ initially at rest
2. An equation of state $p = \epsilon/3$

Boundary conditions

1. The flow velocity is zero at the center of he slab,
   $v = 0$ at $z = 0$.
2. The Khalatnikov solution need to match to the Riemann solution at the edge of the system.

What is the Riemann solution ?

1. Riemann solution corresponds to the situation when the energy density and the velocity can be expressed as a function of each other in which x and t do not explicitly appear.
2. Because of this mutual dependence, the two independent degrees of freedom is reduced to a single degree of freedom.
3. The Riemann solution gives the superposition of (i) the propagation of the disturbance with the speed of sound and (ii) the propagation of the fluid element itself with the flow velocity

What is the Khalatnikov solution ?

1. Representing density by ζ and velocity by y,
$$\zeta = \frac{1}{4}\ln\left(\frac{\epsilon}{\epsilon_0}\right) = \ln\left(\frac{T}{T_0}\right)$$
$$y = \tanh^{-1} v$$
It is a general explicit solution of $(x = z - l, t)$ as a function of (ζ, y).
2. It is necessary to invert $(x(\zeta, y), t(\zeta, y))$ to yield $(\zeta(x,t), y(x,t))$ for the usual hydrodynamical equation. Inversion can be done only numerically.
3. The Khalatnikov solution cannot be applied to $t/l < \sqrt{3}$.
4. For $t/l < \sqrt{3}$, only the Riemann solution applies.

Explicit form of the solution

Definition of various quantities

$$\zeta = \frac{1}{4}\ln(\epsilon/\epsilon_0) = \ln(T/T_0),$$
$$\epsilon/\epsilon_0 = (T/T_0)^4 = e^{4\zeta},$$
$$s/s_0 = (\epsilon/\epsilon_0)^{3/4} = (T/T_0)^3 = e^{3\zeta}$$

$$y = \tanh^{-1} v,$$
$$v = \tanh y.$$

(i) Riemann solution

$$y = \pm\frac{\zeta}{c_s}.$$

$$\frac{x}{t} = \frac{\tanh(-\zeta/c_s) - c_s}{1 - \tanh(-\zeta/c_s)\ c_s}.$$

(ii) Khalatnikov solution

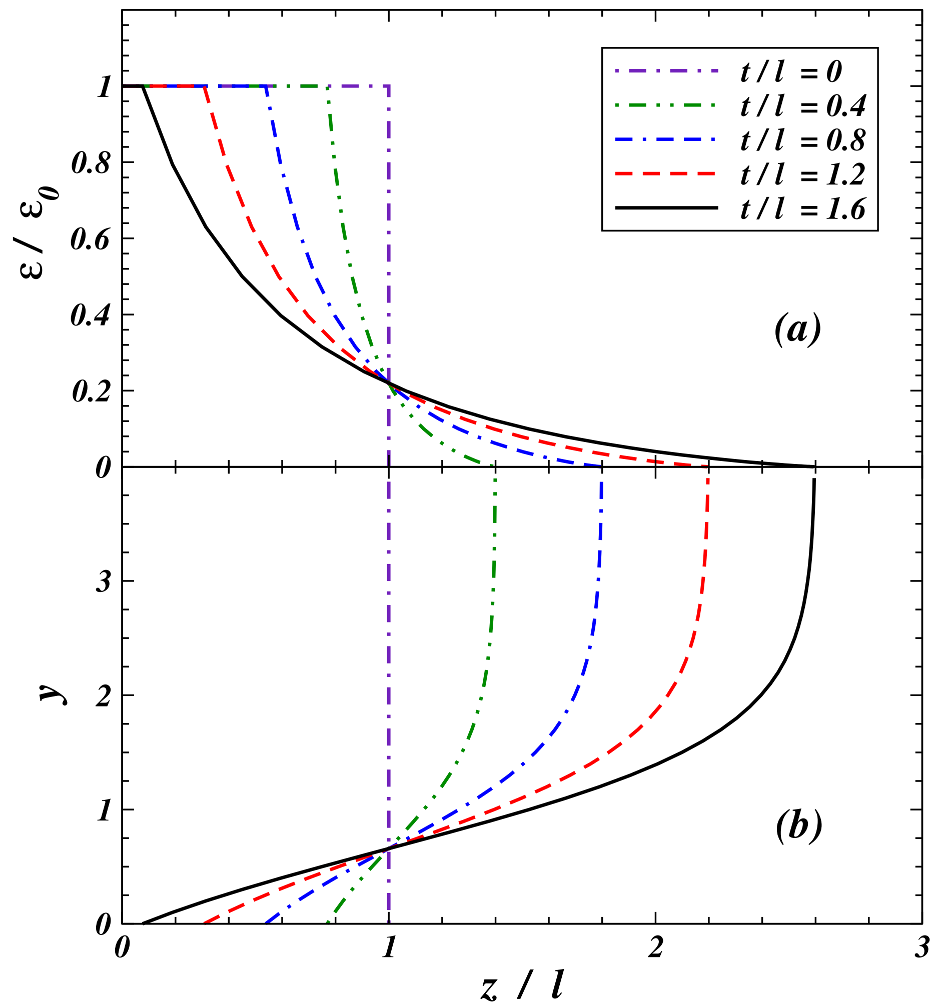
$$t(\zeta, y) = e^{-\zeta}\left(\frac{\partial\chi}{\partial\zeta}\cosh y - \frac{\partial\chi}{\partial y}\sinh y\right),$$
$$x(\zeta, y) = e^{-\zeta}\left(\frac{\partial\chi}{\partial\zeta}\sinh y - \frac{\partial\chi}{\partial y}\cosh y\right).$$

$$\chi(\zeta, y) = -l\sqrt{3}e^{\zeta}\int_{y/\sqrt{3}}^{-\zeta} e^{2\zeta'} I_0\left[\sqrt{\zeta'^2 - \frac{1}{3}y^2}\right] d\zeta',$$

where $I_0(z)$ is the Bessel function of purely imaginary argument,

$$I_\alpha(z) = e^{-i\alpha\pi/2}J_\alpha(iz)$$

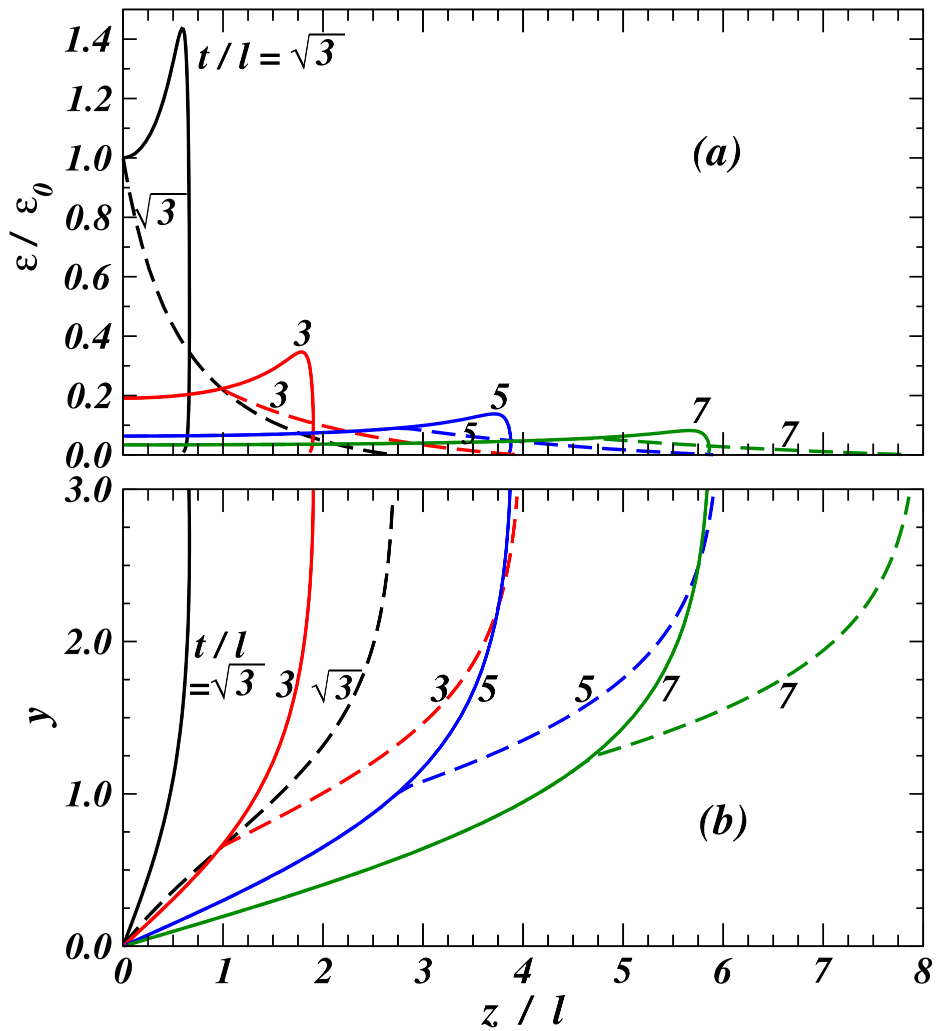
Riemann solution only for $t/l < \sqrt{3}$,



Wong,Sen,Gerhard,Torrieri,Read,PRC90,064907('14)

Solid lines, Khalatinkov solution.

Dashed line, Riemann solution.



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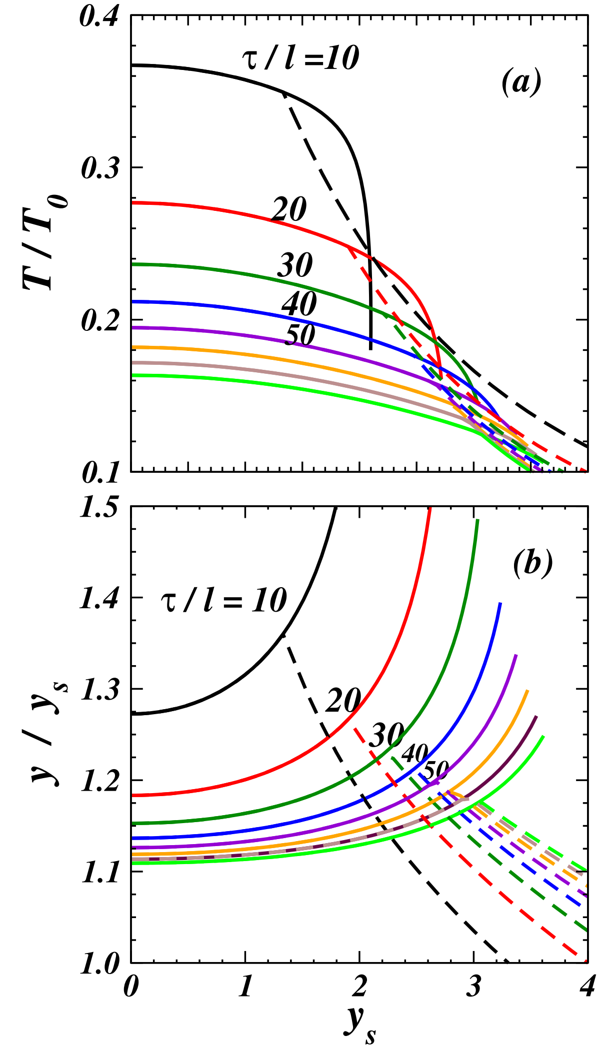

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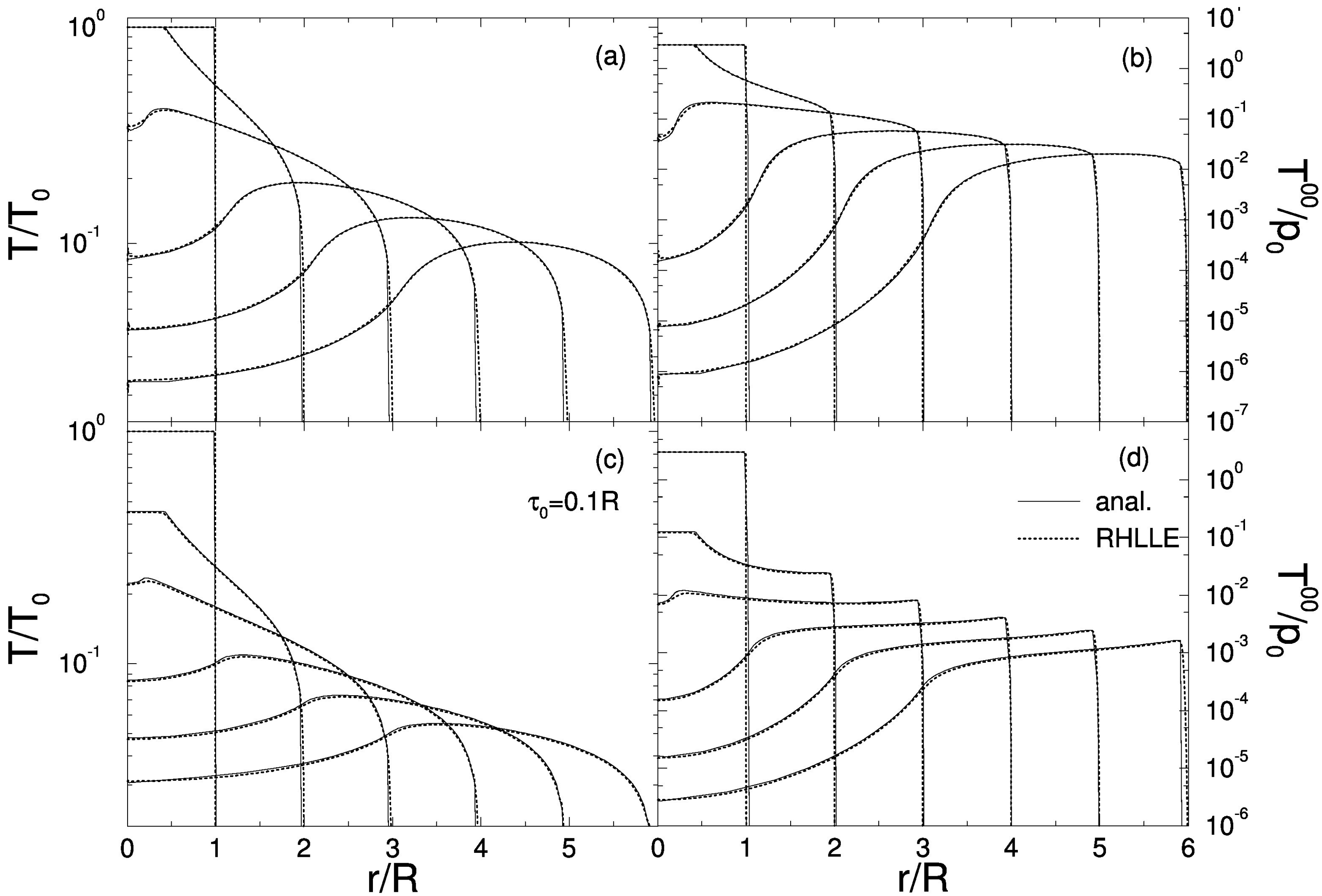
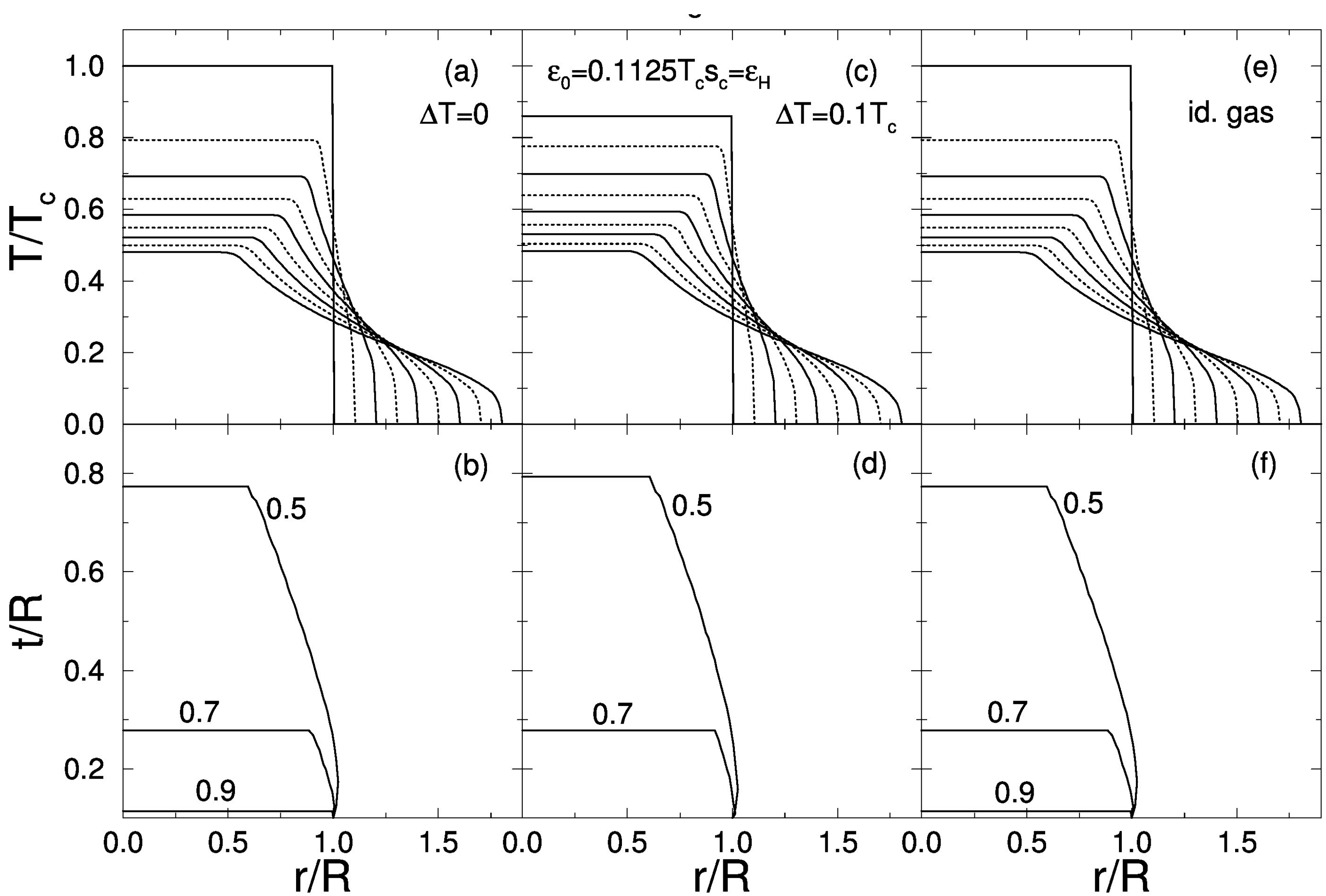


Rischke and Gyulassy,NPA597,701(1996)



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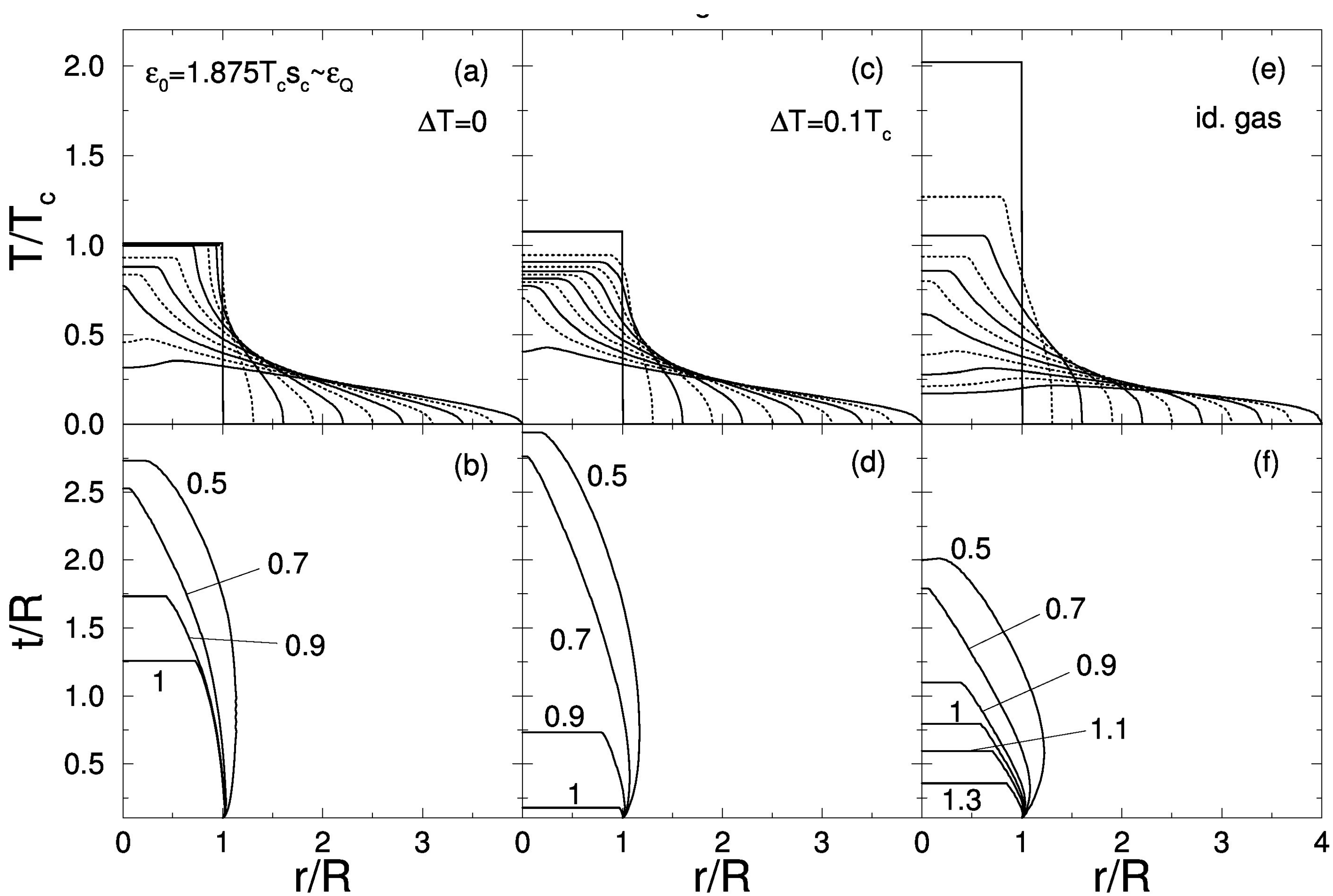


Rischke and Gyulassy,NPA597,701(1996)

Riemann solution

Method of finding the Rieman solution is from Landau and Lifshitz page 503 *Fluid Mechnics*

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{01}}{\partial x} = 0$$
$$\frac{\partial T^{10}}{\partial t} + \frac{\partial T^{11}}{\partial x} = 0$$

Because T^{ik} depend on each other, we have

$$\frac{\partial T^{00}}{\partial T^{01}} + \frac{\partial t}{\partial x} = 0$$
$$\frac{\partial T^{10}}{\partial T^{11}} + \frac{\partial t}{\partial x} = 0$$

$$\frac{\partial T^{00}}{\partial T^{01}} = \frac{\partial T^{10}}{\partial T^{11}}$$
$$dT^{00}\ dT^{11} = dT^{01}\ dT^{10}$$

$$
\begin{aligned}
T^{00} &= (\epsilon + p)u^0u^0 - pg^{00} = (\epsilon + p)u^0u^0 - p \\
&= \epsilon u^0u^0 + p(u^0u^0 - 1) = \epsilon u^0u^0 + pu^1u^1 \\
T^{01} = T^{10} &= (\epsilon + p)u^0u^1 \\
T^{11} &= (\epsilon + p)u^1u^1 - pg^{11} = (\epsilon + p)u^1u^1 + p \\
&= \epsilon u^1u^1 + p(u^1u^1 + 1) = \epsilon u^1u^1 + pu^0u^0
\end{aligned}
$$

We use

$$
p/\epsilon = c_s^2, \qquad u^0 = \cosh y, \qquad u^1 = \sinh y,
$$

we can rewrite $T^{\mu\nu}$ and $dT^{\mu\nu}$ as

$$
\begin{aligned}
T^{00} &= \epsilon(u^0u^0 + c_s^2u^1u^1) = \epsilon(\cosh^2 y + c_s^2 \sinh^2 y) \\
dT^{00} &= d\epsilon(\cosh^2 y + c_s^2 \sinh^2 y) + 2\epsilon \cosh y \sinh y(1 + c_s^2)dy \\
T^{01} &= (\epsilon + p)u^0u^1 = d\epsilon(1 + c_s^2) \cosh y \sinh y \\
dT^{01} &= d\epsilon(1 + c_s^2) \cosh y \sinh y + \epsilon(1 + c_s^2)[\cosh^2 y + \sinh^2 y]dy \\
T^{11} &= \epsilon u^1u^1 + pu^0u^0 = \epsilon(\sinh^2 y + c_s^2 \cosh^2 y) \\
dT^{11} &= d\epsilon(\sinh^2 y + c_s^2 \cosh^2 y) + 2\epsilon \cosh y \sinh y(1 + c_s^2)dy
\end{aligned}
$$

Cosntruct $dT^{00}\ dT^{11} - dT^{01}\ dT^{10}$, the $d\epsilon d\epsilon$ term is

$$
\begin{aligned}
& d\epsilon d\epsilon[(\cosh^2 y + c_s^2 \sinh^2 y)(\sinh^2 y + c_s^2 \cosh^2 y) \\
& \qquad -(1+c_s^2)\cosh y \sinh y \times (1+c_s^2)\cosh y \sinh y] \\
& = d\epsilon d\epsilon[(1+c_s^4)\cosh^2 y \sinh^2 y + c_s^2(\sinh^4 y + \cosh^4 y) \\
& \qquad -(1+2c_s^2+c_s^4)\cosh^2 y \sinh^2 y] \\
& = d\epsilon d\epsilon[c_s^2(\sinh^4 y - 2\cosh^2 y \sinh^2 y + \cosh^4 y)] \\
& = d\epsilon d\epsilon c_s^2(\cosh^2 y - \sinh^2 y)^2 = c_s^2 (d\epsilon)^2
\end{aligned}
$$

Cosntruct $dT^{00}\ dT^{11} - dT^{01}\ dT^{10}$, the $dydy$ term is

$$
\begin{aligned}
& dydy[2\epsilon \cosh y \sinh y(1+c_s^2)2\epsilon \cosh y \sinh y(1+c_s^2) \\
& \qquad -\epsilon^2(1+c_s^2)^2(\cosh^2 y + \sinh^2 y)^2] \\
& = dydy\epsilon^2(1+c_s^2)^2[4\cosh^2 y \sinh^2 y - (\cosh^4 y + 2\cosh^2 y \sinh^2 y + \sinh^4)] \\
& = dydy\epsilon^2(1+c_s^2)^2[-(\cosh^4 y - 2\cosh^2 y \sinh^2 y + \sinh^4)] \\
& = -dydy\epsilon^2(1+c_s^2)^2(\cosh^2 y - \cosh^2)^2 \\
& = -dydy\epsilon^2(1+c_s^2)^2
\end{aligned}
$$

Cosntruct $dT^{00}\ dT^{11} - dT^{01}\ dT^{10}$, the $d\epsilon dy$ term is

$$\begin{aligned} & d\epsilon dy\{[(\cosh^2 y + c_s^2 \sinh^2 y)2\epsilon \cosh y \sinh y(1 + c_s^2) \\ & \qquad +2\epsilon \cosh y \sinh y(1 + c_s^2)(\sinh^2 y + c_s^2 \cosh^2 y)] \\ & \quad - 2(1 + c_s^2) \cosh y \sinh y\epsilon(1 + c_s^2)[\cosh^2 y + \sinh^2 y]\} \\ & = d\epsilon dy(1 + c_s^2)2 \cosh y \sinh y\epsilon\{(\cosh^2 y + c_s^2 \sinh^2 y) \\ & \qquad +(\sinh^2 y + c_s^2 \cosh^2 y) - (1 + c_s^2)[\cosh^2 y + \sinh^2 y]\} \\ & = d\epsilon dy(1 + c_s^2)2 \cosh y \sinh y \quad \epsilon\{(1 + c_s^2) \cosh^2 y \\ & \qquad +(1 + c_s^2) \sinh^2 y - (1 + c_s^2)[\cosh^2 y + \sinh^2 y]\} \\ & = 0 \end{aligned}$$

Finally, we get

$$dT^{00}\ dT^{11} - dT^{01}\ dT^{10} = c_s^2(d\epsilon)^2 - \epsilon^2(1 + c_s^2)^2(dy)^2 = 0$$

We get

$$c_s^2(\frac{d\epsilon}{\epsilon})^2 = (1 + c_s^2)^2(dy)^2$$

$$dy = \pm\frac{c_s d\epsilon}{(1 + c_s^2)\epsilon} = \pm\frac{c_s d\epsilon}{\epsilon + p}$$

which is equation (2) of Landau and Lifshitz, page 503. We have now

$$dy = \pm\frac{c_s d\epsilon}{(1+c_s^2)\epsilon} = \pm\frac{1}{c_s}\frac{c_s^2 d\epsilon}{(1+c_s^2)\epsilon} = \pm\frac{1}{c_s}\frac{c_s^2 d\ln(\epsilon/\epsilon_0)}{(1+c_s^2)}$$
$$= \pm\frac{1}{c_s}d\{\ln[(\epsilon/\epsilon_0)^{\frac{c_s^2}{(1+c_s^2)}}]\} = \pm\frac{d\zeta}{c_s}$$

The Rieman solution is then

$$\boxed{y = \pm\frac{\zeta}{c_s}}$$

where we introduce

$$\zeta = \{\ln[(\epsilon/\epsilon_0)^{\frac{c_s^2}{(1+c_s^2)}}]\} = \frac{c_s^2}{(1+c_s^2)}\ln[(\epsilon/\epsilon_0)]$$

$$\frac{(1+c_s^2)}{c_s^2}\zeta = \ln[(\epsilon/\epsilon_0)]$$

$$\frac{\epsilon}{\epsilon_0} = e^{\frac{(1+c_s^2)}{c_s^2}\zeta}$$

we can show $(\frac{\epsilon}{\epsilon_0})^{c_s^2/(1+c_s^2)} = \frac{T}{T_0}$

$$\zeta = \ln(\frac{T}{T_0})$$

The entropy conservation equations, along u^i

We take projection along u^i the basic hydro equation:

$$u^i \frac{\partial T^{ik}}{\partial x^k} = 0$$

Upon multiply by u^i and summed, we get

$$\sum_i u_i \frac{\partial T^{ik}}{\partial x^k} = 0 \quad \text{we shall use summed convention}$$

$$u_i \frac{\partial T^{ik}}{\partial x^k} = 0$$

$$u_i \frac{\partial\{(\epsilon + p)u^i u^k - g^{ik} p\}}{\partial x^k} = 0,$$

$$u_i[(\epsilon + p)u^k \frac{\partial u^i}{\partial x^k} + u_i u^i \frac{\partial(\epsilon + p)u^k}{\partial x^k}] - u_i g^{ik} \frac{\partial p}{\partial x^k} = 0,$$

$$\frac{\partial(\epsilon + p)u^k}{\partial x^k}] - u^k \frac{\partial p}{\partial x^k} = 0,$$

$$u^k \frac{\partial(\epsilon + p)}{\partial x^k} + (\epsilon + p)\frac{\partial u^k}{\partial x^k} - u^k \frac{\partial p}{\partial x^k} = 0,$$

$$(\epsilon + p) = Ts, \qquad d(\epsilon + p) = Tds \quad s \text{ is the entropy density},$$

$$u^k T \frac{\partial s}{\partial x^k} + Ts \frac{\partial u^k}{\partial x^k} = 0,$$

$$T \frac{\partial(su^k)}{\partial x^k} = 0, \quad \text{conservation of entropy} \tag{1}$$

$$\boxed{\frac{\partial(su^k)}{\partial x^k} = 0, \quad \text{conservation of entropy}} \tag{2}$$

The equation perpendicular to u^i

We take projection perpendicular to u^i the basic hydro equation:

$$\frac{\partial T^{ik}}{\partial x^k} - u_i u_k \frac{\partial T^{kl}}{\partial x^l} = 0 \tag{3}$$

Upon multiply by u^i and summed, we get

$$\begin{aligned} u^i \frac{\partial T^{ik}}{\partial x^k} - u^i u_i u_k \frac{\partial T^{kl}}{\partial x^l} &= 0 \\ u^i \frac{\partial T^{ik}}{\partial x^k} - u_k \frac{\partial T^{kl}}{\partial x^l} &= 0 \end{aligned} \tag{4}$$

Let us expand out the terms:

$$\begin{aligned} &\frac{\partial T^{ik}}{\partial x^k} - u^i u_k \frac{\partial T^{kl}}{\partial x^l} = 0 \\ &= \frac{\partial}{\partial x^k}[(\epsilon + p)u^i u^k - g^{ik}p] - u^i u_k \frac{\partial}{\partial x^l}[(\epsilon + p)u^k u^l - g^{kl}p] \\ &= \frac{\partial}{\partial x^k}[(\epsilon + p)u^i u^k - g^{ik}p] - u^i \frac{\partial}{\partial x^l}(\epsilon + p)u^k u^l u_k + u^i(\epsilon + p)u^k u^l \frac{\partial}{\partial x^l} u_k + g^{kl} u_i u_k \frac{\partial}{\partial x^l} p \\ &= [(\epsilon + p)u^k \frac{\partial}{\partial x^k} u^i + u^i \frac{\partial}{\partial x^k}(\epsilon + p)u^k - g^{ik} \frac{\partial}{\partial x^k} p] - u^i \frac{\partial}{\partial x^l}(\epsilon + p)u^k u^l u_k + u^i(\epsilon + p)u^k u^l \frac{\partial}{\partial x^l} u_k + g^{kl} u^i u_k \frac{\partial}{\partial x^l} p \\ &= [(\epsilon + p)u^k \frac{\partial}{\partial x^k} u^i - g^{ik} \frac{\partial}{\partial x^k} p] + g^{kl} u^i u_k \frac{\partial}{\partial x^l} p = 0 \end{aligned} \tag{5}$$

Now,

$$dp = s dT, \quad (\epsilon + p) = Ts$$

So, we have

$$
\begin{aligned}
&[(\epsilon+p)u^k\frac{\partial}{\partial x^k}u^i - g^{ik}\frac{\partial}{\partial x^k}p] + g^{kl}u^iu_k\frac{\partial}{\partial x^l}p = 0\\
&[Tsu^k\frac{\partial}{\partial x^k}u^i - g^{ik}s\frac{\partial}{\partial x^k}T] + g^{kl}su^iu_k\frac{\partial}{\partial x^l}T = 0\\
&Tu^k\frac{\partial}{\partial x^k}u_i + u^iu^l\frac{\partial}{\partial x^l}T - g^{ik}\frac{\partial}{\partial x^k}T = 0\\
&u^k\frac{\partial u^iT}{\partial x^k} - g^{ik}\partial_kT = 0\\
&u^k\frac{\partial u^iT}{\partial x^k} - \partial^iT = 0, \quad \text{as in Landau's (4.5)}
\end{aligned} \tag{6}
$$

For $i=1$, we have

$$
\begin{aligned}
&u^k\frac{\partial u^1T}{\partial x^k} - \partial^1T = 0,\\
&u^0\frac{\partial u^1T}{\partial x^0} + u^1\frac{\partial u^1T}{\partial x^1} + \partial_1T = 0, \quad \text{using } \ \partial^1 = g^{11}\partial_1 = -\partial_1\\
&u^0\frac{\partial u^1T}{\partial x^0} + \frac{\partial(u^1Tu_1)}{\partial x^1} - u^1T\frac{\partial u_1}{\partial x^1} + \partial_1T = 0, \quad \text{using } \ \partial^1 = g^{11}\partial_1 = -\partial_1\\
&u^0\frac{\partial u^1T}{\partial x^0} + \frac{\partial[T(u^0u^0-1)]}{\partial x^1} - u^1T\frac{\partial u_1}{\partial x^1} + \partial_1T = 0,\\
&u^0\frac{\partial u^1T}{\partial x^0} + u^0\frac{\partial Tu^0}{\partial x^1} + Tu^0\frac{\partial u^0}{\partial x^1} - u^1T\frac{\partial u_1}{\partial x^1} = 0,\\
&u^0\{\frac{\partial u^1T}{\partial x^0} + \frac{\partial Tu^0}{\partial x^1}\} + \frac{T}{2}\frac{\partial(u^0)^2-(u^1)^2)}{\partial x^1} = 0
\end{aligned}
$$

$$\boxed{\frac{\partial u^1 T}{\partial x^0} + \frac{\partial T u^0}{\partial x^1} = 0}$$

the solution is

$$\begin{aligned} Tu^1 &= T_0 \frac{\partial \phi}{\partial x^1} \\ Tu^0 &= -T_0 \frac{\partial \phi}{\partial x^0} \\ d\phi &= -\frac{T}{T_0} u^0 dx^0 + \frac{T}{T_0} u^1 dx^1 \end{aligned} \tag{7}$$

Introduce χ, we have

$$\chi = \phi + Tu^0 x^0 - Tu^1 x^1; \quad u^0 = \cosh y, \quad u^1 = \sinh y \tag{8}$$

Then

$$\begin{aligned} d\chi &= tu^0 d(T/T_0) + t(T/T_0)\cosh y dy - xu^1 d(T/T_0) - x(T/T_0)\cosh y dy \\ &= (tu^0 - xu^1) d(T/T_0) + (T/T_0)(t \sinh y - x \cosh y) dy \end{aligned}$$

$$\boxed{d\chi = (t \cosh - x \sinh) d(T/T_0) + (T/T_0)(t \sinh y - x \cosh y) dy}$$

Therefore,

$$\begin{aligned} \frac{\partial \chi}{\partial (T/T_0)} &= t \cosh y - x \sinh y) \\ \frac{\partial \chi}{\partial y} &= (T/T_0)(t \sinh y - x \cosh y) \end{aligned} \tag{9}$$

The inverse solution is

$$t = T_0 \frac{\partial \chi}{\partial T} \cosh y - \frac{T_0}{T} \frac{\partial \chi}{\partial y} \sinh y$$
$$x = T_0 \frac{\partial \chi}{\partial T} \sinh y - \frac{T_0}{T} \frac{\partial \chi}{\partial y} \cosh y$$

The Khalatnikov Equation for χ

We can get an equation for χ.
We have the equation along u^i, the entropy conservation equation

$$\frac{\partial su^0}{\partial x^0} + \frac{\partial su^1}{\partial x^1} = 0$$

Note that

$$\frac{\partial su^0}{\partial x^0} = \frac{\partial(su^0, x)}{\partial(t,x)} = \begin{vmatrix} \frac{\partial su^0}{\partial(t)} & \frac{\partial su^0}{\partial(x)} \\ \frac{\partial(x)}{\partial(t)} & \frac{\partial(x)}{\partial(x)} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial su^0}{\partial(t)} & \frac{\partial su^0}{\partial(x)} \\ 0 & \frac{\partial(x)}{\partial(x)} \end{vmatrix} = \frac{\partial(su^0)}{\partial(t)}$$

Note that

$$\frac{\partial su^1}{\partial x^1} = \frac{\partial(su^1, t)}{\partial(t,x)} = \begin{vmatrix} \frac{\partial(su^1)}{\partial(t)} & \frac{\partial(su^1)}{\partial(x)} \\ \frac{\partial(t)}{\partial(t)} & \frac{\partial(t)}{\partial(x)} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial(su^1}{\partial(t)} & \frac{\partial(su^1)}{\partial(x)} \\ 1 & 0 \end{vmatrix} = -\frac{\partial(su^1)}{\partial(x)}$$

Entropy equation is

$$\frac{\partial(su^0, x)}{\partial(t,x)} - \frac{\partial(su^1, t)}{\partial(t,x)} = 0$$

We have therefore

$$\frac{\partial(t,x)}{\partial(T,y)} \left\{ \frac{\partial(su^0, x)}{\partial(t,x)} - \frac{\partial(su^1, t)}{\partial(t,x)} \right\} = 0$$

$$\left\{ \frac{\partial(su^0, x)}{\partial(T,y)} - \frac{\partial(su^1, t)}{\partial(T,y)} \right\} = 0$$

Let us set T_0 as a unit for T and add this unit factor to T back later on

$$\frac{\partial(su^0,x)}{\partial(T,y)}-\frac{\partial(su^1,t)}{\partial(T,y)}=0$$
$$\frac{\partial(s\cosh y,x)}{\partial(T,y)}-\frac{\partial(s\sinh y,t)}{\partial(T,y)}=0$$

$$\begin{vmatrix}\frac{\partial s\cosh y}{\partial(T)} & \frac{\partial s\cosh y}{\partial(y)}\\ \frac{\partial(x)}{\partial(T)} & \frac{\partial(x)}{\partial(y)}\end{vmatrix}-\begin{vmatrix}\frac{\partial s\sinh y}{\partial(T)} & \frac{\partial s\sinh y}{\partial(y)}\\ \frac{\partial(t)}{\partial(T)} & \frac{\partial(t)}{\partial(y)}\end{vmatrix}=0$$
$$\left\{\frac{\partial s\cosh y}{\partial(T)}\frac{\partial(x)}{\partial(y)}-\frac{\partial s\sinh y}{\partial(T)}\frac{\partial(t)}{\partial(y)}\right\}-\frac{\partial s\cosh y}{\partial(y)}\frac{\partial(x)}{\partial(T)}+\frac{\partial s\sinh y}{\partial(y)}\frac{\partial(t)}{\partial(T)}=0$$

this leads to

$$\frac{\partial s}{\partial T}\{\frac{\partial}{\partial(y)}(x\cosh y-t\sinh y)-(x\sinh y-t\cosh y)\}$$
$$-\frac{\partial s\cosh y}{\partial(y)}\frac{\partial(x)}{\partial(T)}+\frac{\partial s\sinh y}{\partial(y)}\frac{\partial(t)}{\partial(T)}=0$$
$$\frac{\partial s}{\partial T}\{\frac{\partial}{\partial(y)}(x\cosh y-t\sinh y)-(x\sinh y-t\cosh y)\}$$
$$-s\sinh y\frac{\partial(x)}{\partial(T)}+s\cosh y\frac{\partial(t)}{\partial(T)}=0$$
$$\frac{\partial s}{s\partial T}\{\frac{\partial}{\partial(y)}(x\cosh y-t\sinh y)-(x\sinh y-t\cosh y)\}$$
$$-\frac{\partial}{\partial T}(x\sinh y-t\cosh y)=0,$$

which is Landau's (4.13).

We introduce the transformation

$$t = T_0 \frac{\partial \chi}{\partial T} \cosh y - \frac{T_0}{T} \frac{\partial \chi}{\partial y} \sinh y$$
$$x = T_0 \frac{\partial \chi}{\partial T} \sinh y - \frac{T_0}{T} \frac{\partial \chi}{\partial y} \cosh y$$

$$(x \cosh y - t \sinh y) = -\frac{T_0}{T} \frac{\partial \chi}{\partial y} (\cosh^2 y - \sinh^2 y) = -\frac{T_0}{T} \frac{\partial \chi}{\partial y}$$
$$(x \sinh y - t \cosh y) = T_0 \frac{\partial \chi}{\partial T} (\sinh^2 y - \cosh^2 y) = -T_0 \frac{\partial \chi}{\partial T}$$

Hydro equation becomes

$$\frac{\partial s}{s \partial T} \{ -\frac{T_0}{T} \frac{\partial^2 \chi}{\partial y^2} + T_0 \frac{\partial \chi}{\partial T} \} + T_0 \frac{\partial^2 \chi}{\partial T^2} = 0$$
$$T_0 \frac{\partial s}{s \partial T} \{ -\frac{T_0}{T} \frac{\partial^2 \chi}{\partial y^2} + T_0 \frac{\partial \chi}{\partial T} \} + T_0^2 \frac{\partial^2 \chi}{\partial T^2} = 0$$

$$\frac{s \partial T}{T \partial s} = \frac{1}{T} \frac{\partial p}{\partial s} = \frac{1}{T} \frac{\partial p}{\partial \epsilon} \frac{\partial \epsilon}{\partial s} = c_s^2$$
$$\frac{s \partial T}{\partial s} = T c_s^2$$
$$\frac{\partial s}{s \partial T} = \frac{1}{T c_s^2}$$

$$\{ -\frac{T_0}{T} \frac{\partial^2 \chi}{\partial y^2} + T_0 \frac{\partial \chi}{\partial T} \} + T_0 T c_s^2 \frac{\partial^2 \chi}{\partial T^2} = 0$$

$$\zeta = \ln T/T_0, \quad (T/T_0) = e^{\zeta}$$
$$d\zeta = \frac{T_0 dT}{T}\frac{1}{T_0} = \frac{dT}{T}$$
$$\frac{\partial \chi}{\partial T} = \frac{\partial \chi}{\partial \zeta}\frac{\partial \zeta}{\partial T} = \frac{\partial \chi}{\partial \zeta}\frac{1}{T}$$
$$\frac{\partial^2 \chi}{\partial T^2} = \frac{\partial \zeta}{\partial T}\frac{\partial}{\partial \zeta}[\frac{\partial \chi}{\partial T}] = \frac{1}{T}\frac{\partial}{\partial \zeta}[\frac{\partial \chi}{\partial T}]$$
$$= \frac{1}{T}[\frac{\partial^2 \chi}{\partial \zeta^2}\frac{1}{T} - \frac{\partial \chi}{\partial \zeta}\frac{1}{T^2}\frac{\partial T}{\partial \zeta}] = \frac{\partial^2 \chi}{\partial \zeta^2}\frac{1}{T^2} - \frac{\partial \chi}{\partial \zeta}\frac{1}{T^2}$$

We get

$$\{-\frac{T_0}{T}\frac{\partial^2 \chi}{\partial y^2} + T_0\frac{\partial \chi}{\partial \zeta}\frac{1}{T}\} + T_0 T c_s^2(\frac{\partial^2 \chi}{\partial \zeta^2}\frac{1}{T^2} - \frac{\partial \chi}{\partial \zeta}\frac{1}{T^2}) = 0$$

We get finally

$$\frac{\partial^2 \chi}{\partial y^2} - \frac{\partial \chi}{\partial \zeta} - c_s^2\frac{\partial^2 \chi}{\partial \zeta^2} + c_s^2\frac{\partial \chi}{\partial \zeta} = 0$$
$$\boxed{\frac{\partial^2 \chi}{\partial y^2} - c_s^2\frac{\partial^2 \chi}{\partial \zeta^2} + (c_s^2 - 1)\frac{\partial \chi}{\partial \zeta} = 0}$$

This is just Landau's (4.15a), Khalatnikov equation.

Khalatnikov solution

First, we need to eliminate the first order term by the transformation

$$\chi(\zeta,y) = f(\zeta)Z(\zeta,y),$$
$$\frac{\partial\chi}{\partial\zeta} = f'(\zeta)Z + f(\zeta)Z'(\zeta,y)$$
$$\frac{\partial^2\chi}{\partial\zeta^2} = f''(\zeta)Z + 2f'(\zeta)Z' + f(\zeta)Z''(\zeta,y)$$

We choose f according to

$$-c_s^2(2f') + (c_s^2-1)f = 0$$
$$\frac{f'}{f} = \frac{c_s^2-1}{2c_s^2}$$
$$f(\zeta) = e^{\frac{c_s^2-1}{2c_s^2}\zeta} = e^{-\frac{1-c_s^2}{2c_s^2}\zeta}$$

$$f'(\zeta) = -\frac{1-c_s^2}{2c_s^2}f$$
$$f''(\zeta) = (\frac{1-c_s^2}{2c_s^2})^2 f$$
$$\chi(\zeta,y) = e^{-\frac{1-c_s^2}{2c_s^2}\zeta}Z(\zeta,y)$$

The Khaltnikov equation becomes, prime is for $d/d\zeta$

$$\chi(\zeta,y) = e^{-\frac{1-c_s^2}{2c_s^2}\zeta} Z(\zeta,y)$$
$$\frac{\partial^2\chi}{\partial y^2} - c_s^2\frac{\partial^2\chi}{\partial\zeta^2} + (c_s^2-1)\frac{\partial\chi}{\partial\zeta} = 0$$
$$f\frac{\partial^2 Z}{\partial y^2} - c_s^2(f''Z + 2f'Z' + fZ'') + (c_s^2-1)(f'Z + fZ') = 0$$
$$f\frac{\partial^2 Z}{\partial y^2} + Z''[-c_s^2 f] + Z'[-c_s^2 2f' + (c_s^2-1)f] + Z[c_s^2 f'' + (c_s^2-1)f'] = 0$$
$$f\frac{\partial^2 Z}{\partial y^2} + Z''[-c_s^2 f] + Z[-c_s^2 f'' + (c_s^2-1)f'] = 0$$
$$\frac{\partial^2 Z}{\partial y^2} - c_s^2 Z'' + Z\frac{[-c_s^2 f'' + (c_s^2-1)f']}{f} = 0$$
$$\frac{\partial^2 Z}{\partial y^2} - c_s^2 Z'' + Z[-c_s^2(\frac{1-c_s^2}{2c_s^2})^2 + (-\frac{1-c_s^2}{2c_s^2})(c_s^2-1)] = 0$$
$$\frac{\partial^2 Z}{\partial y^2} - c_s^2 Z'' + Z[-(\frac{(1-c_s^2)^2}{4c_s^2}) + (\frac{(1-c_s^2)^2}{2c_s^2})] = 0$$
$$\frac{\partial^2 Z}{\partial y^2} - c_s^2 Z'' + Z[(\frac{(1-c_s^2)^2}{4c_s^2})] = 0$$
$$\frac{\partial^2 Z}{\partial y^2} - c_s^2\frac{\partial^2 Z}{\partial\zeta^2} + Z[(\frac{(1-c_s^2)^2}{4c_s^2})] = 0 \tag{10}$$

We write it as

$$\frac{\partial^2 Z}{\partial\zeta^2} - \frac{\partial^2 Z}{c_s^2\partial y^2} - \left[\frac{(1-c_s^2)}{2c_s^2})\right]^2 Z = 0 \tag{11}$$

This is the equation we would like to solve. We make a change of variables. We could use $+\zeta$ or $-\zeta$ below, but because our ζ is always negative, so we choose to use $-\zeta$ below, as argued by Beuf *et al.* [**?**]

$$\alpha = -\zeta + c_s y, \qquad \beta = -\zeta - c_s y, \tag{12}$$

$$\zeta = -\frac{1}{2}(\alpha+\beta), \quad c_s y = \frac{1}{2}(\alpha-\beta), \tag{13}$$

$$\frac{\partial}{\partial\zeta} = \frac{\partial\alpha}{\partial\zeta}\frac{\partial}{\partial\alpha} + \frac{\partial\beta}{\partial\zeta}\frac{\partial}{\partial\beta} = -\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta}$$

$$d\alpha d\beta = 2d\zeta c_s dy$$

$$\left(\frac{\partial}{\partial\zeta}\right)^2 = \left(-\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta}\right)^2$$

$$\frac{\partial}{c_s\partial y} = \frac{\partial\alpha}{c_s\partial y}\frac{\partial}{\partial\alpha} + \frac{\partial\beta}{c_s\partial y}\frac{\partial}{\partial\beta} = \frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta}$$

$$\left(\frac{\partial}{c_s\partial y}\right)^2 = \left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta}\right)^2$$

$$\left(\frac{\partial}{\partial\zeta}\right)^2 - \left(\frac{\partial}{c_s\partial y}\right)^2 = \left(-\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta}\right)^2 - \left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta}\right)^2 = 4\frac{\partial^2}{\partial\alpha\partial\beta} \tag{14}$$

$$\left[\frac{\partial^2}{\partial\zeta^2} - \frac{\partial^2}{c_s^2\partial y^2}\right]Z - \left[\frac{(1-c_s^2)}{2c_s^2}\right]^2 Z = 0$$

$$4\frac{\partial^2}{\partial\alpha\partial\beta}Z(\alpha,\beta) - \left[\frac{(1-c_s^2)}{2c_s^2}\right]^2 Z(\alpha,\beta) = 0$$

$$\frac{\partial^2}{\partial\alpha\partial\beta}Z(\alpha,\beta) - \left[\frac{(1-c_s^2)}{4c_s^2}\right]^2 Z(\alpha,\beta) = 0 \tag{15}$$

We solve this equation by using green's function (Beuf et al (2008))

$$\frac{\partial^2}{\partial\alpha\partial\beta}G(\alpha,\beta) - \left[\frac{(1-c_s^2)}{4c_s^2}\right]^2 G(\alpha,\beta) = \delta(\alpha)\delta(\beta) \tag{16}$$

Consider the following and we would like to choose a to make it work.

$$
\begin{aligned}
G(\alpha,\beta) &= \Theta(\alpha)\Theta(\beta)I_0(a\sqrt{\alpha\beta}) \\
\partial_\alpha G(\alpha,\beta) &= \delta(\alpha)\Theta(\beta)I_0(a\sqrt{\alpha\beta}) + \Theta(\alpha)\Theta(\beta)I_0' a\frac{1}{2}\alpha^{-1/2}\sqrt{\beta}, \\
&\qquad (\text{ where } \quad I'(z) = dI_0(z)/dz,) \\
&= \delta(\alpha)\Theta(\beta)I_0(a\sqrt{\alpha\beta})) + \Theta(\alpha)\Theta(\beta)I_0' a(\frac{1}{2})\sqrt{\frac{\beta}{\alpha}} \\
\partial_{\alpha\beta} G(\alpha,\beta) &= \delta(\alpha)\delta(\beta)I_0(a\sqrt{\alpha\beta}) + \delta(\alpha)\Theta(\beta)I_0' a\frac{1}{2}\beta^{-1/2}\alpha + \Theta(\alpha)\delta(\beta)I_0' a(\frac{1}{2})\sqrt{\frac{\beta}{\alpha}} \\
&\quad + \Theta(\alpha)\Theta(\beta)I_0'' a(\frac{1}{2})\sqrt{\frac{\beta}{\alpha}}a(\frac{1}{2})\sqrt{\frac{\alpha}{\beta}} + \Theta(\alpha)\Theta(\beta)I_0' a(\frac{1}{2})\sqrt{\frac{1}{\alpha}}\frac{1}{2}\beta^{-1/2} \\
&= \delta(\alpha)\delta(\beta)I_0(0) + \Theta(\alpha)\Theta(\beta)I_0'' a(\frac{1}{2})\sqrt{\frac{\beta}{\alpha}}a(\frac{1}{2})\sqrt{\frac{\alpha}{\beta}} + \Theta(\alpha)\Theta(\beta)I_0' a(\frac{1}{4})\sqrt{\frac{1}{\alpha\beta}} \\
&= \delta(\alpha)\delta(\beta) + \Theta(\alpha)\Theta(\beta)I_0'' \times (\frac{a^2}{4}) + \Theta(\alpha)\Theta(\beta)I'\frac{a}{4}\sqrt{\frac{1}{\alpha\beta}} \qquad (17)
\end{aligned}
$$

So, we have

$$
\begin{aligned}
&\partial_{\alpha\beta}G(\alpha,\beta) - \left[\frac{(1-c_s^2)}{4c_s^2}\right]^2 G(\alpha,\beta) \\
&= \delta(\alpha)\delta(\beta) + \frac{a^2}{4}\Theta(\alpha)\Theta(\beta)I_0'' + \frac{a}{4}\sqrt{\frac{1}{\alpha\beta}}\Theta(\alpha)\Theta(\beta)I'(z) - \left[\frac{(1-c_s^2)}{4c_s^2}\right]^2 G(\alpha,\beta) \\
&= \delta(\alpha)\delta(\beta) + \frac{a^2}{4}\Theta(\alpha)\Theta(\beta)I_0'' + \frac{a}{4}\sqrt{\frac{1}{\alpha\beta}}\Theta(\alpha)\Theta(\beta)I'(z) - \left[\frac{(1-c_s^2)}{4c_s^2}\right]^2 \Theta(\alpha)\Theta(\beta)I(z)
\end{aligned}
$$

Bessel equation for $I_0(z)$ is (Abramowitz page 374 9.6.1)

$$
z^2\frac{d^2 I_\nu}{dz^2} + z\frac{dI_\nu}{dz} - (z^2+\nu^2)I_\nu = 0 \qquad (18)
$$

We choose

$$z = \frac{(1-c_s^2)}{2c_s^2}\sqrt{\alpha\beta}$$
$$a = \frac{(1-c_s^2)}{2c_s^2}$$
$$z = a\sqrt{\alpha\beta} \tag{19}$$

$$G(\alpha,\beta) = \Theta(\alpha)\Theta(\beta)I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{\alpha\beta}) \tag{20}$$

The last three terms in (18) give

$$\alpha\beta\left[+\frac{a^2}{4}\Theta(\alpha)\Theta(\beta)I_0'' + \frac{a}{4}\sqrt{\frac{1}{\alpha\beta}}\Theta(\alpha)\Theta(\beta)I'(z) - \left[\frac{(1-c_s^2)}{4c_s^2}\right]^2\Theta(\alpha)\Theta(\beta)I(z)\right]$$
$$= \frac{z^2}{4}\Theta(\alpha)\Theta(\beta)I_0'' + \frac{z}{4}\Theta(\alpha)\Theta(\beta)I_0'(z) - \frac{z^2}{4}\Theta(\alpha)\Theta(\beta)I_0(z) \tag{21}$$

which gives zero if I_0 is the solution. So, we have

Then

$$\partial_{\alpha\beta}G(\alpha,\beta) - \left[\frac{(1-c_s^2)}{4c_s^2}\right]^2 G(\alpha,\beta) = \delta(\alpha)\delta(\beta) \tag{22}$$

with

$$G(\alpha,\beta) = \Theta(\alpha)\Theta(\beta)I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{\alpha\beta}) \tag{23}$$

A general solution is

$$\chi(\alpha,\beta) = e^{-\frac{1-c_s^2}{2c_s^2}\zeta}Z(\alpha,\beta)$$
$$Z(\alpha,\beta) = \int d\alpha' d\beta' G(\alpha-\alpha',\beta-\beta')F(\alpha',\beta') \tag{24}$$

$$\partial_{\alpha\beta}Z(\alpha,\beta) - \left[\frac{(1-c_s^2)}{4c_s^2}\right]^2 Z(\alpha,\beta) = \int d\alpha' d\beta' \delta(\alpha-\alpha')\delta(\beta-\beta')F(\alpha',\beta') = F(\alpha,\beta)$$

as required.

$$\chi(\alpha,\beta) = e^{-\frac{1-c_s^2}{2c_s^2}\zeta} \int d\alpha' d\beta' G(\alpha-\alpha',\beta-\beta') F(\alpha',\beta') \tag{25}$$

We change variables:

$$d\alpha d\beta = 2d\zeta c_s dy$$
$$\int \delta(\alpha)\delta(\beta) d\alpha d\beta = \int \delta(\zeta)\delta(c_s y) d\zeta c_s dy = 1$$
$$\int \delta(\alpha)\delta(\beta) d\alpha d\beta = \int \frac{\delta(\zeta)\delta(c_s y)}{2} 2d\zeta c_s dy$$

$$\delta(\alpha)\delta(\beta) = \frac{\delta(\zeta)\delta(c_s y)}{2} \tag{26}$$

Let us write the Green's function for $G(\zeta, y)$: Then

$$\partial_{\alpha\beta} G(\alpha,\beta) - \left[\frac{(1-c_s^2)}{4c_s^2}\right]^2 G(\alpha,\beta) = \delta(\alpha)\delta(\beta) = \frac{\delta(\zeta)\delta(c_s y)}{2} \tag{27}$$

Eq. (16) gives

$$\frac{1}{4}\left\{\left(\frac{\partial}{\partial\zeta}\right)^2 - \left(\frac{\partial}{c_s\partial y}\right)^2\right\} G(\alpha,\beta) - \left[\frac{(1-c_s^2)}{4c_s^2}\right]^2 G(\alpha,\beta) = \frac{\delta(\zeta)\delta(c_s y)}{2}$$
$$\left\{\left(\frac{\partial}{\partial\zeta}\right)^2 - \left(\frac{\partial}{c_s\partial y}\right)^2\right\} G(\alpha,\beta) - 4\left[\frac{(1-c_s^2)}{4c_s^2}\right]^2 G(\alpha,\beta) = 2\delta(\zeta)\delta(c_s y) \tag{28}$$

$$\chi(\zeta, y) = e^{-\frac{1-c_s^2}{2c_s^2}\zeta} \int d\zeta' c_s dy' G(\zeta-\zeta', y-y') \frac{F(\eta', y')}{2} \tag{29}$$

$$
\begin{aligned}
G(\alpha,\beta) &= \Theta(\alpha)\Theta(\beta)I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{\alpha\beta}) \\
&= \Theta(\alpha)\Theta(\beta)I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{\zeta^2-c_s^2y^2}) \\
&= \Theta(-\zeta+c_sy)\Theta(-\zeta-c_sy)I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{\zeta^2-c_s^2y^2}) \\
&= G(\zeta,y) \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
\chi(\zeta,y) = e^{-\frac{1-c_s^2}{2c_s^2}\zeta}\int d\zeta' c_s dy' \Theta[-(\zeta-\zeta')+c_s(y-y')]\Theta[-(\zeta-\zeta')-c_s(y-y')] \\
\times I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta-\zeta')^2-c_s^2(y-y')^2})\frac{F(\zeta',y')}{2}
\end{aligned}
$$

We introduce

$$
-(\zeta-\zeta')=\zeta'', \quad \zeta'=\zeta''+\zeta, \quad d\zeta'=d\zeta'' \qquad (31)
$$

and choose

$$
F(\zeta',y')=f(\zeta')\delta(c_sy') \qquad (32)
$$

then we have

$$
\chi(\zeta,y)=e^{-\frac{1-c_s^2}{2c_s^2}\zeta}\int d\zeta'\Theta[-(\zeta-\zeta')+c_sy]\Theta[-(\zeta-\zeta')-c_sy]I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta-\zeta')^2-c_s^2y^2})\frac{f(\zeta')}{2} \qquad (33)
$$

We consider only the region of $y>0$. We have from the step functions $\Theta[-(\zeta-\zeta')+c_s(y-y')]\Theta[-(\zeta-\zeta')-c_s(y-y')]$,

$$
\begin{aligned}
&-(\zeta-\zeta')+c_sy>0, \quad -(\zeta-\zeta')>-c_sy \\
&-(\zeta-\zeta')-c_sy>0, \quad -(\zeta-\zeta')>c_sy>-c_sy, \quad \zeta'-\zeta=\zeta''>c_sy
\end{aligned} \qquad (34)
$$

We consider only the region of $y>0$, because there is complete symmetry in y. Then

$$
\begin{aligned}
\Theta[-\zeta''+c_sy]\Theta[-\zeta''-c_sy] &= \Theta[-\zeta''-c_sy+2c_sy]\Theta[-\zeta''-c_sy] \\
&= \Theta[-\zeta''-c_sy] \text{ because } 2c_sy>0,
\end{aligned}
$$

$$\begin{aligned}
\chi(\zeta,y) &= e^{-\frac{1-c_s^2}{2c_s^2}\zeta}\int d\zeta'\Theta[-(\zeta-\zeta')+c_s y]\Theta[-(\zeta-\zeta')-c_s y] \\
&\qquad \times I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta-\zeta')^2-c_s^2y^2})\frac{f(\zeta')}{2} \\
&= e^{-\frac{1-c_s^2}{2c_s^2}\zeta}\int d\zeta'\Theta[-(\zeta-\zeta')-c_s y] I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta-\zeta')^2-c_s^2y^2})\frac{f(\zeta')}{2} \\
&= e^{-\frac{1-c_s^2}{2c_s^2}\zeta}\int d\zeta'\Theta[(\zeta'-\zeta)-c_s y] I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta-\zeta')^2-c_s^2y^2})\frac{f(\zeta')}{2} \\
&= e^{-\frac{1-c_s^2}{2c_s^2}\zeta}\int_{c_s y}^{\infty} d\zeta'' I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta'')^2-c_s^2y^2})\frac{f(\zeta''+\zeta)}{2} \qquad (35)
\end{aligned}$$

We need to look for $f(\zeta''+\zeta)$ to satisfy the boundary condition. the Khalatnikov solution must satisfy two boundary conditions

Boundary conditions

(1) It must match to the Rieman solution when $\zeta = -c_s y$, as given by (1). This occurs when $\chi(\zeta, y)$ is no longer a function of ζ and y, the case when there are two degrees of frrdom. Instead, χ is a consstant. Note that beause x and t are related to χ by derivative, χ is free up to a constant, and we can choose this constant to be zero, when we match to the Riemann solution.

$$(i) \quad \chi(\zeta, y) = 0 \ \text{ at Rieman matching} \tag{36}$$

In order to satisfy this boundary condition, we must have

$$\chi(\zeta, y) = e^{-\frac{1-c_s^2}{2c_s^2}\zeta} \int_{c_s y}^{-\zeta} d\zeta'' I_0\left(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta'')^2 - c_s^2 y^2}\right)\frac{f(\zeta''+\zeta)}{2} \tag{37}$$

which gives zero when

$$-\zeta = c_s y \quad \text{or} \quad \zeta = -c_s y \tag{38}$$

(2) When $y = 0$, whatever is $\epsilon(\zeta)$, the matter is at rest, $y = 0$ and that occurs at $x = -l$. The midpoint is the point of reflection and no motion. Eq.(**??**) gives

$$(ii) \quad \text{at} \ \ y = 0, \ \text{we must have} \tag{39}$$

$$\begin{aligned} x &= e^{-\zeta}\left(\frac{\partial \chi}{\partial \zeta}\sinh y - \frac{\partial \chi}{\partial y}\cosh y\right), \quad \alpha = y \\ x &= -e^{-\zeta}\frac{\partial \chi}{\partial y}\cosh y = -e^{-\zeta}\frac{\partial \chi}{\partial y} \\ &= -l \\ \left.\frac{\partial \chi}{\partial y}\right|_{y=0} &= e^{\zeta} l \end{aligned} \tag{40}$$

We get from (37) that

$$\frac{\partial\chi(\zeta,y)}{\partial y}\bigg|_{y=0} = \frac{\partial}{\partial y}\bigg|_{y=0} e^{-\frac{1-c_s^2}{2c_s^2}\zeta}\int_{c_s y}^{-\zeta} d\zeta'' I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta'')^2-c_s^2y^2})\frac{f(\zeta''+\zeta)}{2}$$
$$= \left(e^{-\frac{1-c_s^2}{2c_s^2}\zeta}(-c_s)I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta'')^2-c_s^2y^2})\bigg|_{\zeta''=c_s,\ \text{and}\ y=0} + y\times\text{integral...}\right)$$
$$\times\frac{f(\zeta''+\zeta)}{2} \tag{41}$$

The term proportional to y inside the bracket is zero at $y=0$. So, we are left with

$$\frac{\partial\chi(\zeta,y)}{\partial y}\bigg|_{y=0} = \left(e^{-\frac{1-c_s^2}{2c_s^2}\zeta}(-c_s)I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta'')^2-c_s^2y^2})\bigg|_{\zeta''=c_s y,\ \text{and}\ y=0}\right)\frac{f(\zeta''+\zeta)}{2}$$
$$= e^{-\frac{1-c_s^2}{2c_s^2}\zeta}(-c_s)\frac{f(\zeta''+\zeta)}{2} = e^{\zeta}l \tag{42}$$

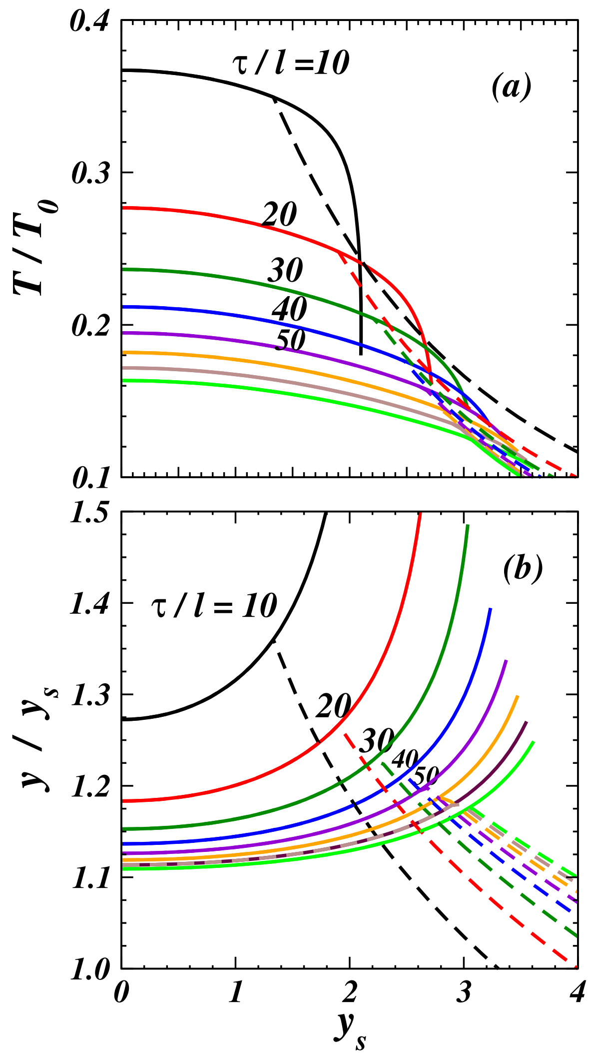
$$\frac{f(\zeta''+\zeta)}{2} = -e^{\zeta}le^{\frac{1-c_s^2}{2c_s^2}\zeta}/c_s \tag{43}$$

We have therefore

$$\frac{f(\zeta''+\zeta)}{2} = -\frac{e^{(\frac{1-c_s^2}{2c_s^2}+1)(\zeta+\zeta'')}}{c_s} = -\frac{e^{(\frac{1-c_s^2}{2c_s^2}+1)\zeta}e^{(\frac{1-c_s^2}{2c_s^2}+1)\zeta'')}}{c_s} = -\frac{le^{(\frac{1-c_s^2}{2c_s^2}+1)\zeta}e^{\frac{1+c_s^2}{2c_s^2}\zeta''}}{c_s} \tag{44}$$

We put this back into χ, and we get from (37),

$$\chi(\zeta,y) = e^{-\frac{1-c_s^2}{2c_s^2}\zeta}\int_{c_s y}^{-\zeta} d\zeta'' I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta'')^2-c_s^2y^2})\frac{f(\zeta''+\zeta)}{2}$$
$$= e^{-\frac{1-c_s^2}{2c_s^2}\zeta}\int_{c_s y}^{-\zeta} d\zeta'' I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta'')^2-c_s^2y^2})(-1)\frac{le^{(\frac{1-c_s^2}{2c_s^2}+1)\zeta}e^{\frac{1+c_s^2}{2c_s^2}\zeta''}}{c_s}$$
$$= -\frac{l}{c_s}e^{\zeta}\int_{c_s y}^{-\zeta} d\zeta'' I_0(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta'')^2-c_s^2y^2})e^{\frac{1+c_s^2}{2c_s^2}\zeta''}, \quad \text{change}\ \ \zeta''\to\zeta'$$
$$= -\frac{l}{c_s}e^{\zeta}\int_{c_s y}^{-\zeta} d\zeta' e^{\frac{1+c_s^2}{2c_s^2}\zeta'} I_0\left(\frac{(1-c_s^2)}{2c_s^2}\sqrt{(\zeta')^2-c_s^2y^2}\right), \quad \text{the Khalatnikov solution} \tag{45}$$



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Conclusions

1. Exact solution of 1+1 dimensional hydrodynamics gives a compact form of the solution with the Landau initial conditions
2. Landau hydrodynamics remain different from Hwa-Bjorken hydrodynamics even at very late time
3. Hwa-Bjorken hydrodynmics must obtain the flow condition not from hydrodynamics but from other types of initial conditions.