On Determining the Running Coupling from the Effective Action

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Abstract

Renormalization scale μ leads to a conformal anomaly. The trace of the energy-momentum tensor is proportional to the renormalization group β-function. The effective action for a gauge theory can be written in terms of the running gauge coupling when considered as a function of a strong background field [Dunne, Gies and Schubert, 2006]. At the same time, the effective action satisfies the renormalization group equation, which leads to explicit summation of all its leading-log (LL), next-to-leading-log (NLL) etc. contributions [McKeon, 2011]. We compare these two different expressions for the effective action to obtain a novel expression for the running gauge coupling.

Outline

1. Radiative Corrections to the Scalar Effective Potential
2. Gauge Fields in a strong background constant field

O(N)-symmetric scalar model with a massless potential $V_{\rm cl} = \lambda\phi^4$, radiative corrections [Coleman and Weinberg, 1973]

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \lambda^{n+1} T_{n,m} L^m \phi^4 \tag{1}$$

$L = \ln(\phi^2/\mu^2)$ and the CW RG condition

$$\left.\frac{\mathrm{d}^4 V}{\mathrm{d}\phi^4}\right|_{\phi=\mu} = 24\lambda \tag{2}$$

The n-loop contribution to V fix the coefficients $T_{n,m}$ $(m \leq n)$. V is independent of the unphysical renormalization scale parameter μ if

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \gamma(\lambda)\phi\frac{\partial}{\partial\phi}\right) V(\lambda,\phi,\mu) = 0. \tag{3}$$

Example

$$V = \lambda\phi^4 + \lambda^2 \left(T_{10} + T_{11} L\right) + \lambda^3 \left(T_{20} + T_{21} L + T_{22} L^2\right) \phi^4 \cdots \quad (4)$$

The RG functions $\beta(\lambda) = \mu\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}$, $\gamma(\lambda) = \frac{\mu}{\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\mu}$ are needed to find the implicit dependence of V on μ. Upon expanding

$$\beta(\lambda) = \sum_{k=2}^{\infty} b_k \lambda^k, \tag{5}$$

$$\gamma(\lambda) = \sum_{k=1}^{\infty} g_k \lambda^k \tag{6}$$

b_{k-1} and g_k come from k-loop considerations
$V = \sum_{n=0}^{\infty} \lambda^{n+1} S_n(\xi)\phi^4$ with

$$S_n(\xi) = \sum_{m=0}^{\infty} T_{n+m,m}\xi^m, \qquad \xi = \lambda L \tag{7}$$

then eq. [3] is satisfied at order λ^{n+2} if

$$\left[(-2+b_2\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}+(b_2+4g_1)\right]S_0(\xi)=0,$$

and

$$\left[(-2+b_2\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}+(n+1)b_2+4g_1\right]S_n(\xi)$$

$$+\sum_{m=0}^{n-1}\left[(2g_{n-m}+b_{n-m+2}\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}+(m+1)b_{n+2-m}+4g_{n+1-m}\right]S_m(\xi)$$

These nested equations can be solved in turn for $S_0,\ S_1,\ S_2,\ \cdots,$ with the boundary conditions $S_n(0) = T_{n,0}$; in particular

$$S_0 = \frac{T_{0,0}}{w}, \tag{9}$$

$$S_1 = -\frac{4g_2 T_{0,0}}{b_2 w} + \frac{4g_2 T_{0,0} + b_2 T_{1,0}}{b_2 w^2} - \frac{b_3 T_{0,0}}{b_2 w^2} \ln|w| \tag{10}$$

where

$$w = 1 - \frac{b_2}{2}\xi. \tag{11}$$

The sum for $S_n(\xi)$ gives the total $\mathrm{N}^n\mathrm{LL}$ contribution to V; it contains portions of the p-loop contribution to V for all p.

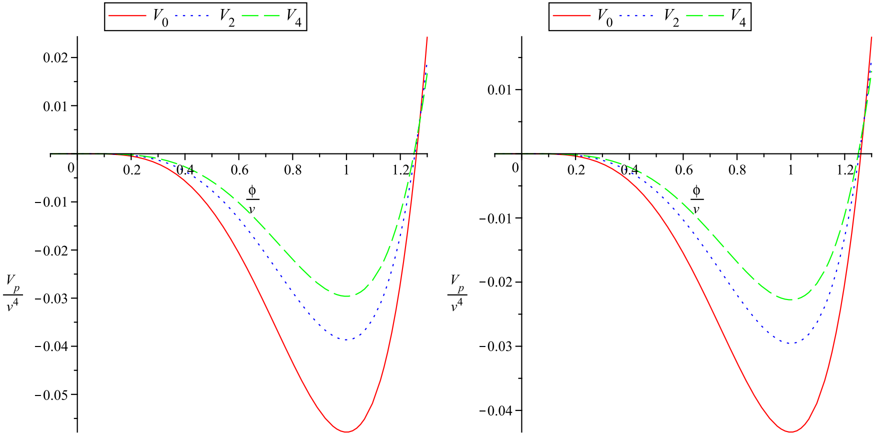

Figure: The effective potential for N=1 (left) and N=4 (right) O(N) $\lambda\phi^4$ theory at different order p in the CW scheme. [1005.1936]

If the effective Lagrangian L is treated as a function of μ (the renormalization scale), $F_{\mu\nu}$ (the constant background field strength) and λ (the gauge coupling), then we have the RG equation:

$$\mu \frac{dL}{d\mu} = \left(\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + \gamma(\lambda) F_{\mu\nu} \frac{\partial}{\partial F_{\mu\nu}} \right) L(\lambda, F_{\mu,\nu}, \mu) = 0. \quad (12)$$

Since $\lambda F_{\mu\nu}$ is not renormalized it follows that $\beta(\lambda) = -\lambda\gamma(\lambda)$ and

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \left(\frac{\partial}{\partial \lambda} - \frac{2}{\lambda} \Phi \frac{\partial}{\partial \Phi} \right) \right] L = 0, \quad (13)$$

where $\Phi = F_{\mu\nu} F^{\mu\nu}$.

For strong background fields ($\lambda\Phi \gg \mu^2$)

$$L = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} T_{n,m} \lambda^{2n} t^m \Phi \qquad (14)$$

where $t = \frac{1}{4} \ln \left(\frac{\lambda^2 \Phi}{\mu^4} \right)$. If $S_n(\lambda^2 t) = \sum_{m=0}^{\infty} T_{n+m,m} (\lambda^2 t)^m$ ($n = 0$ is LL, $n = 1$ is NLL etc.), then eq. [14] leads to the nested equations ($n = 0, 1, 2 \ldots$)

$$-\frac{d}{d\xi} S_n(\xi) + 2 \sum_{\rho=0}^{n} b_{2\rho+3} \left[\xi \frac{d}{d\xi} + (n - \rho - 1) \right] S_{n-\rho} = 0 \qquad (15)$$

where $\beta(\lambda) = \sum_{\rho=0}^{\infty} b_{2\rho+3} \lambda^{2\rho+3}$ and $\xi = \lambda^2 t$. The boundary condition for these equations is $S_n(\xi = 0) = T_{n,0}$.

$$S_0 = -T_{0,0}w \tag{16}$$

$$S_1 = \frac{T_{0,0}b_5}{b_3}\ln|w| + T_{1,0} \tag{17}$$

$w = -1 + 2b_3\xi$

Connection with Conformal Anomaly

$$\langle \Theta^{\mu}_{\mu} \rangle = \frac{\beta(\bar{\lambda}(t))}{2\bar{\lambda}(t)} \frac{\lambda(t)^2}{\bar{\lambda}(t)^2}. \tag{18}$$

Using the effective Lagrangian for a constant background field

$$\langle \Theta^{\mu\nu} \rangle = -\eta^{\mu\nu} L + 2 \frac{\partial L}{\partial \eta_{\mu\nu}} \tag{19}$$

$$L = -\frac{1}{4} \frac{\lambda_0^2}{\bar{\lambda}^2(t)} \Phi \tag{20}$$

where the running coupling $\bar{\lambda}(t)$ satisfies

$$\frac{d\bar{\lambda}(t)}{dt} = \beta(\lambda(\bar{t})) \qquad (\bar{\lambda}(t=0) = \lambda_0) \tag{21}$$

Eq. (20) satisfies the expression for conformal anomaly provided $\mu = \mu_0$ is fixed. (14) and (20) should be consistent. To find the boundary conditions, an extra condition must be found. $L = \sum_{n=0}^{\infty} A_n(\lambda) t^n \Phi$ where $A_n = \sum_{m=n}^{\infty} T_{m,n}\lambda^{2m}$. Eq. (13) is now satisfied at each order in t provided

$$\frac{1}{\lambda^2} A_{n+1}(\lambda) = \frac{1}{n+1}\beta(\lambda)\frac{d}{d\lambda}\left(\frac{1}{\lambda^2}A_n(\lambda)\right). \tag{22}$$

If now $A_n(\lambda) = \lambda^2 \overline{A}_n(\lambda)$ and $\eta = \int_{\lambda_0}^{\lambda(\eta)} \frac{dx}{\beta(x)}$ then

$$\overline{A}_{n+1}(\lambda(\eta)) = \frac{1}{(n+1)!}\frac{d^{n+1}}{d\eta^{n+1}}\overline{A}_0(\lambda(\eta)) \tag{23}$$

so that

$$L = \lambda^2(\eta)\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{d^n}{d\eta^n}\overline{A}_0(\lambda(\eta))\Phi = \lambda^2(\eta)\overline{A}_0(\lambda(\eta+t))\Phi \tag{24}$$

$$L = \frac{\lambda^2(\eta)}{\lambda^2(\eta+t)}A_0(\lambda(\eta+t))\Phi\,. \tag{25}$$

$$T_{n,0} = -\frac{1}{4}\delta_{n,0}. \tag{26}$$

New!

Furthermore

$$\bar{\lambda}^2(t) = \frac{-\lambda_0^2}{4}\left[\sum_{n=0}^{\infty} S_n(\lambda_0^2 t)\lambda_0^{2n}\right]^{-1} . \tag{27}$$

More explicitly

$$\bar{\lambda}^2(t) = \lambda_0^2\Bigg[(1-2b_3\lambda_0^2) + \lambda_0^2\left(\frac{b_5}{b_3}\ln\left|-1+2b_3\lambda_0^2 t\right|\right) \tag{28}$$
$$+\lambda_0^4\left(\frac{b_7}{b_3}\frac{2b_3\lambda_0^2 t}{-1+2b_3\lambda_0^2 t} - \left(\frac{b_5}{b_3}\right)^2\frac{\ln\left|-1+2b_3\lambda_0^2 t\right| + 2b_3\lambda_0^2 t}{-1+2b_3\lambda_0^2 t}\right) + \dots\Bigg]^{-1}$$

By letting $x = \bar{\lambda}^2$ and $2b_{2\rho+3} = \beta_\rho(\rho = 0, 1, 2 \ldots)$ so that eq. (21) becomes

$$\frac{dx}{dt} = x^2(\beta_0 + \beta_1 x + \beta_2 x^2 + \ldots) \tag{29}$$

Rescaling $t \to t/\epsilon, x \to \epsilon x$ $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$ $(x_n(t = 0) = x\delta_{n,0})$ at successive orders in ϵ,

$$\frac{dx_0}{dt} = \beta_0 x_0^2 \tag{30a}$$

$$\frac{dx_1}{dt} = \beta_0 x_0^2 + 2\beta_1 x_0 x_1 \tag{30b}$$

$$\frac{dx_2}{dt} = \beta_0(x_1^2 + 2x_0 x_2) + 3\beta_1 x_1 x_0^2 + \beta_4 x_0^4 \tag{30c}$$

An alternate approach is to systematically solve eq. (21) is to write (in analogy with eq. (14))

$$x(\mu_0) = x(\mu) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tau_{n,m} x^n(\mu) \ln^m \left(\mu^2/\mu_0^2\right) \tag{31a}$$

$$\equiv \sum_{n=0}^{\infty} \sigma_n(\zeta) x^{n+1}(\mu) \qquad (\sigma_n(0) = \delta_{n0}) \tag{31b}$$

where $\zeta = x(\mu) \ln \left(\mu^2/\mu_0^2\right)$. If now $\beta(x) = x^2 \sum_{n=0}^{\infty} \beta_n x^n$ and

$$\mu^2 \frac{d}{d\mu^2} x(\mu_0) = 0 \tag{32a}$$

$$\mu^2 \frac{d}{d\mu^2} x(\mu) = \beta\left(x(\mu)\right) \tag{32b}$$

Using the expansion

$$S_n(\xi) = \sum_{i=0}^{n} \sum_{j=0}^{i} S_{ij}^{n} \frac{L^j}{w^{i-1}}; \tag{33}$$

we find that

$$(j+1)S_{i,j+1}^{n} + (n-i)S_{ij}^{n} + \sum_{\rho=1}^{n-1} \chi_{2\rho+3} \Big[(j+1)S_{i-1,j+1}^{n-\rho} - (i-2)S_{i-1,j}^{n-\rho} + (j+1)S_{i,j+1}^{n-\rho} + (n-\rho-i)S_{ij}^{n-\rho} \Big] = 0, \tag{34}$$

where $\chi_{2\rho+3} = b_{2\rho+3}/b_3 \ (\rho = 1, 2 \ldots)$.

The final result for L/Φ

$$L/\Phi = \frac{1}{4}\left[w - \chi_5 \ln w\lambda^2 + (\chi_5^2 - \chi_7)\left(\frac{1+w}{w}\right)\lambda^4 \right. \tag{35}$$

$$-\frac{\lambda^4}{w}\frac{1}{1-\lambda^2 \ln w/w}\left(\lambda^2 \ln w/w - \ln\left(1-\lambda^2 \ln w/w\right)\right)$$

$$\left. +\frac{\lambda^6}{w}\left(\chi_7\chi_5 - \chi_5^3\right)\left(1-\lambda^2\chi_5 \ln w/w\right)^{-1}\right]$$

where $N^p LL$.

What about other $N^p LL$ corrections?

Thank you

Figure: Your Questions Please