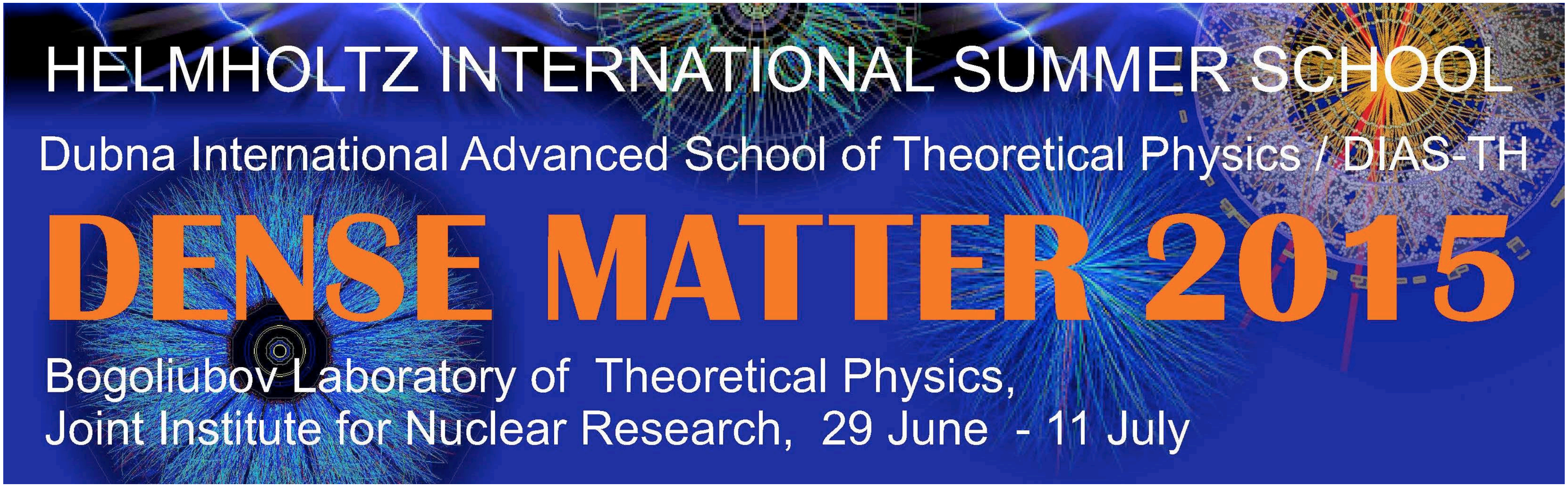
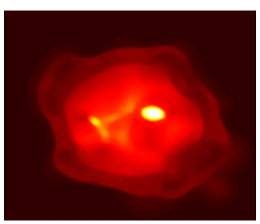


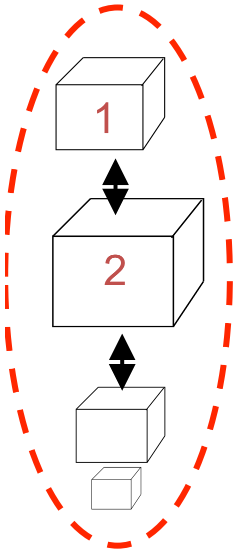
Spinodal Instabilities at the Deconfinement Phase Transition

Jørgen Randrup (Berkeley)

Lecture I:	*Phase coexistence (equilibrium)*	WED 12:00-13:00
Lecture II:	*Phase separation (non-equilibrium)*	THU 16:30-17:30
	Discussion	THU 18:00-19:00
Lecture III:	*Effects on collision dynamics (clumping)*	SAT 11:00-12:00



Basic thermodynamics



$\mathbf{X}_1 = \{E_1, N_1, V_1, \dots\} \Rightarrow S_1(\mathbf{X}_1)$

$\mathbf{X}_2 = \{E_2, N_2, V_2, \dots\} \Rightarrow S_2(\mathbf{X}_2)$

$\mathbf{X} = \{E, N, V, \dots\} = \mathbf{X}_1 + \mathbf{X}_2 + \dots$

$$\begin{cases} E = E_1 + E_2 + \dots \\ N = N_1 + N_2 + \dots \\ V = V_1 + V_2 + \dots \end{cases}$$

The combined system is in equilibrium provided S has a local *maximum* - which requires $\delta S = 0$ and $\delta^2 S < 0$:

$S = S_1 + S_2 + \dots$

δS:

$$0 \doteq \delta S = \sum_i \delta S_i = \sum_i \left(\sum_\ell \frac{\partial S_i}{\partial X_i^\ell} \delta X_i^\ell \right) = \sum_\ell \left(\sum_i \lambda_i^\ell \delta X_i^\ell \right)$$

$$\lambda_i^\ell \equiv \frac{\partial S_i}{\partial X_i^\ell} \begin{cases} \lambda_i^E = \dfrac{\partial S_i}{\partial E_i} = \beta_i = \dfrac{1}{T_i} \\ \lambda_i^N = \dfrac{\partial S_i}{\partial N_i} = \alpha_i = -\dfrac{\mu_i}{T_i} \\ \lambda_i^V = \dfrac{\partial S_i}{\partial V_i} = \pi_i = \dfrac{p_i}{T_i} \end{cases}$$

$$\delta X^\ell = \sum_i \delta X_i^\ell \doteq 0 \begin{cases} \delta E = \sum_i \delta E_i \doteq 0 \\ \delta N = \sum_i \delta N_i \doteq 0 \\ \delta V = \sum_i \delta V_i \doteq 0 \end{cases}$$

=> $\lambda_1^\ell \doteq \lambda_2^\ell \doteq \dots$

$$\begin{cases} T_1 = T_2 = \dots \\ \mu_1 = \mu_2 = \dots \\ p_1 = p_2 = \dots \end{cases}$$

$\delta^2 S$:

$$0 \dot{>} \delta^2 S = \sum_i \delta^2 S_i = \sum_i \left(\sum_{\ell_1 \ell_2} \frac{\partial^2 S_i}{\partial X_i^{\ell_1} \partial X_i^{\ell_2}} \delta X_i^{\ell_1} \delta X_i^{\ell_2} \right)$$

=> The entropy curvature matrices $\dfrac{\partial^2 S_i}{\partial X_i^{\ell_1} \partial X_i^{\ell_2}}$ have only *negative* eigenvalues

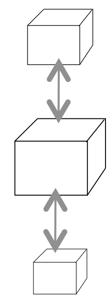
Microcanonical thermodynamics: E and N are specified:

entropy density $\sigma(\varepsilon,\rho)$ $\Rightarrow$
$$\begin{cases} \beta(\varepsilon,\rho) = \partial_\varepsilon\sigma(\varepsilon,\rho) = 1/T(\varepsilon,\rho) & \text{temperature} \\ \alpha(\varepsilon,\rho) = \partial_\rho\sigma(\varepsilon,\rho) = -\mu(\varepsilon,\rho)/T(\varepsilon,\rho) & \text{chemical potential} \end{cases}$$

$$\begin{bmatrix} \partial_\varepsilon^{\,2}\sigma & \partial_\rho\partial_\varepsilon\sigma \\ \partial_\varepsilon\partial_\rho\sigma & \partial_\rho^{\,2}\sigma \end{bmatrix} < 0 \;\Rightarrow\; \textit{stability}$$

$$p(\varepsilon,\rho) = \sigma T - \varepsilon + \mu\rho \quad \textit{pressure}$$
$$h(\varepsilon,\rho) = p + \varepsilon \quad \textit{enthalpy density}$$

Canonical thermodynamics: <E> and N are specified:



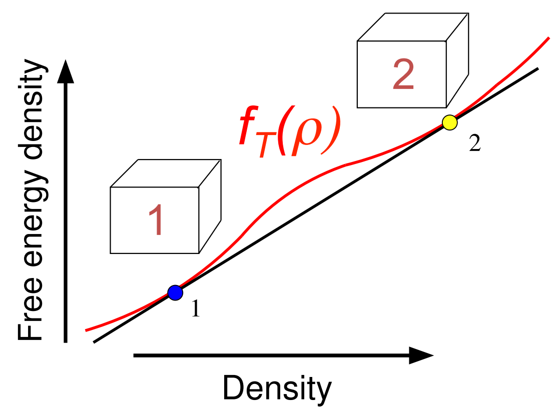
Same:
- Then replace S by $S' = S - \beta E$ and require $\delta S'=0$ & $\delta^2 S'<0$
- or consider $F = -TS' = E - TS$ and require $\delta F=0$ & $\delta^2 F>0$

free energy density
$$f_T(\rho) \equiv \varepsilon_T(\rho) - T\sigma_T(\rho) = \mu_T(\rho)\rho - p_T(\rho)$$

$$\mu_T(\rho) = \partial_\rho f_T(\rho)$$
$$\sigma_T(\rho) = -\partial_T f_T(\rho)$$

$\partial_\rho^{\,2} f_T(\rho) > 0 \;\Rightarrow\;$ *stability*

Phase coexistence $\Leftrightarrow$ *$f_T(\rho)$ has common tangent*



Thermodynamics of non-uniform matter: microcanonical

Non-uniform charge density $\tilde{\rho}(\boldsymbol{r})$

Non-uniform energy density $\tilde{\varepsilon}(\boldsymbol{r})$

Non-uniform entropy density $\tilde{\sigma}[\tilde{\rho}(\boldsymbol{r}),\tilde{\varepsilon}(\boldsymbol{r})](\boldsymbol{r})$

$$N = \int \tilde{\rho}(\boldsymbol{r})d\boldsymbol{r}$$
$$E = \int \tilde{\varepsilon}(\boldsymbol{r})d\boldsymbol{r}$$
$$S = \int \tilde{\sigma}(\boldsymbol{r})d\boldsymbol{r}$$

$$\boxed{\delta S = \int [\tilde{\beta}(\boldsymbol{r})\delta\tilde{\varepsilon}(\boldsymbol{r}) + \tilde{\alpha}(\boldsymbol{r})\delta\tilde{\rho}(\boldsymbol{r})]d\boldsymbol{r}}$$

$$\begin{cases} \tilde{T}(\boldsymbol{r}) = 1/\tilde{\beta}(\boldsymbol{r}) \\ \tilde{\mu}(\boldsymbol{r}) = -\tilde{\alpha}(\boldsymbol{r})\tilde{T}(\boldsymbol{r}) \end{cases}$$

$$\forall\,\delta\tilde{\varepsilon}(\boldsymbol{r}),\forall\,\delta\tilde{\rho}(\boldsymbol{r}):\ 0 \doteq \delta S - \beta_0\delta E - \alpha_0\delta N = \int [\underbrace{(\tilde{\beta}(\boldsymbol{r})-\beta_0)}\delta\tilde{\varepsilon}(\boldsymbol{r}) + \underbrace{(\tilde{\alpha}(\boldsymbol{r})-\alpha_0)}\delta\tilde{\rho}(\boldsymbol{r})]d\boldsymbol{r}$$

Constant temperature: $\forall\,\boldsymbol{r}:\ \tilde{\beta}(\boldsymbol{r}) \doteq \beta_0 \ \Rightarrow\ \boldsymbol{\nabla}\tilde{\beta} \doteq \boldsymbol{0}$

Constant chemical potential: $\forall\,\boldsymbol{r}:\ \tilde{\alpha}(\boldsymbol{r}) \doteq \alpha_0 \ \Rightarrow\ \boldsymbol{\nabla}\tilde{\alpha} \doteq \boldsymbol{0}$

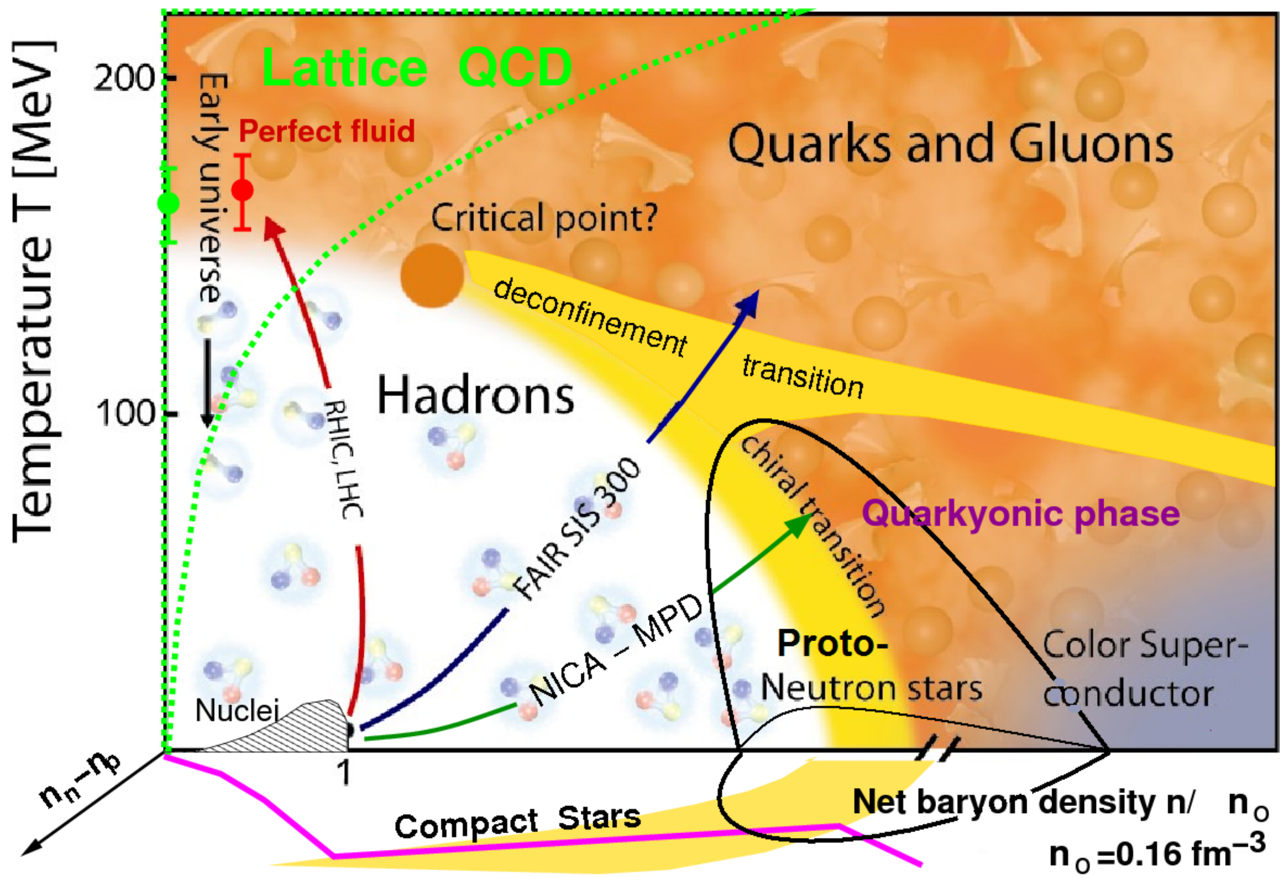
Constant pressure:

$$\delta\pi = -\varepsilon\delta\beta - \rho\delta\alpha \qquad \pi \equiv p/T = \sigma - \beta\varepsilon - \alpha\rho$$

$$\boldsymbol{\nabla}\tilde{\pi} = -\tilde{\varepsilon}\boldsymbol{\nabla}\tilde{\beta} - \tilde{\rho}\boldsymbol{\nabla}\tilde{\alpha}$$ => $$\tilde{p}(\boldsymbol{r}) = p_0$$

Gradient corrections:

$$\begin{cases} \tilde{f}_T(\boldsymbol{r}) = f_T(\tilde{\rho}(\boldsymbol{r})) + \frac{1}{2}C(\boldsymbol{\nabla}\tilde{\rho}(\boldsymbol{r}))^2 \\ p(\boldsymbol{r}) \approx p_0(\varepsilon(\boldsymbol{r}),\rho(\boldsymbol{r})) - C\rho_0\nabla^2\rho(\boldsymbol{r}) \end{cases}$$



Temperature T [MeV]
200
100
Lattice QCD
Early universe
Perfect fluid
Quarks and Gluons
Critical point?
deconfinement
transition
Hadrons
RHIC, LHC
FAIR SIS 300
NICA – MPD
chiral transition
Quarkyonic phase
Proto-
Neutron stars
Color Super-
conductor
Nuclei
1
n_n-n_p
Compact Stars
Net baryon density n/ n_o
n_o=0.16 fm^{-3}

Application to relativistic nuclear collisions

Equations of state

Two-phase EoS => Interface tension

One-phase EoS (Maxwell partner)

Fluid dynamics

Gradient term => Spinodal amplification

Collisions

Density moments: Enhancement

Hadron Gas *versus* *Quark-Gluon Plasma*

$$p^H = p_\pi + p_N + p_{\bar{N}} + p_w$$

Free pions, nucleons, and antinucleons:

$$p_\pi(T) = -g_\pi T \int_{m_\pi}^{\infty} \frac{p\varepsilon d\varepsilon}{2\pi^2} \ln[1 - \mathrm{e}^{-\beta\varepsilon}]$$
$$p_N(T) = -g_N T \int_{m_\pi}^{\infty} \frac{p\varepsilon d\varepsilon}{2\pi^2} \ln[1 + \mathrm{e}^{-\beta(\varepsilon-\mu_0)}]$$
$$p_{\bar{N}}(T) = -g_N T \int_{m_\pi}^{\infty} \frac{p\varepsilon d\varepsilon}{2\pi^2} \ln[1 + \mathrm{e}^{-\beta(\varepsilon+\mu_0)}]$$

+ compressional energy density:

$$w(\rho) = \left[-A\left(\frac{\rho}{\rho_s}\right)^\alpha + B\left(\frac{\rho}{\rho_s}\right)^\beta\right]\rho$$

$$p_w(\rho) = \rho^2\partial_\rho(w(\rho)/\rho)$$

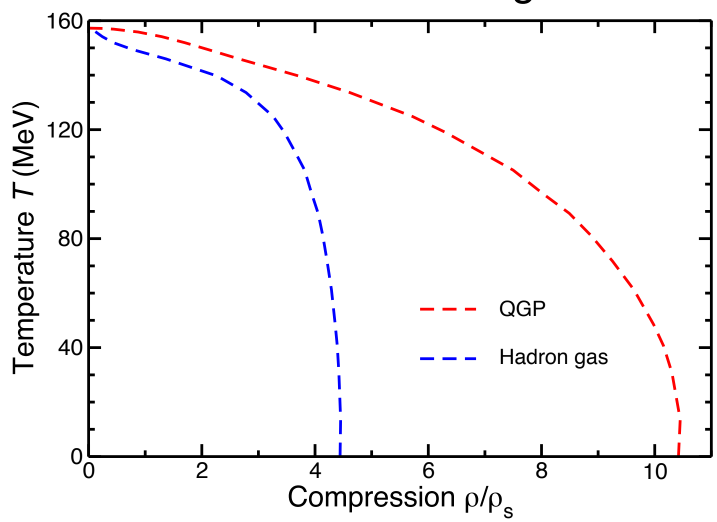
$$\mu = \mu_0 + \partial_\rho w = 3\mu_q$$

$$p^Q = p_g + p_q + p_{\bar{q}} - B$$

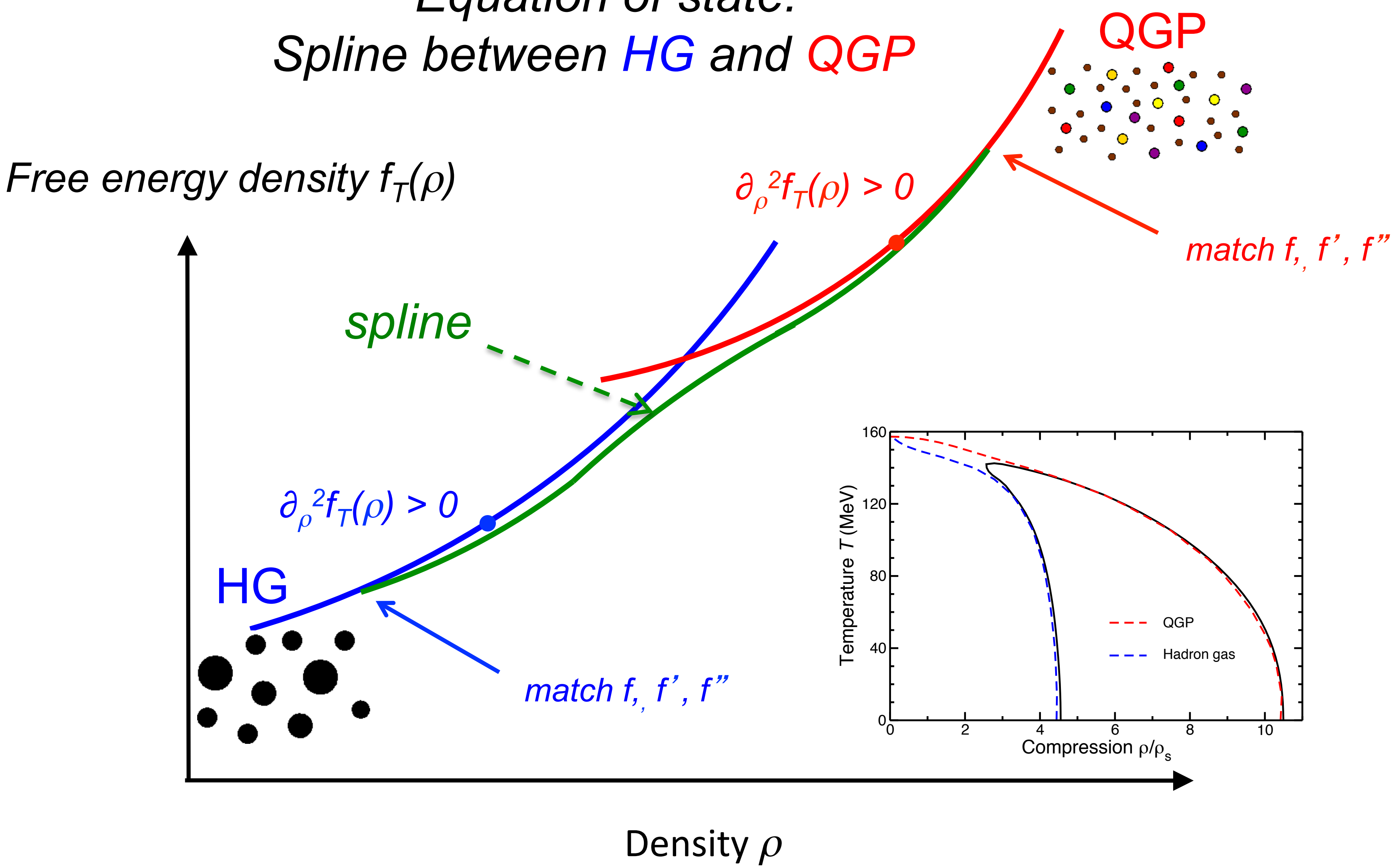
Free gluons, quarks, and antiquarks:

$$p_g = g_g \frac{\pi^2}{90} T^4$$
$$p_q + p_{\bar{q}} = g_q \left[\frac{7\pi^2}{360}T^4 + \frac{1}{12}\mu_q^2 T^2 + \frac{1}{24\pi^2}\mu_q^4\right]$$

Phase crossing:



Equation of state:
Spline between HG and QGP



Isothermal speed of sound

Microcanonical representation: entropy density

$$\sigma(\varepsilon,\rho) \qquad \left.\begin{array}{l} \beta = \partial_\varepsilon \sigma(\varepsilon,\rho) = \sigma_\varepsilon = 1/T \\ \alpha = \partial_\rho \sigma(\varepsilon,\rho) = \sigma_\rho = -\mu/T \end{array}\right\} \quad \pi = \sigma - \beta\varepsilon - \alpha\rho = p/T$$

$$v_T^2 = \frac{\rho}{h}\left(\frac{\partial p}{\partial \rho}\right)_T = -\frac{\rho}{h}\frac{\rho T}{\sigma_{\varepsilon\varepsilon}}[\sigma_{\varepsilon\varepsilon}\sigma_{\rho\rho} - \sigma_{\varepsilon\rho}^2]$$

What happens at T=0?

Canonical representation: free-energy density

$$f = \varepsilon - T\sigma \quad \left\{\begin{array}{lll} \mu_T(\rho) = \partial_\rho f_T(\rho) & \longrightarrow & p_T(\rho) = \rho\partial_\rho f_T(\rho) - f_T(\rho) \\ \sigma_T(\rho) = -\partial_T f_T(\rho) & \longrightarrow & \varepsilon_T(\rho) = f_T(\rho) - T\partial_T f_T(\rho) \end{array}\right.$$

Use $\partial_\rho p_T(\rho) = \partial_\rho f_T(\rho) + \rho\partial_\rho^2 f_T(\rho) - \partial_\rho f_T(\rho) = \rho\partial_\rho\mu_T(\rho)$ to get $v_T^2 = \frac{\rho^2}{h}\partial_\rho\mu_T$:

Example: Free gluons, quarks, and antiquarks

Gluons:

(g_g = 16)

$$\begin{aligned}
p_T^g &= g_g \frac{\pi^2}{90} T^4 = \frac{8}{45}\pi^2 T^4 , \\
\varepsilon_T^g &= 3p_T^g = g_g \frac{\pi^2}{30} T^4 = \frac{8}{15}\pi^2 T^4 , \\
h_T^g &= p_T^g + \varepsilon_T^g = 4p_T^g = g_g \frac{2}{45}\pi^2 T^4 = \frac{32}{45}\pi^2 T^4 , \\
\sigma_T^g &= \partial_T p_T^g = \frac{p_T^g + \varepsilon_T^g}{T} = \frac{4p_T^g}{T} = g_g \frac{2}{45}\pi^2 T^3 = \frac{32}{45}\pi^2 T^3 , \\
f_T^g &= \varepsilon_T^g - T\sigma_T^g = 3p_T^g - 4p_T^g = -p_T^g .
\end{aligned}$$

Quarks:

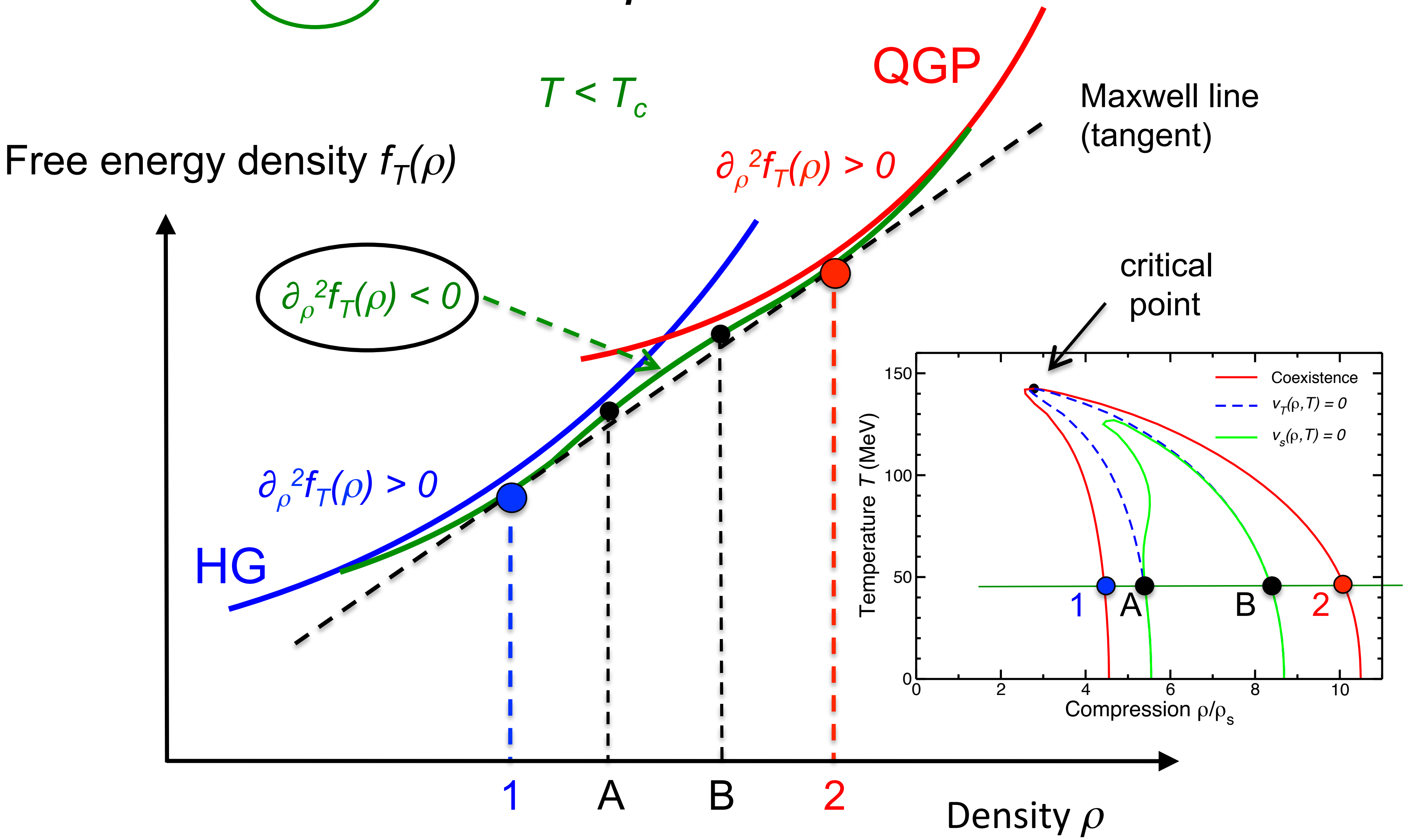
(g_q = 12)

$$\begin{aligned}
\rho_T^q &= \tfrac{1}{3} \times g_q \frac{\mu_q}{6}\left[T^2 + \frac{\mu_q^2}{\pi^2}\right] = \frac{1}{54} g_q \mu_B \left[T^2 + \frac{\mu_B^2}{9\pi^2}\right] = \frac{2}{9}\mu T^2 + \frac{2}{81\pi^2}\mu^3 , \\
p_T^q &= g_q \left[\frac{7\pi^2}{360} T^4 + \frac{1}{12}\mu_q^2 T^2 + \frac{1}{24\pi^2}\mu_q^4\right] = \frac{7}{30}\pi^2 T^4 + \frac{1}{9}\mu^2 T^2 + \frac{\mu^4}{162\pi^2} , \\
\varepsilon_T^q &= g_q \left[\frac{7\pi^2}{120} T^4 + \frac{1}{4}\mu_q^2 T^2 + \frac{1}{8\pi^2}\mu_q^4\right] = \frac{7}{10}\pi^2 T^4 + \frac{1}{3}\mu^2 T^2 + \frac{\mu^4}{54\pi^2} , \\
h_T^q &= p_T^q + \varepsilon_T^q = g_q \left[\frac{7\pi^2}{90} T^4 + \frac{1}{3}\mu_q^2 T^2 + \frac{1}{6\pi^2}\mu_q^4\right] = \frac{14}{15}\pi^2 T^4 + \frac{4}{9}\mu^2 T^2 + \frac{2}{81\pi^2}\mu^4 , \\
\sigma_T^q &= \partial_T p_T^q = \frac{1}{T}[h_T^q - \mu_B \rho_T^q] = g_q \frac{T}{6}\left[\frac{7\pi^2}{15} T^2 + \mu_q^2\right] = \frac{14}{15}\pi^2 T^3 + \frac{2}{9}\mu^2 T , \\
f_T^q &= \varepsilon_T^q - T\sigma_T^q = g_q\left[-\frac{7\pi^2}{360} T^4 + \frac{1}{12}\mu_q^2 T^2 + \frac{1}{8\pi^2}\mu_q^4\right] = -\frac{7}{30}\pi^2 T^4 + \frac{1}{9}\mu^2 T^2 + \frac{\mu^4}{54\pi^2} .
\end{aligned}$$

$$v_T^2 = \frac{\rho^2}{h}\partial_\rho \mu_T = \frac{(g_q[T^2\mu + \mu^3/9\pi^2]/54)^2 \;/\; (g_q[T^2 + \mu^2/3\pi^2]/54)}{g_g 2\pi^2 T^4/45 + g_q[7\pi^2 T^4/90 + \mu^2 T^2/27 + \mu^4/486\pi^2]}$$

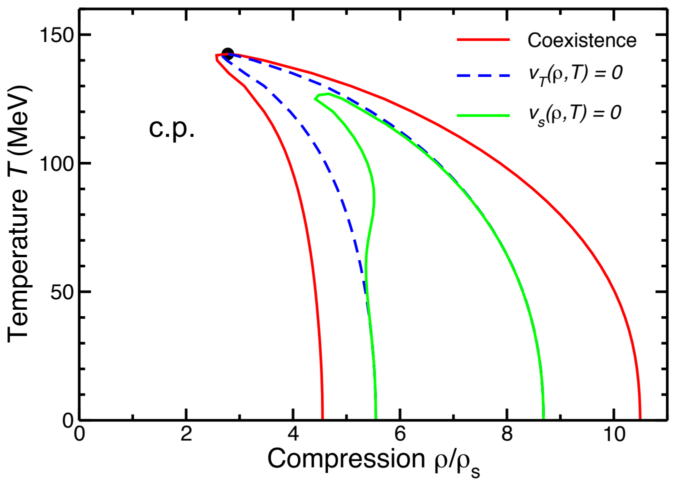
= ⅓ for $T = 0$

Two-Phase Equation of State

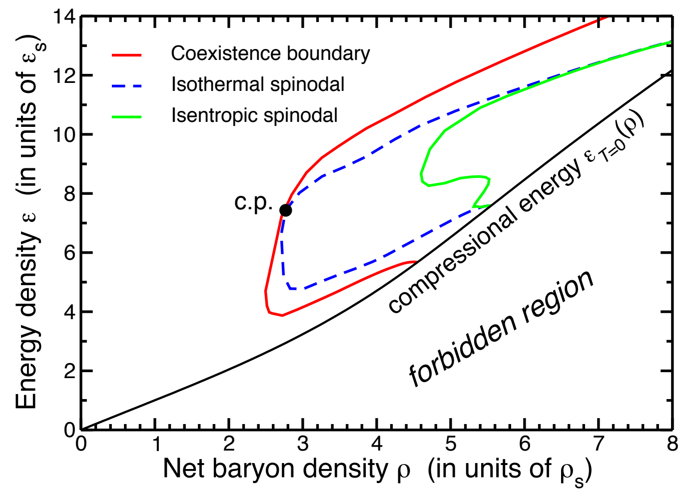


Phase diagrams

(ρ,T)



(ρ,ε)



The spline construction yields $f_T(\rho)$ ⇨ Fluid dynamics needs $p(\varepsilon,\rho)$

First tabulate $p_T(\rho)$ and $\varepsilon_T(\rho)$,
then get $p(\rho,\varepsilon)$ by interpolation

$$p_T(\rho) = \rho\partial_\rho f_T(\rho) - f_T(\rho)$$
$$\varepsilon_T(\rho) = f_T(\rho) - T\partial_T f_T(\rho)$$

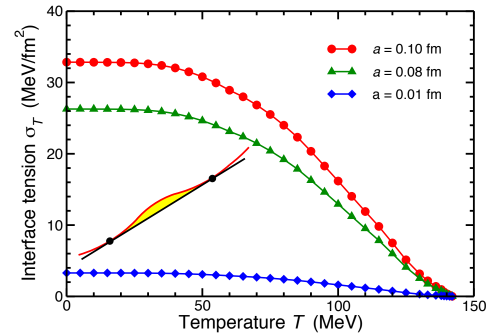
Finite-range (ideal) fluid dynamics

Gradient term in free energy density:

$$\tilde{f}_T(\boldsymbol{r}) \;=\; f_T(\tilde{\rho}(\boldsymbol{r})) + \tfrac{1}{2}C(\boldsymbol{\nabla}\tilde{\rho}(\boldsymbol{r}))^2$$

$$\gamma_T = \int_{\rho_1}^{\rho_2} \left\{2C[f_T(\rho) - f_T^M(\rho)]\right\}^{1/2} d\rho$$

Interface tension $\gamma(T)$



=> gradient term in the pressure:

Laplace operator

$$\tilde{p}(\boldsymbol{r}) = p_0(\tilde{\rho}(\boldsymbol{r}), \tilde{\varepsilon}(\boldsymbol{r})) - C\tilde{\rho}(\boldsymbol{r})\boldsymbol{\Delta}\tilde{\rho}(\boldsymbol{r})$$

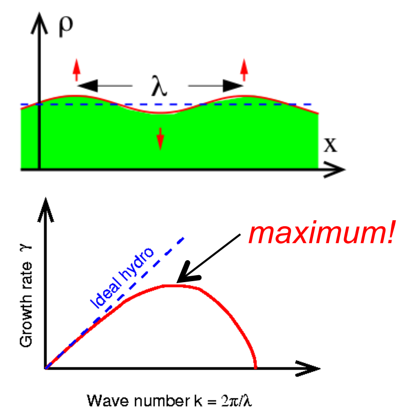
$$\rho(r,t) = \rho_0 + \delta\rho(x,t) \doteq \;\rho_0 + \rho_k \mathrm{e}^{ikx - i\omega t}$$

$$p_k \;\longrightarrow\; p_k + C\rho_0 k^2 \rho_k$$

$$\gamma_k^2 = |v_s^2|k^2 - C\frac{\rho_0^2}{\varepsilon_0 + p_0}k^4$$

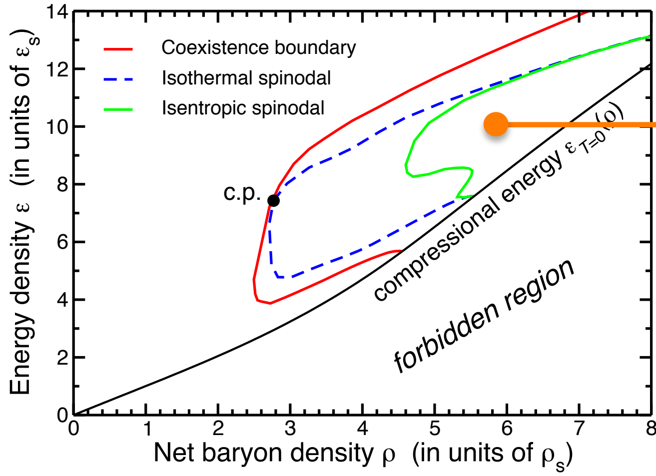
- easy to insert into a fluid dynamic transport code

Spinodal dispersion relation



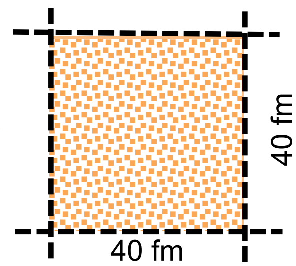
Unstable matter in a box

Phase diagram in (ρ,ε) plane



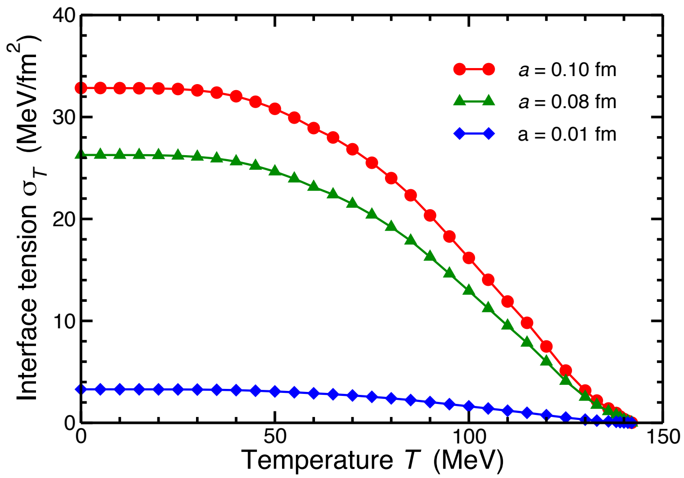
$$\tilde{f}_T(\boldsymbol{r}) \;=\; f_T(\tilde{\rho}(\boldsymbol{r})) + \tfrac{1}{2}C(\boldsymbol{\nabla}\tilde{\rho}(\boldsymbol{r}))^2 \quad \Rightarrow$$

$$\tilde{p}(\boldsymbol{r}) = p_0(\tilde{\rho}(\boldsymbol{r}), \tilde{\varepsilon}(\boldsymbol{r})) - C\tilde{\rho}(\boldsymbol{r})\boldsymbol{\Delta}\tilde{\rho}(\boldsymbol{r})$$



$$C = \frac{\varepsilon_s}{\rho_s^2}\, a^2$$

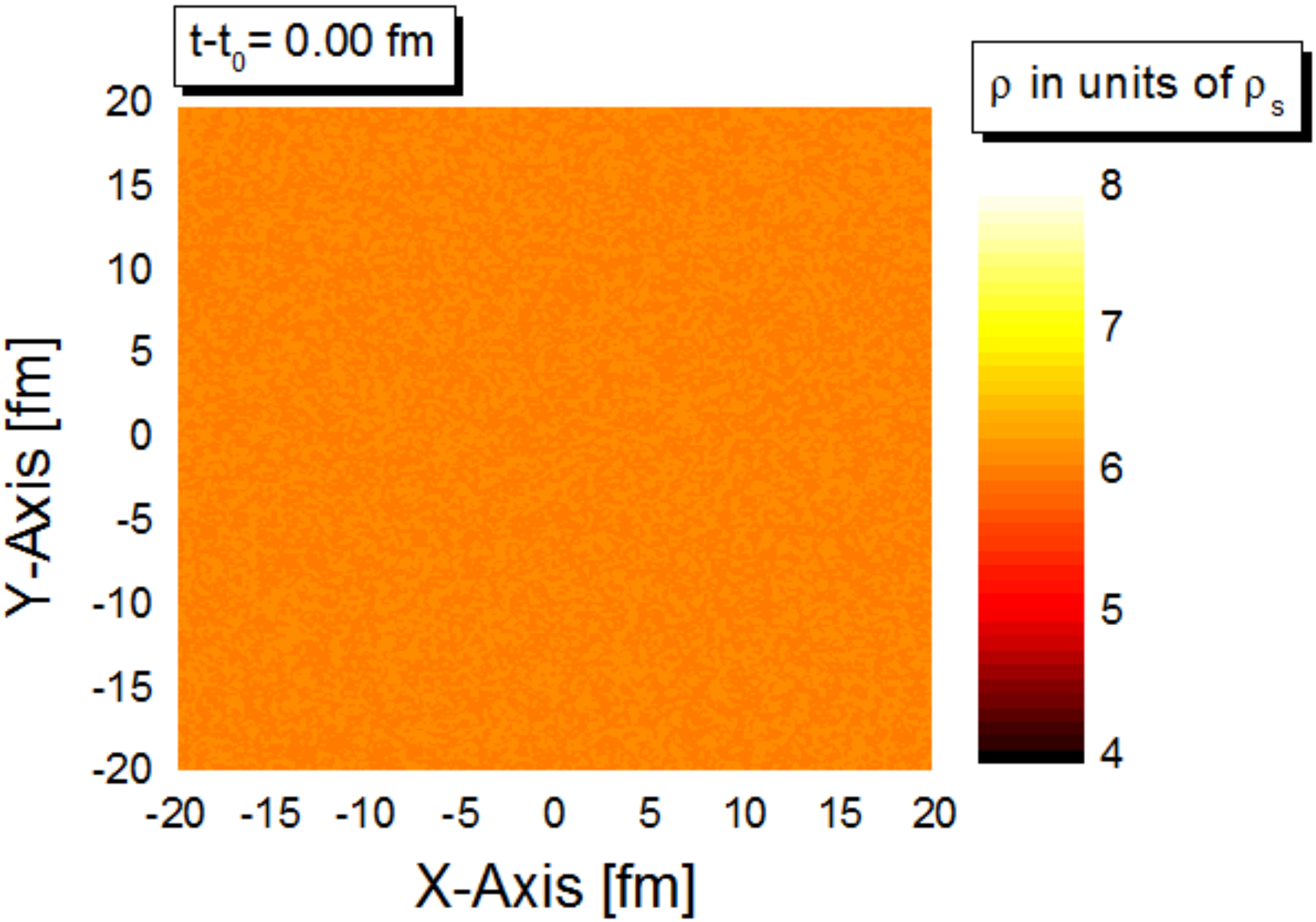
Interface tension



Amplification of irregularities?

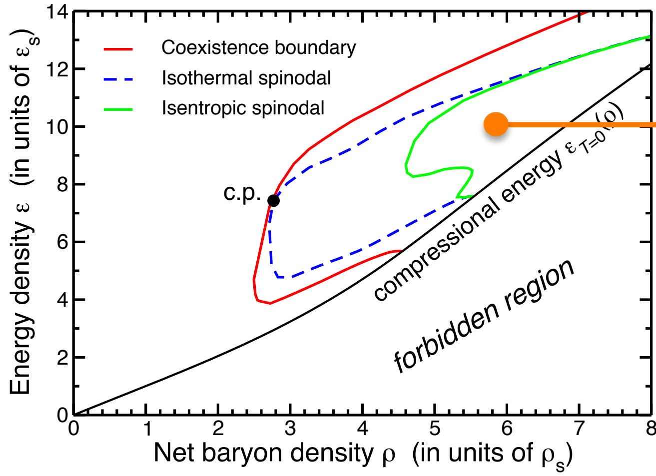
J. Steinheimer & JR, PRL 109, 212301 (2012)

J. Steinheimer & JR, PRC 87, 054903 (2013)


t-t0= 0.00 fm
ρ in units of ρs
20
15
10
5
0
-5
-10
-15
-20
Y-Axis [fm]
-20 -15 -10 -5 0 5 10 15 20
X-Axis [fm]
8
7
6
5
4

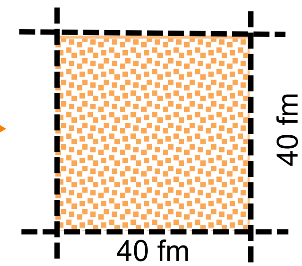
Unstable matter in a box

Phase diagram in (ρ,ε) plane



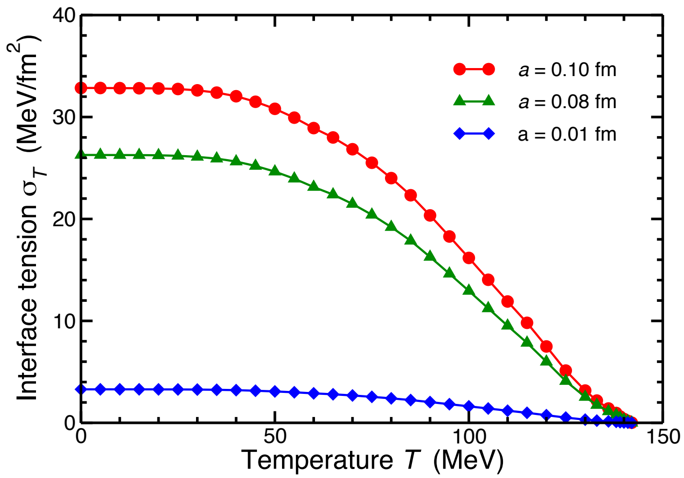
$$\tilde{f}_T(\boldsymbol{r}) = f_T(\tilde{\rho}(\boldsymbol{r})) + \tfrac{1}{2}C(\boldsymbol{\nabla}\tilde{\rho}(\boldsymbol{r}))^2 \quad \Rightarrow$$

$$\tilde{p}(\boldsymbol{r}) = p_0(\tilde{\rho}(\boldsymbol{r}),\tilde{\varepsilon}(\boldsymbol{r})) - C\tilde{\rho}(\boldsymbol{r})\boldsymbol{\Delta}\tilde{\rho}(\boldsymbol{r})$$

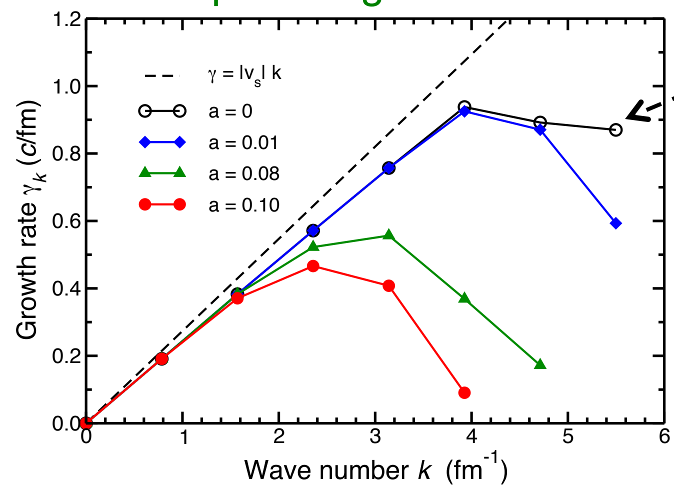


$$C = \frac{\varepsilon_s}{\rho_s^2}\, a^2$$

Interface tension

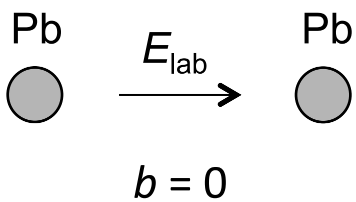


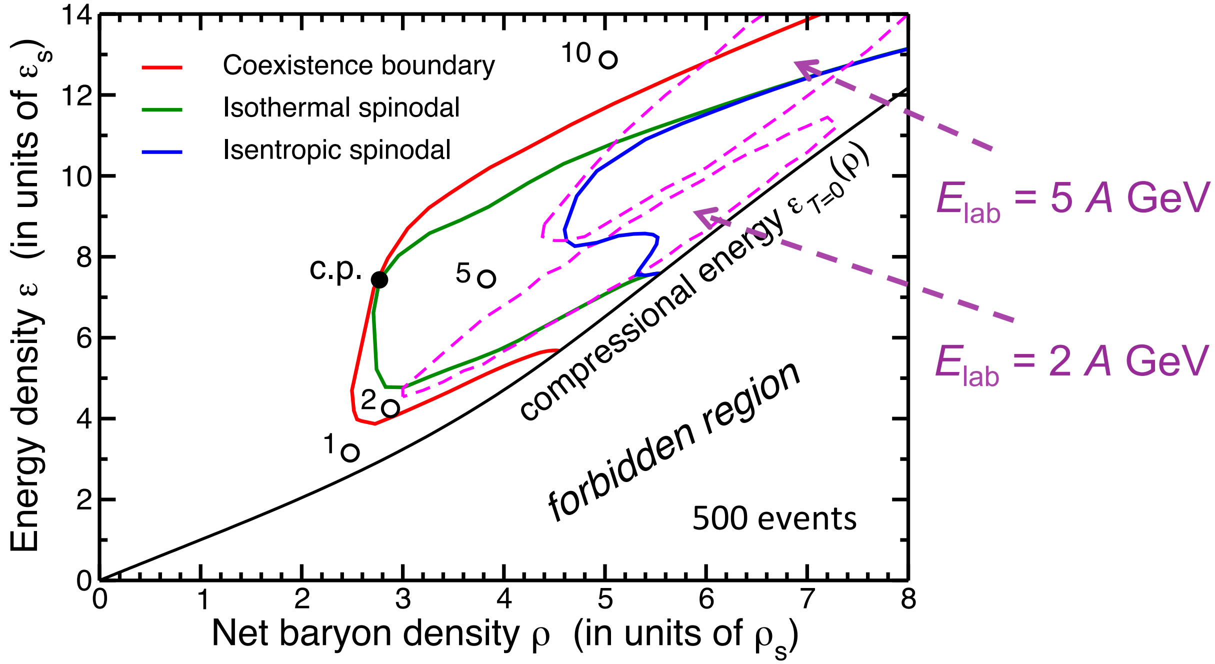
Spinodal growth rate



numerical viscosity

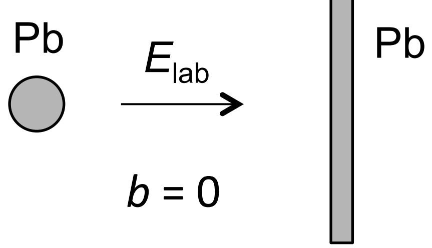
Maximum compression in Pb+Pb collisions

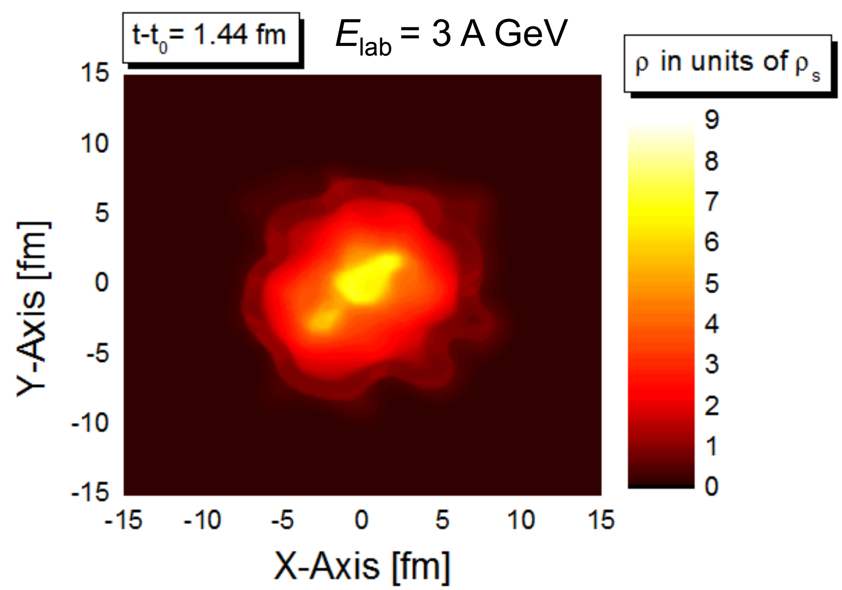




=> Optimal energy range: 2-4 *A* GeV?

Evolution of the (net) baryon density $\rho(\boldsymbol{r},t)$



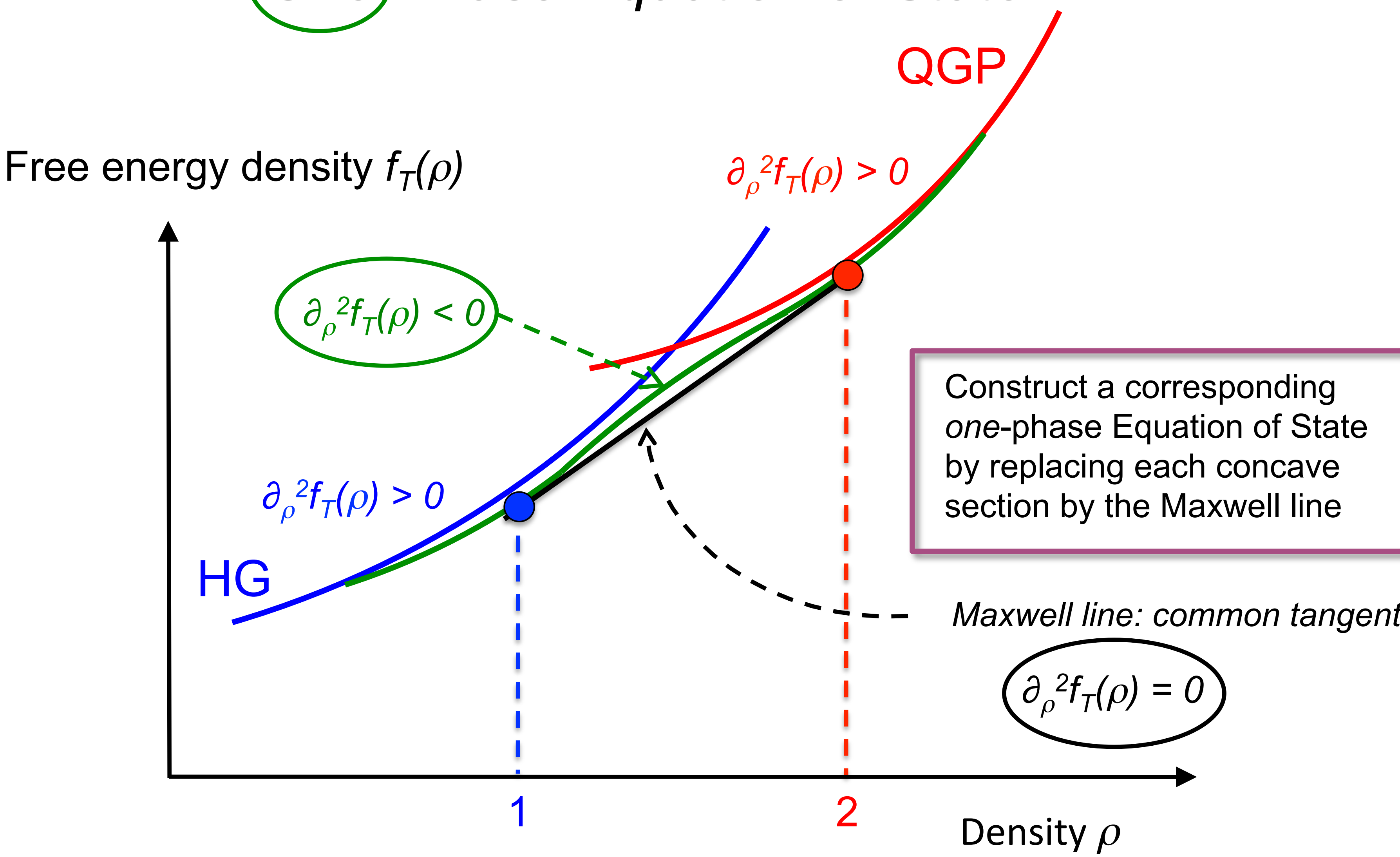


Total baryon number: $$A = \int \rho(\boldsymbol{r})\, d^3\boldsymbol{r}$$

N^{th} density moment: $$\langle \rho^N \rangle \equiv \frac{1}{A^N} \int \rho(\boldsymbol{r})^N \rho(\boldsymbol{r})\, d^3\boldsymbol{r}$$

Mean baryon density: $$\langle \rho \rangle \equiv \frac{1}{A} \int \rho(\boldsymbol{r})\, \rho(\boldsymbol{r})\, d^3\boldsymbol{r} = \langle \rho^{N=1} \rangle$$

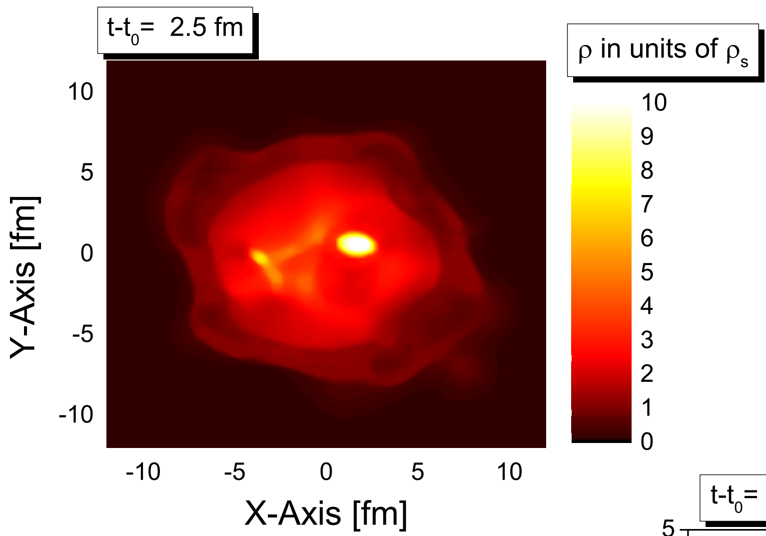
One-Phase Equation of State



3 *A* GeV Pb + Pb (*b*=0)

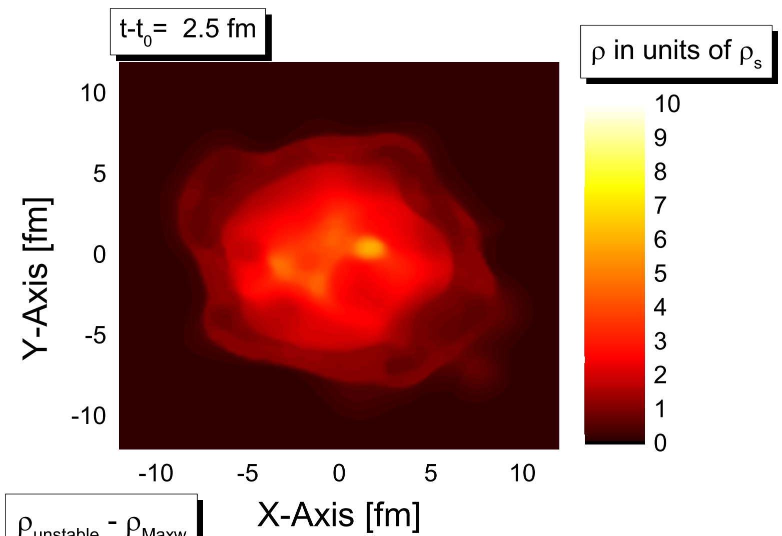
Identical initial conditions

Two-phase EoS:
unstable

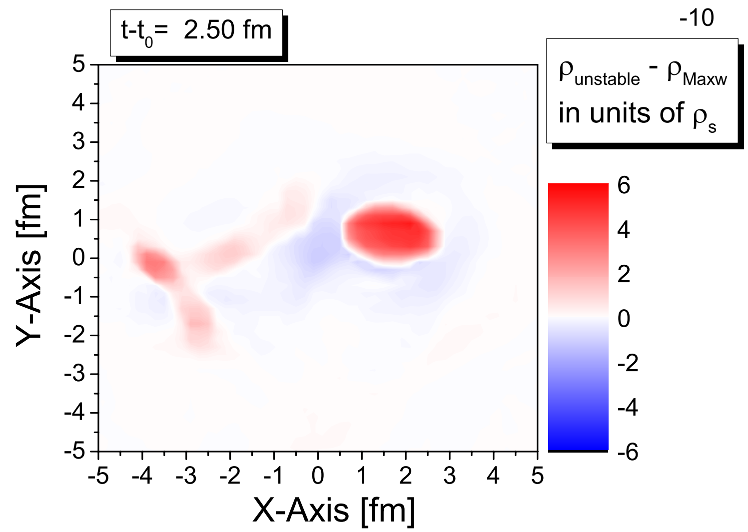


Density ρ(x,y,z=0)

Maxwell EoS:
stable



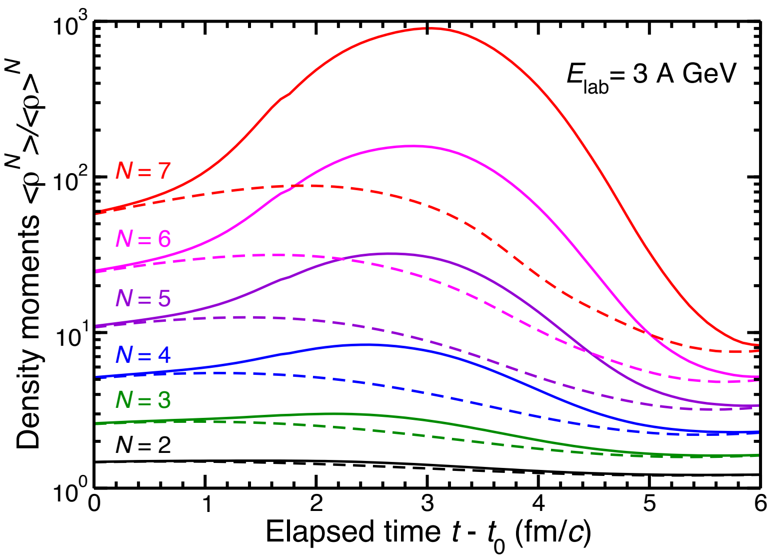
Difference:
unstable - stable

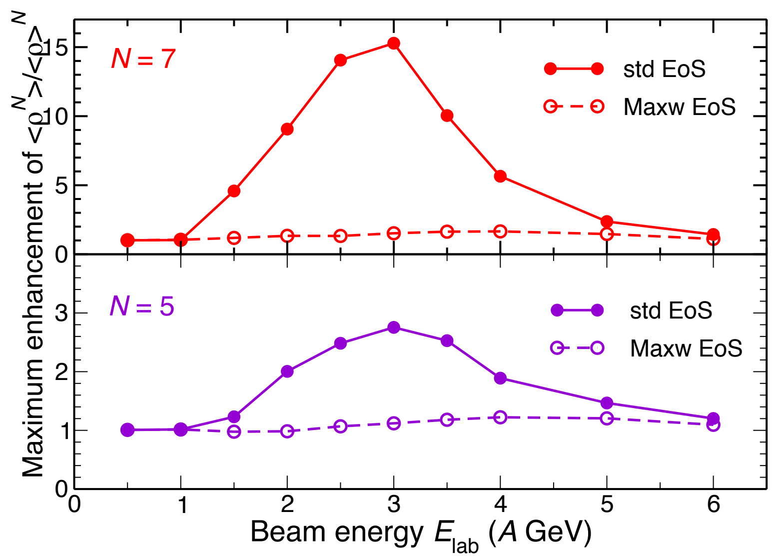


Evolution of the density moments

Density moment: $$\langle \rho^N \rangle \equiv \frac{1}{A^N} \int \rho(\boldsymbol{r})^N \rho(\boldsymbol{r})\, d^3\boldsymbol{r} \qquad A = \int \rho(\boldsymbol{r})\, d^3\boldsymbol{r}$$

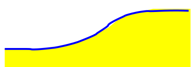
Normalized moment: $\langle \rho^N \rangle / \langle \rho \rangle^N$ (dimensionless)





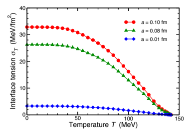
Spinodal Instabilities at the Deconfinement Phase Transition

SUMMARY



I: Phase coexistence

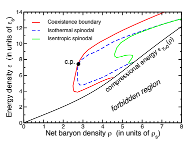
$$\tilde{f}_T(\boldsymbol{r}) \;=\; f_T(\tilde{\rho}(\boldsymbol{r})) + \tfrac{1}{2}C(\boldsymbol{\nabla}\tilde{\rho}(\boldsymbol{r}))^2$$



Thermodynamics of non-uniform systems

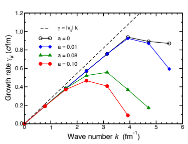
Gradient correction

Interface tension between coexisting phases



II: Phase separation

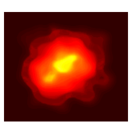
Two-phase equation of state ← *Needed:* theoretical equation of state

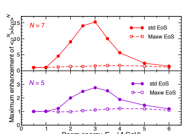


Growth rates of spinodal instabilities

III: Collision dynamics

$$\tilde{p}(\boldsymbol{r}) = p_0(\tilde{\rho}(\boldsymbol{r}), \tilde{\varepsilon}(\boldsymbol{r})) - C\tilde{\rho}(\boldsymbol{r})\boldsymbol{\Delta}\tilde{\rho}(\boldsymbol{r})$$





Finite-range fluid dynamics ← *Needed:* finite-range dissipative fluid dynamics, several flavors

Collisions: density enhancements, optimal energy

Big question: Is the presence of a phase transition experimentally visible?