Geometric approach to asymptotic expansion of Feynman integrals

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Let us consider a Feynman integral

$$\mathcal{F}(a_1, \ldots, a_n) = \int \cdots \int \frac{\mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h}{E_1^{a_1} \ldots E_n^{a_n}}$$

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It is incorrect to take the limit and evaluate the integral afterwards.

For limits typical of Euclidean space (for example, the off-shell large-momentum limit or the large-mass limit), one can write down the corresponding asymptotic expansion in terms of a sum over certain subgraphs of a given graph (Chetyrkin, Gorishnii, V.Smirnov). This prescription of expansion by subgraphs has been mathematically proven.
For other limits (general case) no known formula exists.

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- Set to zero any scaleless integral.
- Take the sum of non-zero integrals.

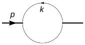
Example

A family of one-loop propagator-type integrals in the Euclidean space:

$$\mathcal{F}(a_1, a_2; p^2, m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{E_1^{a_1} E_2^{a_2}},$$
$$E_1 = k^2 + m^2, \quad E_2 = (k+p)^2 + m^2.$$

Consider the asymptotics of $I_1(a_1, a_2; p^2, m^2)$ in the limit when $|p^2| \gg m^2$, or $t = |m^2/p^2| \ll 1$. The naive Taylor expansion does not capture the complete asymptotic behaviour since the integration variables (components of k) span all values from $-\infty$ to $+\infty$, and in particular may be as small as m or as large as $\sqrt{|p^2|}$.

We need to analyze 3 regions (a-c):



$$\begin{aligned} E_1 &= k^2 + m^2 \\ E_2 &= (k+p)^2 + m^2 \\ t &= |m^2/p^2| \ll 1 \end{aligned}$$

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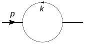
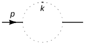
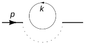
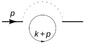
	$E_1 = k^2 + m^2$ $E_2 = (k+p)^2 + m^2$ $t = \lvert m^2/p^2 \rvert \ll 1$
(a) 	$\lvert k^2 \rvert \sim \lvert p^2 \rvert \gg m^2$ $E_1^{(a)} = k^2$ $E_2^{(a)} = (k+p)^2$

(b) 	$\lvert k^2\rvert \sim m^2$ $$E_1^{(b)} = k^2 + m^2$$ $$E_2^{(b)} = p^2$$
(c) 	$\lvert (k+p)^2\rvert \sim m^2$ $$E_1^{(c)} = p^2$$ $$E_2^{(c)} = (k+p)^2 + m^2$$

The sum of integrals of expansions in these 3 regions equals to the expansion of the initial integral.

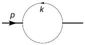
Alpha-representation

The alpha-representation for Feynman integrals has a general structure

$$\mathcal{F}(a_1, ..., a_n) =$$
$$c \int_0^1 dx_1 ... dx_n \; \delta(1 - x_1 - ... - x_n) x_1^{a_1 - 1} ... x_n^{a_n - 1} U^a F^b,$$

where coefficient c and exponents a and b depend only on d, and a_i. U and F are homogeneous polynomials (of order l and $l + 1$, respectively) of integration variables x_i, and F also depends on the kinematic invariants.
The strategy of expansion in kinematic regions may also be formulated in the alpha-representation (V.Smirnov).

Now we append the alpha-representation of the initial integral and its expansion in three regions (a-c):

	$E_1 = k^2 + m^2$ $E_2 = (k+p)^2 + m^2$ $t = \|m^2/p^2\| \ll 1$	$U = x_1 + x_2$ $F = x_1 x_2 (p^2 + 2m^2)$ $\quad + x_1^2 m^2 + x_2^2 m^2$

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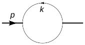
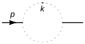
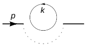
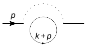
 	$E_1 = k^2 + m^2$ $E_2 = (k+p)^2 + m^2$ $t = \|m^2/p^2\| \ll 1$	$U = x_1 + x_2$ $F = x_1 x_2 (p^2 + 2m^2) + x_1^2 m^2 + x_2^2 m^2$
(a)  	$\|k^2\| \sim \|p^2\| \gg m^2$ $E_1^{(a)} = k^2$ $E_2^{(a)} = (k+p)^2$	$x_1, x_2 \sim t^0$ $U^{(a)} = x_1 + x_2$ $F^{(a)} = x_1 x_2 p^2$

(b) 	$\lvert k^2\rvert \sim m^2$ $E_1^{(b)} = k^2 + m^2$ $E_2^{(b)} = p^2$	$x_1 \sim t^{-1},\ x_2 \sim t$ $U^{(b)} = x_1$ $F^{(b)} = x_1 x_2 p^2 + x_1^2 m^2$
(c) 	$\lvert (k+p)^2\rvert \sim m^2$ $E_1^{(c)} = p^2$ $E_2^{(c)} = (k+p)^2 + m^2$	$x_1 \sim t,\ x_2 \sim t^{-1}$ $U^{(c)} = x_2,$ $F^{(c)} = x_1 x_2 p^2 + x_2^2 m^2$

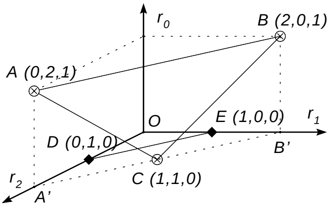
Let us present a method revealing those regions automatically.

Geometric approach

Each term in F corresponds to a vector of $n+1$ exponents:

$$t^{r_0} x_1^{r_1} \ldots x_n^{r_n} \to (r_1, \ldots, r_n, r_0),$$

and F corresponds to a set $w(F)$ of points in $(n+1)$-dimensional vector space.



Graphical representation of sets $w(F = x_1 x_2 (p^2 + 2m^2) + x_1^2 m^2 + x_2^2 m^2)$ (crossed points) and $w(U = x_1 + x_2)$ (diamonds) corresponding to the integral.

If we fix the scales of alpha-parameters as $x_i \sim t^{v_i}$, then the scale of a monomial can be found as $t^{r_0} x_1^{r_1} \ldots x_n^{r_n} \sim t^{r_0 + v_1 r_1 + \ldots + r_n v_n} \sim t^{\vec{r}\vec{v}}$, with $\vec{r} = (r_1, \ldots, r_n, r_0)$ from $w(F)$ and $\vec{v} = (v_1, \ldots, v_n, 1)$.

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Any choice of $\vec{v}$ corresponds to some hierarchy between x_i, but most of such choices lead to zero (scaleless) integrals. The few directions that lead to scaleful integrals determine the regions of expansion that we seek.

The terms in F that remain after the expansion are all characterized by the same scale in powers of t. In terms of the corresponding subset of points $w(F')$, they feature the same value of the projection on $\vec{v}$. In other words, all these points lie in the same hyperplane orthogonal to $\vec{v}$.

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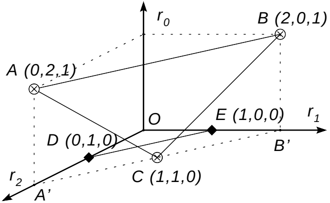
Revealing of regions (=non-trivial scalings) turns to finding the convex hull of points corresponding to monomials.



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In this example we find the corresponding points and vectors (points as denoted in the figure):

a : $\vec{v} = (1, 0, 0)$, $w(F') = (C)$, $w(U') = (D, E)$,

b : $\vec{v} = (1, 1, -1)$, $w(F') = (A, C)$, $w(U') = (D)$,

c : $\vec{v} = (1, -1, 1)$, $w(F') = (B, C)$, $w(U') = (E)$.

Together with Pak we presented (2010) a code `Asy.m` revealing the regions.
The main function is $\mathit{AlphaRepExpand}[ks, ds, cs, hi]$, where
ks is the list of loop momenta (e.g., $\{k\}$),
ds is the list of denominators (e.g., $\{k^2 + m^2, (k+p)^2 + m^2\}$),
cs is the list of constraints (e.g., $\{p^2 \to M^2\}$) and
hi is the list of scalings of kinematic invariants with respect to the small parameter x (e.g., $\{M \to x^0, m \to x^1\}$).

Preresolution

Let us consider the one-loop propagator diagram with two massive lines in the threshold limit, i.e. when $t = m^2 - q^2/4 \to 0$ with q being the external momentum:

$$\mathcal{F}(q^2, m^2) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)\left((k-q)^2 - m^2\right)} ,$$

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we have

$$\begin{aligned} U &= x_1 + x_2 \\ F &= \frac{q^2}{4}(x_1 - x_2)^2 + t(x_1 + x_2)^2 \end{aligned}$$

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In this example let us decompose the integration domain into two subdomains, $x_1 \leq x_2$ and $x_2 \leq x_1$. The two resulting integrals are equal to each other, but such an equality will not generally take place for any integral.

In the first domain we turn to new variables by $x_1 = x_1'/2,\ x_2 = x_2' + x_1'/2$, remove the primes at x_i and obtain the new functions

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The goal of this trick is to make the line $x_1 = x_2$ (in the old variables) the border of an integration domain which turned out to be (in the new variables) $x_2 = 0$.

Such an approach (recursive variable replacements in order to get rid of cancelling terms) was developed together with Jantzen and V.Smirnov, now we can even reveal Glauber regions automatically.

We call this method "preresolution of singularities".

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Step 1. We try all pairs of variables and for each of those try to divide the integration domain into two (either $x_i > cx_j$ or $x_i < cx_j$). We have some methods of finding the best possible constant c for each pair of variables.

For each of those pairs we make an appropriate variable replacement, so that in new coordinates the integration domain is equal to the initial one. We calculate the error level for both new "sectors".

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For all physical examples we tried we result in getting rid of terms that can get cancelled out.

Original paper with Pak: Geometric approach to asymptotic expansion of Feynman integrals, Eur. Phys. J. C71 2011, 1626; arXiv: 1011.4863.

New paper with Jantzen and V.Smirnov: Expansion by regions: revealing potential and Glauber regions automatically; arXiv: 1206.0546.

Url: `http://science.sander.su/`.