# An Introduction to the Thermal Model. II. Theory

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# Outline

# Excluded Volume Corrections. Motivation using Bag Model.

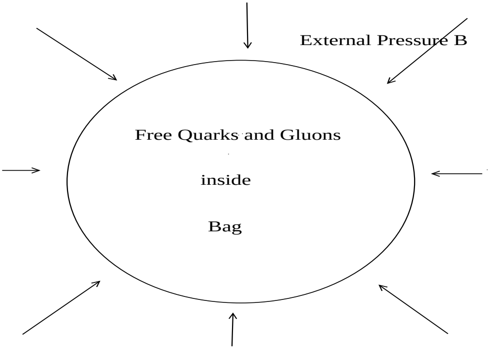
In the bag model the energy density of quarks and gluons is given by

$$\epsilon = \epsilon(\text{free}) + B$$

while the pressure is given by

$$P = P(\text{free}) - B$$

# Phase transition in the Bag



Consider two limiting cases:

- Finite $T$ and zero $\mu$
- Zero $T$ and finite $\mu$

# Bag Model with zero $\mu$ and finite $T$

In the hadronic phase, for a gas of massless pions:

$$P_h = 3\frac{\pi^2}{90}T^4$$

In the QGP phase, for a gas of massless quarks and gluons:

$$\begin{aligned} P_{qgp} &= \left\{2 \times 8 + \frac{7}{8}(3 \times 2 \times 2 \times 2)\right\}\frac{\pi^2}{90} - B \\ &= 37\frac{\pi^2}{90}T^4 - B \end{aligned}$$

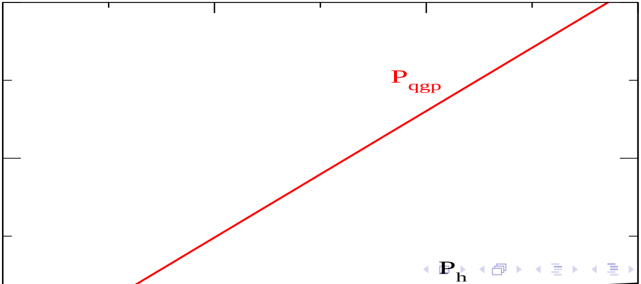
Crossing point is given by:

$$T_c^4 = \frac{1}{34}\frac{90}{\pi^2}B$$

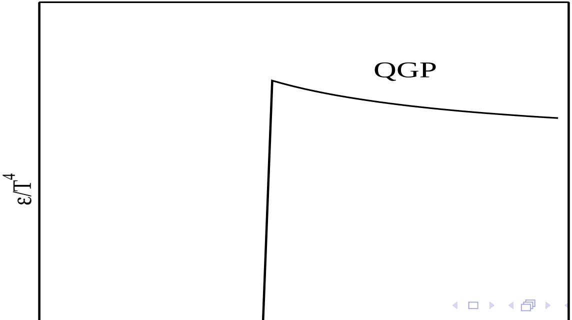
From hadron spectroscopy the bag pressure is given by $B^{1/4} \approx$ 0.2 GeV, so that

$$T_c \approx 145 \quad \text{MeV}$$

# Phase transition in the Bag Model at zero $\mu$



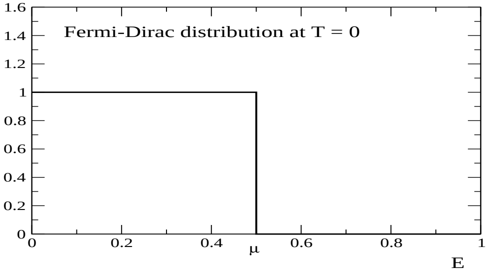
# Phase transition in the Bag Model at zero $\mu$



# Bag Model with zero $\mu$ and finite $T$

Look at the case where the temperature goes to zero

$$\begin{aligned}\lim_{T\to 0}\frac{1}{e^{(E-\mu)/T}+1} &= \frac{1}{e^{(E-\mu)/0}+1}\\ &= \frac{1}{e^{\infty}+1}=0 \quad \textit{if} \quad \mu > E\\ &= \frac{1}{e^{-\infty}+1}=1 \quad \textit{if} \quad \mu < E\end{aligned}$$



The pressure at zero temperature is given by

$$P = g \int_0^{\sqrt{\mu^2-m^2}} \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E}$$

For massless particles, or at very high chemical potential (high density)

$$\begin{aligned} P &= g \frac{4\pi}{(2\pi)^3} \int_0^{\mu} p^2 dp \, \frac{p}{3} \\ P &= g \frac{4\pi}{(2\pi)^3} \frac{\mu^4}{12} \\ &= g \frac{1}{24\pi^2} \mu^4 \end{aligned}$$

at large values of $\mu$ this leads to:

$$\begin{aligned}
\lim_{\mu\to\infty} P(\text{quarks}) &= 2x2x3x\frac{1}{24\pi^2}\left(\frac{\mu}{3}\right)^4 - B \\
&= \frac{4}{27}\frac{1}{24\pi^2}\mu^4
\end{aligned}$$

and

$$\lim_{\mu\to\infty} P(\text{nucleons}) = 4\frac{1}{24\pi^2}\mu^4$$

i.e. the hadronic phase dominates at very high values of $\mu$. This is not acceptable physically.

At large values of $\mu$

$$\lim_{\mu\to\infty} P(\text{quarks}) < P(\text{nucleons})$$

i.e. the system reverts back to the nucleon phase at very high densities.
Nucleon Phase -> Quark Phase -> Nucleon Phase
Excluded volume corrections prevent this from happening.
This has been implemented in all the thermal model codes.

Relation between grand canonical and canonical ensembles:

$$Z_{GC}(T,V,\mu) = \sum_{N=0}^{\infty} e^{\frac{\mu N}{T}} Z_C(T,V,N)$$

Relation between grand canonical and pressure ensembles:

$$Z_p(T,P,\mu) = \int_0^{\infty} dV \, e^{\frac{PV}{T}} Z_{GC}(T,V,\mu)$$

# Excluded Volume Corrections.

$$
\begin{aligned}
Z &= \exp\left\{ V \int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}+\frac{\mu}{T}} \right\} \\
&= \sum_{N=0}^{\infty} \frac{V^N}{N!} e^{\mu N/T} \left[ \int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N
\end{aligned}
$$

with excluded volume corrections

$$
\begin{aligned}
Z \rightarrow & \sum_{N=0}^{\infty} \frac{(V - V_0 N)^N}{N!} e^{\mu N/T} \\
& \left[ \int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N \theta(V - V_0 N)
\end{aligned}
$$

# Excluded Volume Corrections.

It is more convenient to consider these corrections in the pressure ensemble:

$$Z_p \equiv \int_0^\infty dV\, e^{-PV/T} \sum_{N=0}^\infty \frac{V^N}{N!} e^{\mu N/T} \left[ \int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N$$

$$Z_p \rightarrow \sum_{N=0}^\infty \int_0^\infty dV\, e^{-PV/T} \frac{(V - V_0 N)^N}{N!} e^{\mu N/T} \left[ \int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N \theta(V - V_0 N)$$

introduce $x \equiv V - V_0 N$.

# Excluded Volume Corrections.

$$
\begin{aligned}
Z_p &= \sum_{N=0}^{\infty} \int_0^{\infty} dx \; e^{-Px/T} \\
&\quad \frac{x^N}{N!} e^{-PV_0 N/T} e^{\mu N/T} \\
&\quad \left[ \int \frac{d^3 p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N
\end{aligned}
$$

a new variable $\bar{\mu} \equiv \mu - PV_0$

# Excluded Volume Corrections.

$$Z_p \rightarrow \sum_{N=0}^{\infty} \int_0^{\infty} dx\ e^{-Px/T}$$
$$\frac{x^N}{N!} e^{\bar{\mu}N/T} \left[ \int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N$$

which is the original partition function with the replacement

$$\boxed{\bar{\mu} = \mu - P\ V_0}$$

D.H. Rischke, M.I. Gorenstein, H. Stöcker, W. Greiner, ZfP C51, 485 (1991). J. C., M.I. Gorenstein, J. Stålnacke and E. Suhonen P. S. 48 277-280 (1993).

# Excluded Volume Corrections.

The particle number density now becomes:

$$
\begin{aligned}
n &= \frac{T}{V}\frac{\partial}{\partial\mu}\ln\ Z \\
&= \frac{T}{V}\frac{\partial\bar{\mu}}{\partial\mu}\frac{\partial}{\partial\bar{\mu}}\ln\ Z \\
&= \frac{\partial\bar{\mu}}{\partial\mu}n_0 \\
&= [1 - V_0\ n]\,n_0
\end{aligned}
$$

$$
\boxed{n = \frac{n_0}{1 + V_0 n_0}}
$$

Effects Cancel Out in Ratios.

J.C., K. Redlich, H. Satz, E. Suhonen, ZfP C33, 151, (1986)
D.H. Rischke, M.I. Gorenstein, H. Stöcker, W. Greiner, ZfP C51, 485 (1991).

# Canonical Corrections.

# Exact Strangeness Conservation.

For a small system at low temperatures ($T \approx 50$ MeV) canonical corrections are necessary.
This was first noted by Hagedorn who argued that instead of

$$N_K \approx \exp -M_K/T$$

one needs in a small system

$$N_K \approx \exp -2M_K/T$$

because of pair production. The extra suppression is due to strangeness conservation and lack of a large heat bath. This correction disappears quickly for large systems and at higher energies.

# Exact Strangeness Conservation.

Impose exact strangeness conservation by inserting a Kronecker delta in the trace:

$$\sum_i n_i(S=1) + 2\sum_j n_j(S=2) + 3\sum_k n_k(3) =$$

$$\sum_i \bar{n}_i(S=-1) + 2\sum_j \bar{n}_j(S=-2) + 3\sum_k \bar{n}_k(S=-3)$$

and rewrite it as

$$\delta\left(\sum_i n_i(S=1) + ...., \sum_i \bar{n}_i(S=-1) + .....\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi$$

$$\exp\left(i\phi\sum_i n_i(S=1) + .. - i\phi\sum_i \bar{n}_i(S=-1)\right)$$

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# Exact Strangeness Conservation.

For simplicity, we reduce the discussion to a gas with only strangeness $\pm 1$ particle present.

$$
\begin{aligned}
&\delta\left(\sum_i n_i(S=1), \sum_i \bar{n}_i(S=-1)\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi \\
&\exp\left(i\phi \sum_i n_i(S=1) \; - i\phi \sum_i \bar{n}_i(S=-1)\right)
\end{aligned}
$$

$$
\begin{aligned}
Z &= \frac{1}{2\pi}\int_0^{2\pi} d\phi \exp\left\{Z_1 e^{i\phi} + Z_{-1} e^{-i\phi}\right\} \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi \exp\left\{\sqrt{Z_1 Z_{-1}}\left[\sqrt{\frac{Z_1}{Z_{-1}}} e^{i\phi} + \sqrt{\frac{Z_{-1}}{Z_1}} e^{-i\phi}\right]\right\}
\end{aligned}
$$

$Z_1$: sum of all particles with strangeness 1, e.g. $K^+$

$Z_{-1}$: sum of all particles with strangeness -1, e.g. $\Lambda$

# Exact Strangeness Conservation.

Use

$$\exp\left\{\frac{x}{2}\left(t+\frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} I_m(x)t^m$$

to obtain

$$Z = \frac{1}{2\pi}\int_0^{2\pi} e^{ip\phi} \sum_{p=-\infty}^{\infty} I_p(x_1)y_1^p$$

where

$$y_1 = \sqrt{\frac{Z_1}{Z_{-1}}} \qquad x_1 = 2\sqrt{Z_1 Z_{-1}}$$

$$Z = I_0(x_1)$$

# Exact Strangeness Conservation.

In more detail, e.g. the multiplicity of $K^+$

$$N_{K^+} = \frac{T}{Z} \frac{\partial I_0(x_1)}{\partial \mu_{K^+}} \bigg|_{\mu_{K^+}=0}$$

Use

$$\frac{d}{dz} I_0(z) = I_1(z)$$

# Exact Strangeness Conservation.

$$\begin{aligned} N_{K^+} &= \frac{T}{Z}\frac{\partial}{\partial \mu_{K^+}} I_0(x_1) \\ &= \frac{T}{I_0(x_1)} I_1(x_1) \frac{\partial x_1}{\partial \mu_{K^+}} \\ &= \frac{T}{I_0(x_1)} I_1(x_1) \frac{\partial 2\sqrt{Z_1 Z_{-1}}}{\partial \mu_{K^+}} \\ &= \frac{I_1(x_1)}{I_0(x_1)} \sqrt{\frac{Z_{-1}}{Z_1}} N^0_{K^+} \end{aligned}$$

where $N^0_{K^+}$ refers to the "unmodified" kaon multiplicity.

# Exact Strangeness Conservation.

In the small volume limit this becomes

$$\lim_{z\to 0} I_0(z) = 1$$

and

$$\lim_{z\to 0} I_1(z) = \frac{z}{2}$$

$$\begin{aligned} \lim_{V\to 0} &= N^0_{K^+} Z_{-1} \\ \lim &= N^0_{K^+} Z_{-1} \\ &= N^0_{K^+}\left[N^0_{K^-} + N^0_{\Lambda} + \cdots\right] \end{aligned}$$

i.e., the particle multiplicity is

- proportional to $V^2$, and not $V^1$.
- proportional to $\exp(-2m_K/T)$ or to $\exp(-(m_K + m_\Lambda)/T)$ and not simply $\exp(-m_K/T)$, i.e. there is additional suppression of strange particles.

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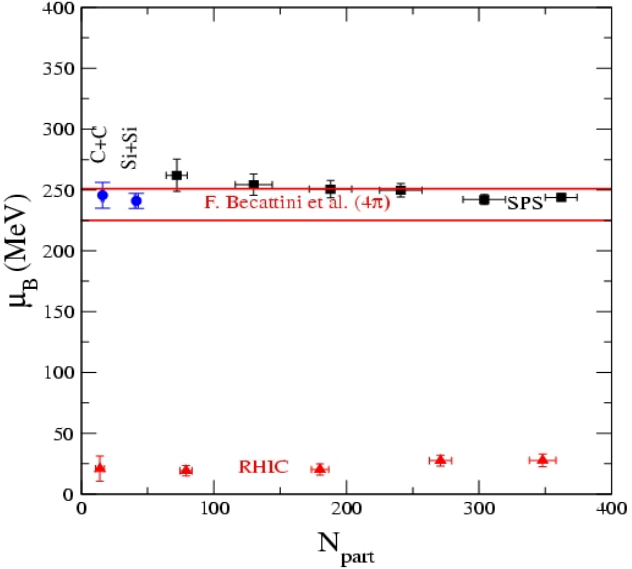
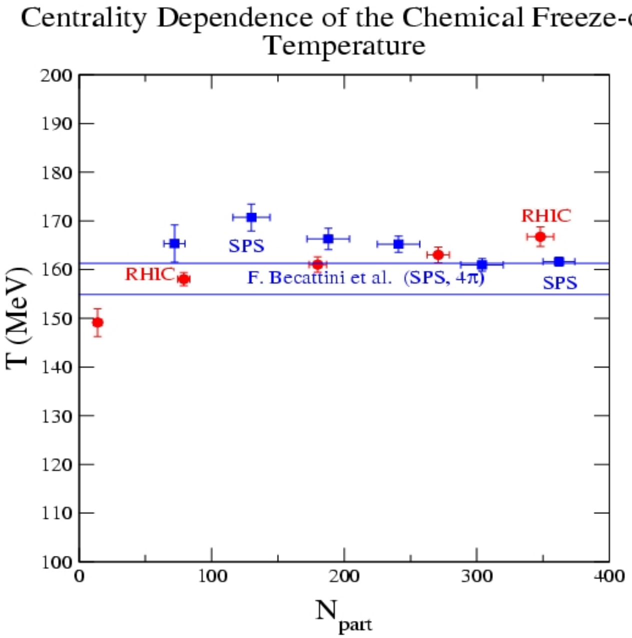
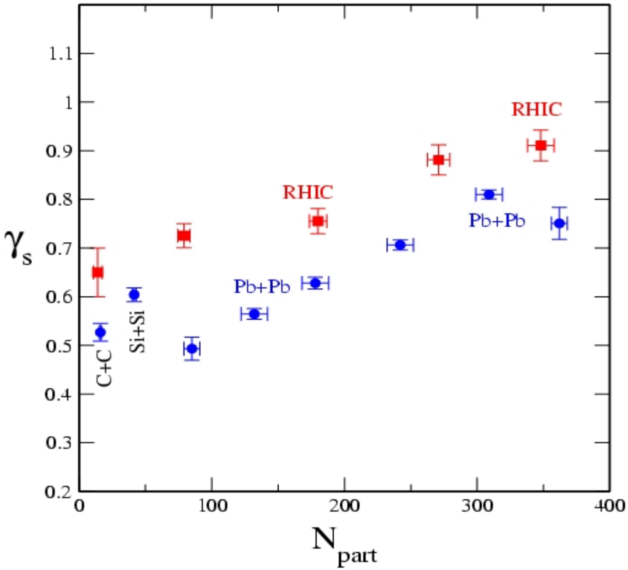
# Centrality Dependence of the Baryon Chemical Potential




Centrality Dependence of the Chemical Freeze-out Temperature
T (MeV)
200
190
180
170
160
150
140
130
120
110
100
0
100
200
300
400
N part
RHIC
SPS
RHIC
F. Becattini et al. (SPS, 4π)
SPS



$\gamma_s$
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
100
200
300
400
$N_{part}$
RHIC
RHIC
Pb+Pb
Pb+Pb
C+C
Si+Si

Hence: strange particles show a very clear dependence on the number of participants.
Introduce a strangeness equilibration radius $R_c$.