Muon $g-2$ from τ and e^+e^- data

– a simple exercise in effective field theory –

Fred Jegerlehner
HU Berlin/DESY Zeuthen,
fjeger@physik.hu-berlin.de

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Abstract

The energy dependence of the $\rho-\gamma$ mixing in the 2×2 $\gamma-\rho$ propagator matrix, is shown to be able to account for the e^+e^- vs. τ spectral function discrepancy. Consequences for the muon $g-2$ are discussed.

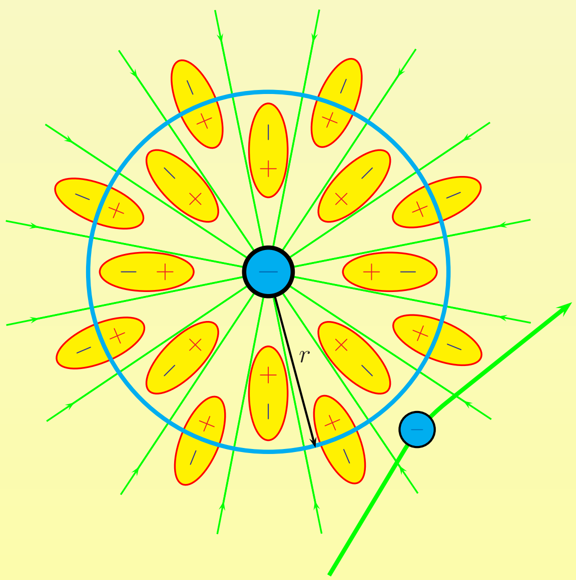
Outline of Talk:

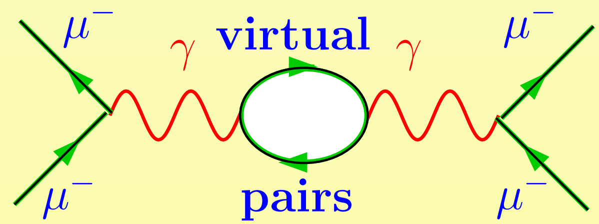
- Prelude: The hadronic vacuum polarization contribution to the muon $g-2$
- The τ vs. e^+e^- problem
- A minimal model for $\gamma-\rho$ mixing: VMD + sQED
- $F_\pi(s)$ with $\rho-\gamma$ mixing at one-loop
- Applications: a_μ and $B^{\mathrm{CVC}}_{\pi\pi^0} = \Gamma(\tau \to \nu_\tau \pi\pi^0)/\Gamma_\tau$
- Summary and Conclusions
- Future

Prelude

❒ Vacuum Polarization, Charge Screening and Running $\alpha_{\rm em}(E)$

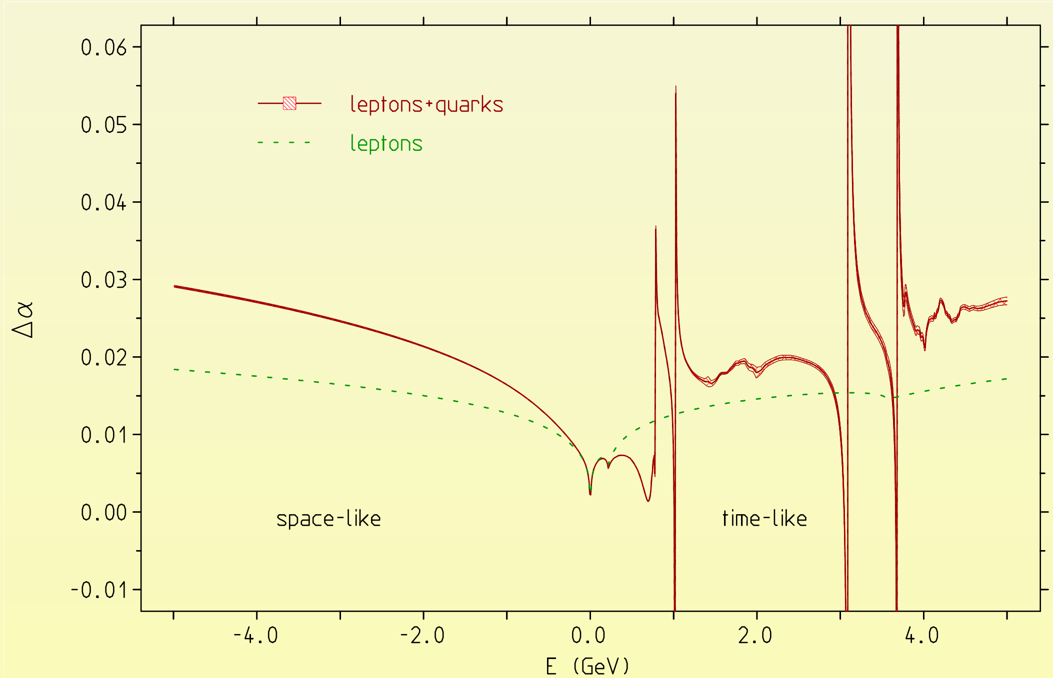
In any QFT quantum vacuum fluctuation:





$\gamma^* \to e^+e^-, \mu^+\mu^-, \tau^+\tau^-, u\bar{u}, d\bar{d}, \cdots \to \gamma^*$

Vacuum polarization causing charge screening by virtual pair creation and re-annihilation



Shift of the effective fine structure constant $\Delta\alpha$ as a function of the energy scale in the time–like region $s > 0$ $(E = \sqrt{s})$ vs the space–like region $-s > 0$ $(E = -\sqrt{-s})$. The band indicates the uncertainties

- leptonic loops calculable in perturbation theory
- quark loops at low energy in effect are hadronic fluctuation like pion pair creation resonances like $\rho, \omega, \phi, J/\psi, \Upsilon$ etc.

Working in physical on-shell renormalization scheme

➠ bottom - up approach

❒ OS vs $\overline{\mathrm{MS}}$ scheme discussed by Misha Kalmykov and Andrei Kataev

Photon propagator: on-shell $s = q^2 = 0$

- vacuum polarization affects $\rightarrow$ dressed propagator

$- \rightarrow$ geometrical progression of self–energy insertions $-\mathrm{i}\Pi_\gamma(q^2)$ $\leftarrow -$

- corresponding Dyson summation: free propagator $\rightarrow$ dressed

$$\mathrm{i}D_\gamma^{\mu\nu}(q) = \frac{-\mathrm{i}g^{\mu\nu}}{q^2 + \mathrm{i}\varepsilon} \rightarrow \mathrm{i}D_\gamma^{\prime\,\mu\nu}(q) = \frac{-\mathrm{i}g^{\mu\nu}}{q^2 + \Pi_\gamma(q^2) + \mathrm{i}\varepsilon}$$

modulo unphysical gauge dependent terms.

- $U(1)_{\mathrm{em}}$ gauge invariance $\rightarrow$ photon remains massless: $\Pi_\gamma(0) \equiv 0$

$$\Pi_\gamma(q^2) = \Pi_\gamma(0) + q^2\,\Pi'_\gamma(q^2) = q^2\,\Pi'_\gamma(q^2)$$

$$\mathrm{i}D_\gamma^{\prime\mu\nu}(q) = \frac{-\mathrm{i}g^{\mu\nu}}{q^2\,(1+\Pi'_\gamma(q^2))} + \text{ gauge terms}$$

“gauge terms” will not contribute to gauge invariant physical quantities, and need not be considered further.

Including a factor e^2 and considering the renormalized propagator (wave function renormalization factor Z_γ) we have

$$\mathrm{i}\,e^2\,D_\gamma^{\prime\mu\nu}(q) = \frac{-\mathrm{i}g^{\mu\nu}\,e^2\,Z_\gamma}{q^2\,\left(1+\Pi'_\gamma(q^2)\right)} + \text{ gauge terms}$$

which in effect means that the charge has to be replaced by a **running charge**

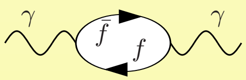
$$e^2 \to e^2(q^2) = \frac{e^2 Z_\gamma}{1+\Pi'_\gamma(q^2)}\quad .$$

The wave function renormalization factor Z_γ is fixed by the condition that at $q^2 \to 0$ one obtains the classical charge (charge renormalization in the *Thomson limit*.

⇒renormalized charge

$$e^2 \to e^2(q^2) = \frac{e^2}{1 + (\Pi'_\gamma(q^2) - \Pi'_\gamma(0))}$$

where the lowest order diagram in perturbation theory which contributes to $\Pi'_\gamma(q^2)$ is



➠ fine structure constant $\alpha = \frac{e^2}{4\pi}$

$$\alpha(q^2) = \frac{\alpha}{1 - \Delta\alpha} \quad ; \quad \Delta\alpha = -\mathrm{Re}\left(\Pi'_\gamma(q^2) - \Pi'_\gamma(0)\right) \ .$$

agrees with solution of Renormalization Group equations in leading log approximation!

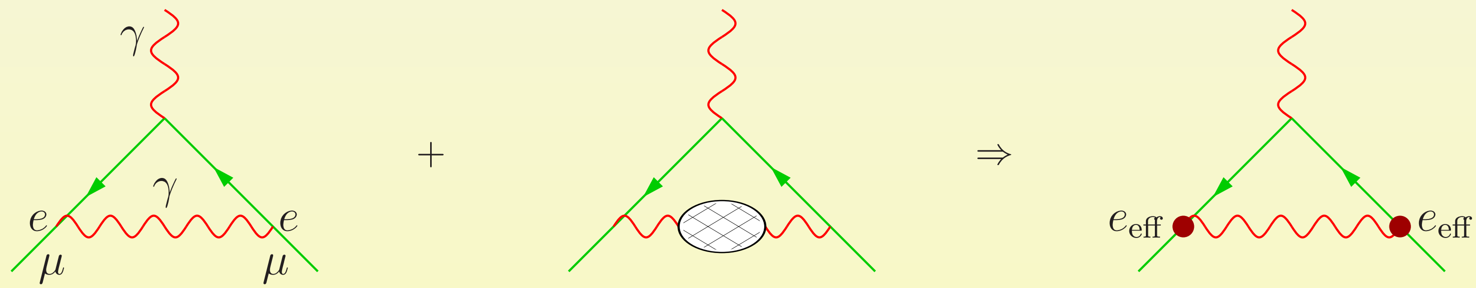
● various contributions to the shift in the fine structure constant come from the leptons (lep = e, μ and τ) the 5 light quarks (u, d, s, c, and b and the corresponding hadrons = had) and from the top quark:

$$\Delta\alpha = \Delta\alpha_{\rm lep} + \Delta^{(5)}\alpha_{\rm had} + \Delta\alpha_{\rm top} + \cdots$$

focus in following:

$$\boxed{\Delta^{(5)}\alpha_{\rm had}}$$

Muon $g-2$: $\alpha \to \alpha_{\rm eff}(m_\mu)$

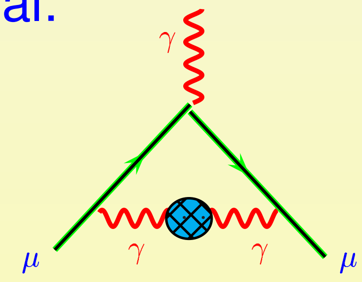


- at scale m_μ the dominant hadronic contribution is the $\pi\pi$ channel
- it is non-perturbative; pions do not exist in perturbative QCD, they are the quasi Nambu-Goldstone bosons and a consequence of spontaneous chiral symmetry breaking
- photon vacuum polarization is related to cross-section of
$$e^+e^- \to \gamma^* \to \text{hadrons}$$
by analyticity (causality) and the optical theorem (unitarity)
- also high quality $\tau \to \nu_\tau \pi^+ \pi^0$ may be used but how precisely?

How to Evaluate $a_\mu^{\rm had}$

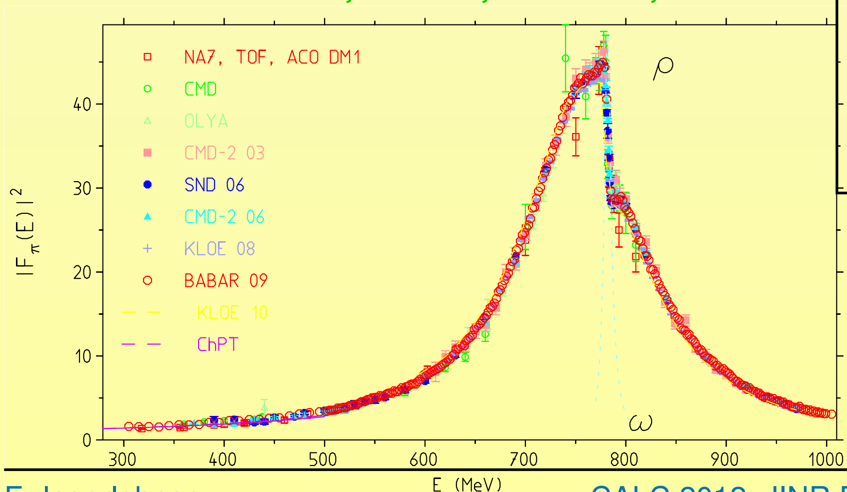
Leading non-perturbative hadronic contributions $a_\mu^{\rm had}$ can be obtained in terms of $R_\gamma(s) \equiv \sigma^{(0)}(e^+e^- \to \gamma^* \to \text{hadrons})/\frac{4\pi\alpha^2}{3s}$ data via dispersion integral:

$$a_\mu^{\rm had} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left(\int_{4m_\pi^2}^{E^2_{\rm cut}} ds \frac{R_\gamma^{\rm data}(s)\ \hat{K}(s)}{s^2} + \int_{E^2_{\rm cut}}^{\infty} ds \frac{R_\gamma^{\rm pQCD}(s)\ \hat{K}(s)}{s^2} \right)$$



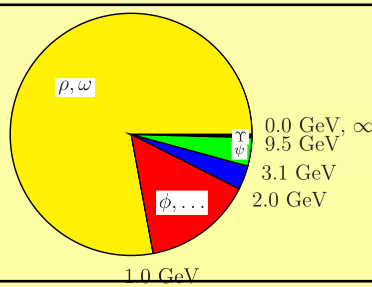
- Experimental error implies theoretical uncertainty!
- Low energy contributions enhanced: $\sim 75\%$ come from region $4m_\pi^2 < m_{\pi\pi}^2 < M_\Phi^2$

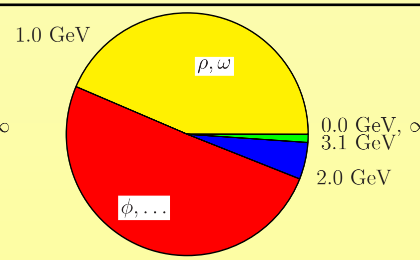
Data: CMD-2, SND, KLOE, BaBar



$$a_\mu^{\rm had(1)} = (690.7 \pm 4.7)[695.5 \pm 4.1]\ 10^{-10}$$

e^+e^-–data based [incl. BaBar MD09]

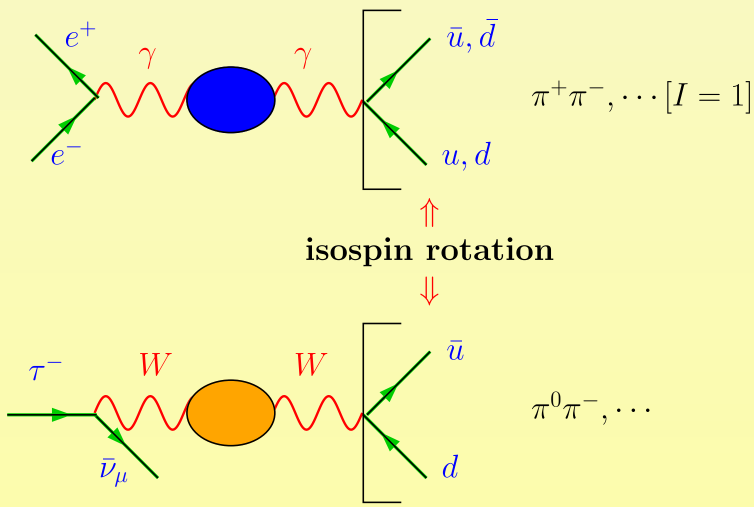




❑ The τ vs. e^+e^- problem

Concerns: calculation of hadronic vacuum polarization from appropriate hadron production data.

① A good idea: enhance e^+e^-–data by isospin rotated/corrected τ–data + CVC



ALEPH–Coll., (OPAL, CLEO), Alemany, Davier, Höcker 1996,
Belle–Coll. Fujikawa, Hayashii, Eidelman 2008

$$\tau^- \to X^- \nu_\tau \quad \leftrightarrow \quad e^+e^- \to X^0$$

where X^- and X^0 are hadronic states related by isospin rotation. The e^+e^- cross–section is then given by

$$\sigma^{I=1}_{e^+e^-\to X^0} = \frac{4\pi\alpha^2}{s} \frac{\beta_0^3(s)}{\beta_-^3(s)} v_{1,X^-} \quad , \quad \sqrt{s} \leq M_\tau$$

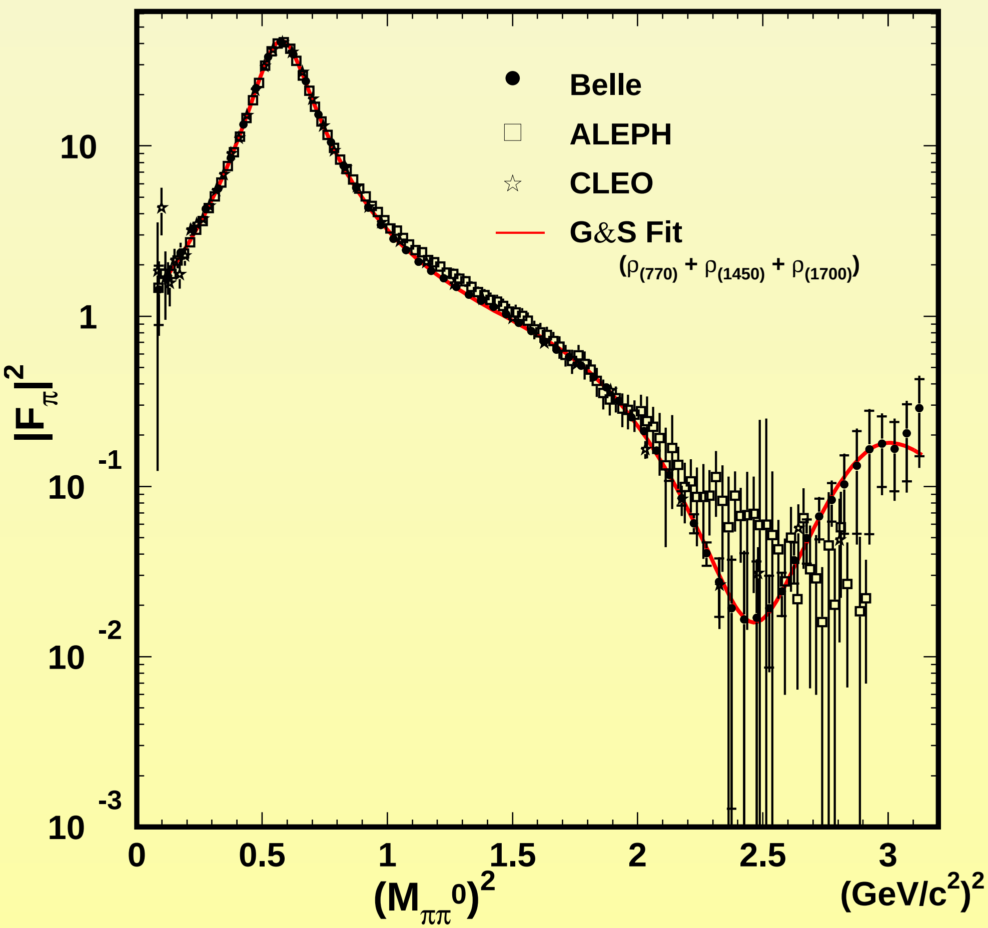
in terms of the τ spectral function v_1.

- mainly improves the knowledge of the $\pi^+\pi^-$ channel (ρ–resonance contribution)
- which is dominating in $a_\mu^{\rm had}$ (72%)

$I = 1 \sim 75\%$; $I = 0 \sim 25\%$ τ–data cannot replace e^+e^-–data

$$\begin{array}{lcccc} \delta a_\mu & : & 15.6 \times 10^{-10} & \to & 10.2 \times 10^{-10} & \\ \delta\Delta\alpha & : & 0.00067 & \to & 0.00065 & (ADH1997) \end{array}$$

Data: ALEPH 97, ALEPH 05, OPAL, CLEO and
most recent measurement from *Belle* (2008):

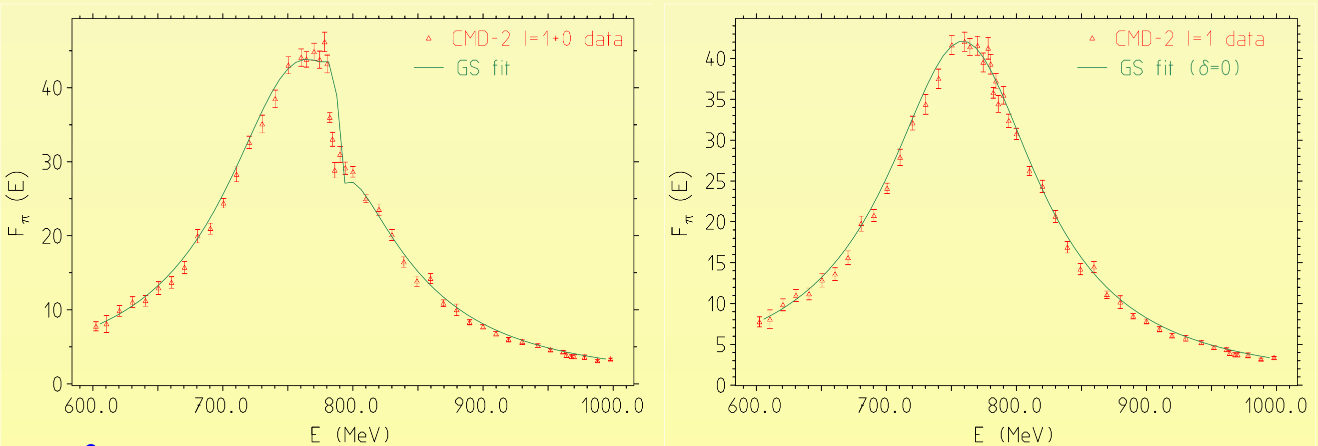


e^+e^-–data*= data corrected for isospin violations: In e^+e^- (neutral channel) $\rho - \omega$ mixing due isospin violation be quark mass difference $m_u \neq m_d$ $\Rightarrow$ I=0 component; to be subtracted for comparison with τ data

- Use Gounaris-Sakurai ansatz

$$F_\pi(s) = \frac{\mathrm{BW}^{\mathrm{GS}}_{\rho(770)}(s) \cdot \left(1 + \delta \frac{s}{M_\omega^2} \mathrm{BW}_\omega(s)\right) + \beta\, \mathrm{BW}^{\mathrm{GS}}_{\rho(1450)}(s) + \gamma\, \mathrm{BW}^{\mathrm{GS}}_{\rho(1700)}(s)}{1 + \beta + \gamma} \,,$$

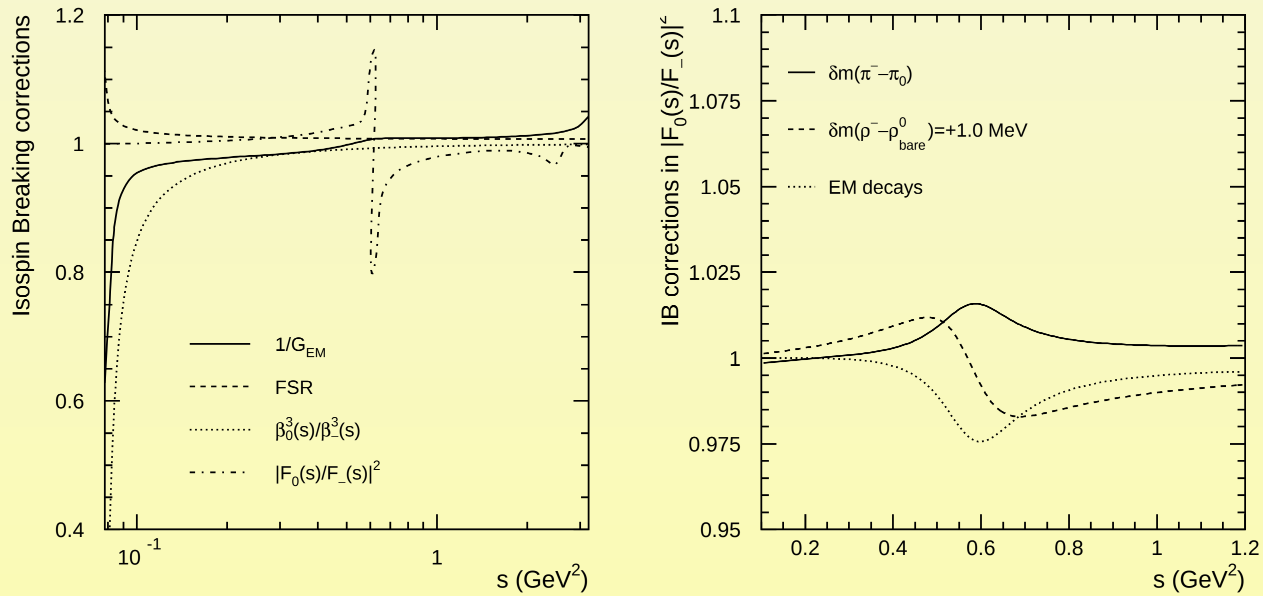
- Fit e^+e^--data for $|F_\pi(s)|^2$ ➠ $\delta_{\rho\omega}$ (complex) and set $\delta = 0$ to obtain $|F_\pi^{I=1}(s)|^2$



CMD-2 data for $|F_\pi|^2$ in $\rho - \omega$ region together with Gounaris-Sakurai fit. Left before subtraction right after subtraction of the ω.

I=0 component to be added to τ data for calculating a_μ^{had} !

Other isospin-breaking corrections Cirigliano et al. 2002, López Castro el al. 2007



Left: Isospin-breaking corrections $G_{\rm EM}$, FSR, $\beta_0^3(s)/\beta_-^3(s)$ and $|F_0(s)/F_-(s)|^2$.
Right: Isospin-breaking corrections in $I = 1$ part of ratio $|F_0(s)/F_-(s)|^2$:

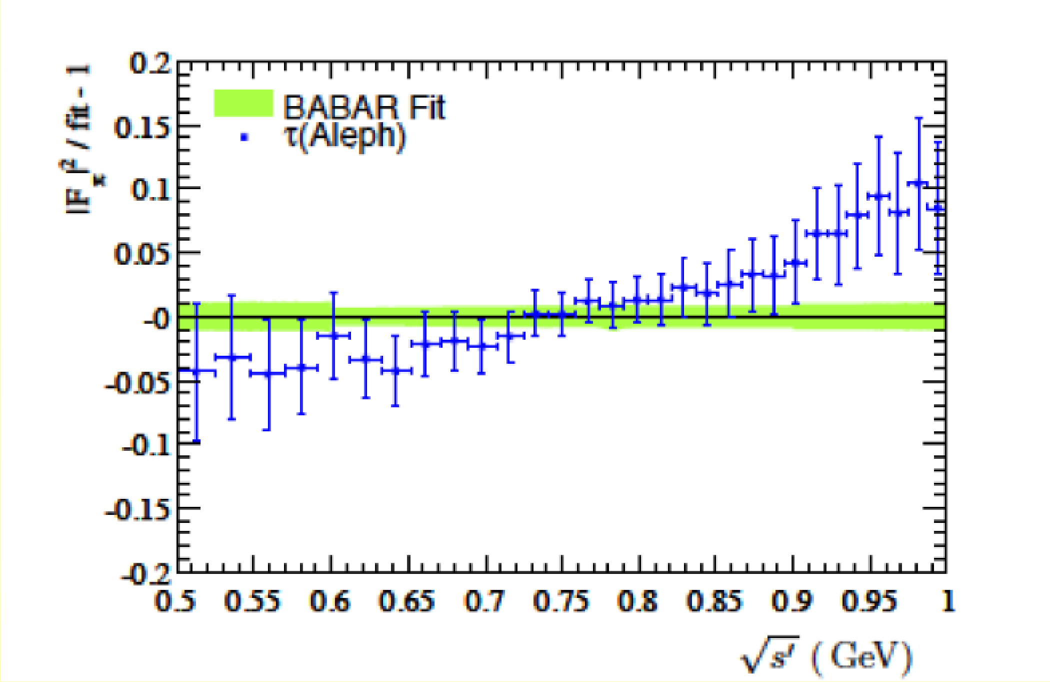
– π mass splitting $\delta m_\pi = m_{\pi^\pm} - m_{\pi^0}$,
– ρ mass splitting $\delta m_\rho = m_{\rho^\pm} - m_{\rho^0_{\rm bare}}$, and
– ρ width splitting $\delta\Gamma_\rho = \Gamma_{\rho^\pm} - \Gamma_{\rho^0}$.

- persisting discrepancy between the corrected τ and e^+e^- data, except very recent BABAR ISR data which agree well with Belle spectrum

- we find that unaccounted $\gamma-\rho$ mixing effects can account for the discrepancies (this talk)

- more elaborate HLS-model incl. mixings and self-energy corrections allows for consistent simultaneous fit of e^+e^- and τ spectra

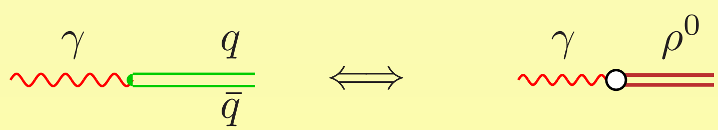
Recent comparison of Data relative to BaBar fit:



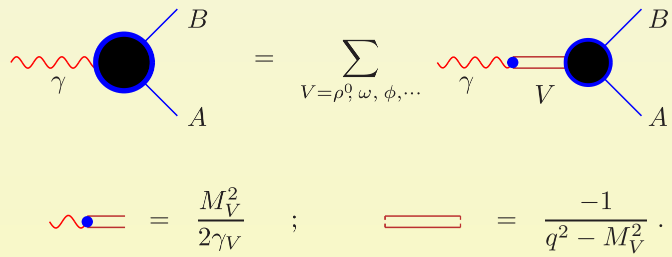
❑ What VMD model?

How do hadrons couple to photons? [a serious question before quarks were known]

- If they have net charge seemingly no problem
- What about neutral hadrons, like the ρ^0?
- In quark model (QCD, SM) photons couple to hadrons via the charged quark: obviously



Vector meson dominance (VMD) model: describes coupling of ρ^0 to the photon (Nambu, Sakurai, Gell-Mann et al 1962) when quark structure is not resolved:



The vector meson dominance model. A and B hadronic states

Original (standard) version: VDM I characterized by an effective Lagrangian

$$\mathcal{L}_{\gamma\rho} = -\frac{e\,M_\rho^2}{g_\rho}\,\rho_\mu A^\mu \ .$$

- not manifestly gauge invariant, photon acquires a mass, which must be renormalized away be hand i.e. by a photon mass counter term (fine tuning problem)

❒ The pion form factor here takes the form

$$F_\pi(s) = -\frac{M_\rho^2}{s - M_\rho^2}\,\frac{g_{\rho\pi\pi}}{g_\rho}$$

and the condition of electromagnetic current conservation $F_\pi(0) = 1$ is satisfied only if $g_{\rho\pi\pi} = g_\rho$, which is called universality condition or complete ρ dominance

Manifest electromagnetic gauge invariance can be implemented by writing the effective VMD Lagrangian in the form

$$\mathcal{L}_{\gamma\rho} = \frac{e}{2\,g_\rho}\,\rho_{\mu\nu}F^{\mu\nu}\,,$$

in terms of the field strength tensors (Kroll, Lee, Zumino 1967)

● The VDM II kinetic term transformes into the field mixing form VMD I plus a mass term by $\rho_\mu \to \rho_\mu + (e/g_\rho)\,A_\mu$, which can be considered as a photon mass counter term.

Often used (PDG, fits of data) $F_\pi(s)$ model is the Gounaris-Sakurai model (Gounaris, Sakurai 1968) based on VMD I (not gauge invariant, no decoupling)

❒ Note that physics is invariant under field redefinition in a properly formulated quantum field theory

As it satisfies gauge invariance, the form factor calculated with VMD II reads

$$F_\pi(s) = 1 - \frac{s}{s - M_\rho^2} \frac{g_{\rho\pi\pi}}{g_\rho}$$

and satisfies the current conservation condition $F_\pi(0) = 1$ in any case, irrespective of the universality constraint $g_{\rho\pi\pi} = g_\rho$.

❒ another essential difference: if $g_{\rho\pi\pi} \neq g_\rho$ different high energy behavior!

VDM I $F_\pi(s) \sim 1/s$ any $g_{\rho\pi\pi}$; VMD II $F_\pi(s) \sim 1 - g_{\rho\pi\pi}/g_\rho + O(1/s)$

● Whether $g_{\rho\pi\pi} = g_\rho$ is a phenomenological question, in fact experimentally $g_{\rho\pi\pi} \neq g_\rho$ (see below).

Generalized VMD I:
constraints $F_\pi(0) = 1$ and $F_\pi(s) \sim 1/s$ can be accomplished simultaneously by ansatz

$F_\pi^{\mathrm{VMD}}(s) = \sum_V \frac{-M_V^2}{s-M_V^2} \frac{g_{V\pi\pi}}{g_V}$ with a few resonances V satisfying $\sum_V g_{V\pi\pi}/g_V = 1$.

However, in generalized VMD ansatz,

- heavy states can not decouple sensitive to heavier states and truncation
- high energy behavior different form what QCD suggests

What does QCD require?

- $F_\pi(s) \overset{s\to\infty}{\gtrsim}$ "constant up to logs" is well compatible with QCD (as VMD II)
- for large s we have

$$|F_\pi(s)|^2 \sim 4R^{(\pi\pi)}(s)$$

Note

$$R^{\rm had}(s) = \sigma(e^+e^- \to {\rm hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$$

is given by perturbative QCD

$$R^{\rm had} \sim N_c \sum_f Q_f^2 \, \{1 + O(\alpha_s)\}$$

6 to which the $\pi\pi$ channel also contributes some amount as pions are made of quarks

In any case the low energy effective theory has to be matched to perturbative QCD at some intermediate scale like ~ 2 GeV, typically.

Generalized VMD model also contradicts large $N_c \to \infty$ limit where quark-hadron duality is exact i.e. infinite series of vector resonances has to reproduce QCD in limit $N_c \to \infty$.

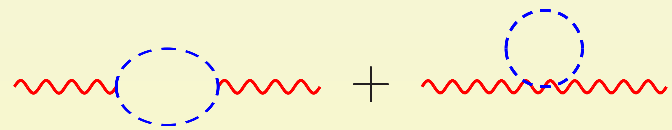
❑ A minimal model: VMD + sQED

Effective Lagrangian $\mathcal{L} = \mathcal{L}_{\gamma\rho} + \mathcal{L}_\pi$

$$\begin{aligned}
\mathcal{L}_\pi &= D_\mu \pi^+ D^{+\mu} \pi^- - m_\pi^2 \pi^+ \pi^- \; ; \quad D_\mu = \partial_\mu - \mathrm{i}\, e\, A_\mu - \mathrm{i}\, g_{\rho\pi\pi}\, \rho_\mu \\
\mathcal{L}_{\gamma\rho} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \rho_{\mu\nu} \rho^{\mu\nu} + \frac{M_\rho^2}{2} \rho_\mu \rho^\mu + \frac{e}{2\, g_\rho} \rho_{\mu\nu} F^{\mu\nu}
\end{aligned}$$

Self-energies: pion loops to photon-rho vacuum polarization

$$-\mathrm{i}\,\Pi^{\mu\nu\,(\pi)}_{\gamma\gamma}(q) =$$



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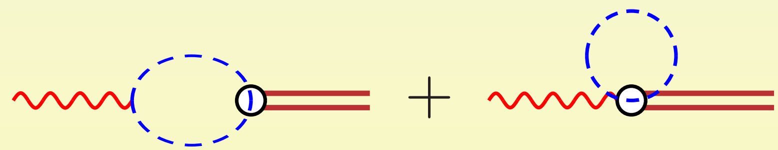
$$-\mathrm{i}\,\Pi^{\mu\nu\,(\pi)}_{\gamma\rho}(q) =$$



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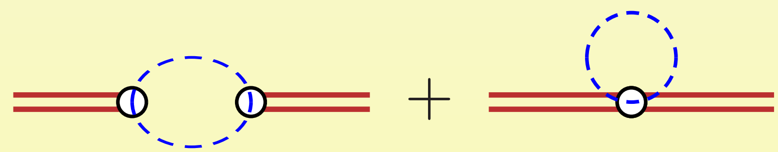
$$-\mathrm{i}\,\Pi^{\mu\nu\,(\pi)}_{\rho\rho}(q) =$$



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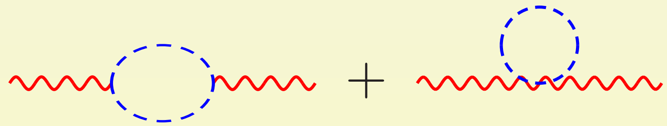
Irreducible self-energy contribution at one-loop

bare $\gamma - \rho$ transverse self-energy functions

$$\Pi_{\gamma\gamma} = \frac{e^2}{48\pi^2}\, f(q^2)\,, \quad \Pi_{\gamma\rho} = \frac{e g_{\rho\pi\pi}}{48\pi^2}\, f(q^2) \quad \text{and} \quad \Pi_{\rho\rho} = \frac{g^2_{\rho\pi\pi}}{48\pi^2}\, f(q^2)\,,$$

$$-\mathrm{i}\,\Pi^{\mu\nu\,(\pi)}_{\gamma\gamma}(q) =$$

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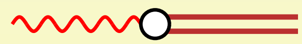
$$-\mathrm{i}\,\Pi^{\mu\nu\,(\pi)}_{\gamma\rho}(q) =$$

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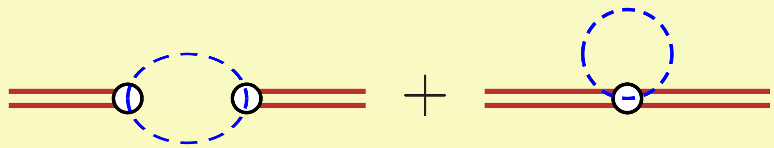
$$-\mathrm{i}\,\Pi^{\mu\nu\,(\pi)}_{\rho\rho}(q) =$$

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Previous calculations, consider mixing term to be constant

bare $\gamma - \rho$ transverse self-energy functions

$$\Pi_{\gamma\gamma} = \frac{e^2}{48\pi^2}\,f(q^2)\;,\quad \Pi_{\gamma\rho} = q^2(e/g_\rho) \quad \text{and} \quad \Pi_{\rho\rho} = \frac{g^2_{\rho\pi\pi}}{48\pi^2}\,f(q^2)\;,$$

This lowest order mixing term does not affect the renormalized self-energies:

$$\delta\Pi^{\mathrm{ren}}_{\gamma\rho} = q^2\frac{e}{g} - \frac{q^2}{M_\rho^2}\,M_\rho^2\,\frac{e}{g} = 0$$

Propagators = inverse of symmetric 2×2 self-energy matrix

$$\hat{D}^{-1} = \left(\begin{array}{cc} q^2 + \Pi_{\gamma\gamma}(q^2) & \Pi_{\gamma\rho}(q^2) \\ \Pi_{\gamma\rho}(q^2) & q^2 - M_\rho^2 + \Pi_{\rho\rho}(q^2) \end{array} \right)$$

inverted $\Rightarrow$

$$\begin{aligned} D_{\gamma\gamma} &= \frac{1}{q^2 + \Pi_{\gamma\gamma}(q^2) - \frac{\Pi_{\gamma\rho}^2(q^2)}{q^2 - M_\rho^2 + \Pi_{\rho\rho}(q^2)}} \\ D_{\gamma\rho} &= \frac{-\Pi_{\gamma\rho}(q^2)}{(q^2 + \Pi_{\gamma\gamma}(q^2))(q^2 - M_\rho^2 + \Pi_{\rho\rho}(q^2)) - \Pi_{\gamma\rho}^2(q^2)} \\ D_{\rho\rho} &= \frac{1}{q^2 - M_\rho^2 + \Pi_{\rho\rho}(q^2) - \frac{\Pi_{\gamma\rho}^2(q^2)}{q^2 + \Pi_{\gamma\gamma}(q^2)}} \,. \end{aligned}$$

Resonance parameters $\Leftrightarrow$ location s_P of the pole of the propagator

$$s_P - m^2_{\rho^0} + \Pi_{\rho^0\rho^0}(s_P) - \frac{\Pi^2_{\gamma\rho^0}(s_P)}{s_P + \Pi_{\gamma\gamma}(s_P)} = 0 \;,$$

with $s_P = \tilde{M}^2_{\rho^0}$ complex.

$$\tilde{M}^2_\rho \equiv \left(q^2\right)_{\mathrm{pole}} = M^2_\rho - \mathrm{i}\; M_\rho \, \Gamma_\rho$$

Diagonalization $\Rightarrow$ physical ρ acquires a direct coupling to the electron

$$\begin{aligned} \mathcal{L}_{\mathrm{QED}} &= \bar{\psi}_e \gamma^\mu (\partial_\mu - \mathrm{i}\, e_b \, A_{b\mu}) \, \psi_e \\ &\Downarrow \\ \mathcal{L}_{\mathrm{QED}} &= \bar{\psi}_e \gamma^\mu (\partial_\mu - \mathrm{i}\, e \, A_\mu + \mathrm{i}\, g_{\rho ee} \rho_\mu) \, \psi_e \end{aligned}$$

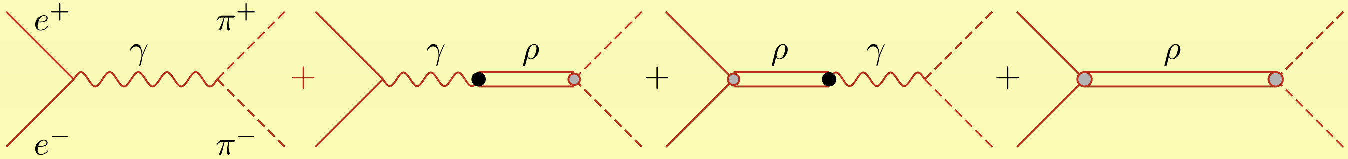
with $g_{\rho ee} = e\,(\Delta_\rho + \Delta_0)$, where in our case $\Delta_0 = 0$.

❑ $F_\pi(s)$ with $\rho - \gamma$ mixing at one-loop

The $e^+e^- \to \pi^+\pi^-$ matrix element in sQED is given by

$$\mathcal{M} = -\mathrm{i}\, e^2\, \bar{v}\gamma^\mu u\, (p_1 - p_2)_\mu\, F_\pi(q^2)$$

with $F_\pi(q^2) = 1$. In our extended VMD model we have the four terms



Diagrams contributing to the process $e^+e^- \to \pi^+\pi^-$.

$$F_\pi(s) \propto e^2\, D_{\gamma\gamma} + e g_{\rho\pi\pi}\, D_{\gamma\rho} - g_{\rho ee} e D_{\rho\gamma} - g_{\rho ee} g_{\rho\pi\pi}\, D_{\rho\rho}\,,$$

Properly normalized (VP subtraction: $e^2(s) \to e^2$):

$$\boxed{F_\pi(s) = \left[e^2\, D_{\gamma\gamma} + e\,(g_{\rho\pi\pi} - g_{\rho ee})\, D_{\gamma\rho} - g_{\rho ee} g_{\rho\pi\pi}\, D_{\rho\rho}\right] / \left[e^2\, D_{\gamma\gamma}\right]}$$

Typical couplings

$g_{\rho\pi\pi\,\mathrm{bare}} = 5.8935$, $g_{\rho\pi\pi\,\mathrm{ren}} = 6.1559$, $g_{\rho ee} = 0.018149$, $x = g_{\rho\pi\pi}/g_\rho = 1.15128$.

We note that the precise s-dependence of the effective ρ-width is obtained by evaluating the imaginary part of the ρ self-energy:

$$\mathrm{Im}\,\Pi_{\rho\rho} = \frac{g^2_{\rho\pi\pi}}{48\,\pi}\beta^3_\pi\, s \equiv M_\rho\, \Gamma_\rho(s)\ ,$$

which yields

$$\Gamma_\rho(s)/M_\rho = \frac{g^2_{\rho\pi\pi}}{48\,\pi}\beta^3_\pi\,\frac{s}{M^2_\rho}\ ;\quad \Gamma_\rho/M_\rho = \frac{g^2_{\rho\pi\pi}}{48\,\pi}\beta^3_\rho\ ;\quad g_{\rho\pi\pi} = \sqrt{48\,\pi\,\Gamma_\rho/(\beta^3_\rho\, M_\rho)}\ .$$

In our model, in the given approximation, the on ρ-mass-shell form factor reads

$$F_\pi(M_\rho^2) = 1 - \mathrm{i}\,\frac{g_{\rho ee}g_{\rho\pi\pi}}{e^2}\,\frac{M_\rho}{\Gamma_\rho}\;;\;\; |F_\pi(M_\rho^2)|^2 = 1 + \frac{36}{\alpha^2}\frac{\Gamma_{ee}}{\beta_\rho^3\,\Gamma_\rho}\;,$$

$$\Gamma_{\rho ee} = \frac{1}{3}\,\frac{g_{\rho ee}^2}{4\pi}\,M_\rho\;;\;\; g_{\rho ee} = \sqrt{12\pi\,\Gamma_{\rho ee}/M_\rho}\;.$$

Compare: Gounaris-Sakurai (GS) formula

$$F_\pi^{\mathrm{GS}}(s) = \frac{-M_\rho^2 + \Pi_{\rho\rho}^{\mathrm{ren}}(0)}{s - M_\rho^2 + \Pi_{\rho\rho}^{\mathrm{ren}}(s)}\;;\;\; \Gamma_{\rho ee}^{\mathrm{GS}} = \frac{2\alpha^2\,\beta_\rho^3 M_\rho^2}{9\,\Gamma_\rho}\,\left(1 + d\,\Gamma_\rho/M_\rho\right)^2\;.$$

GS does not involve $g_{\rho ee}$ resp. $\Gamma_{\rho ee}$ in a direct way, as normalization is fixed by applying an overall factor $1 + d\,\Gamma_\rho/M_\rho \equiv 1 - \Pi_{\rho\rho}^{\mathrm{ren}}(0)/M_\rho^2 \simeq 1.089$ to enforce $F_\pi(0) = 1$ (in our approach “automatic” by gauge invariance).

Note: no new free parameter!

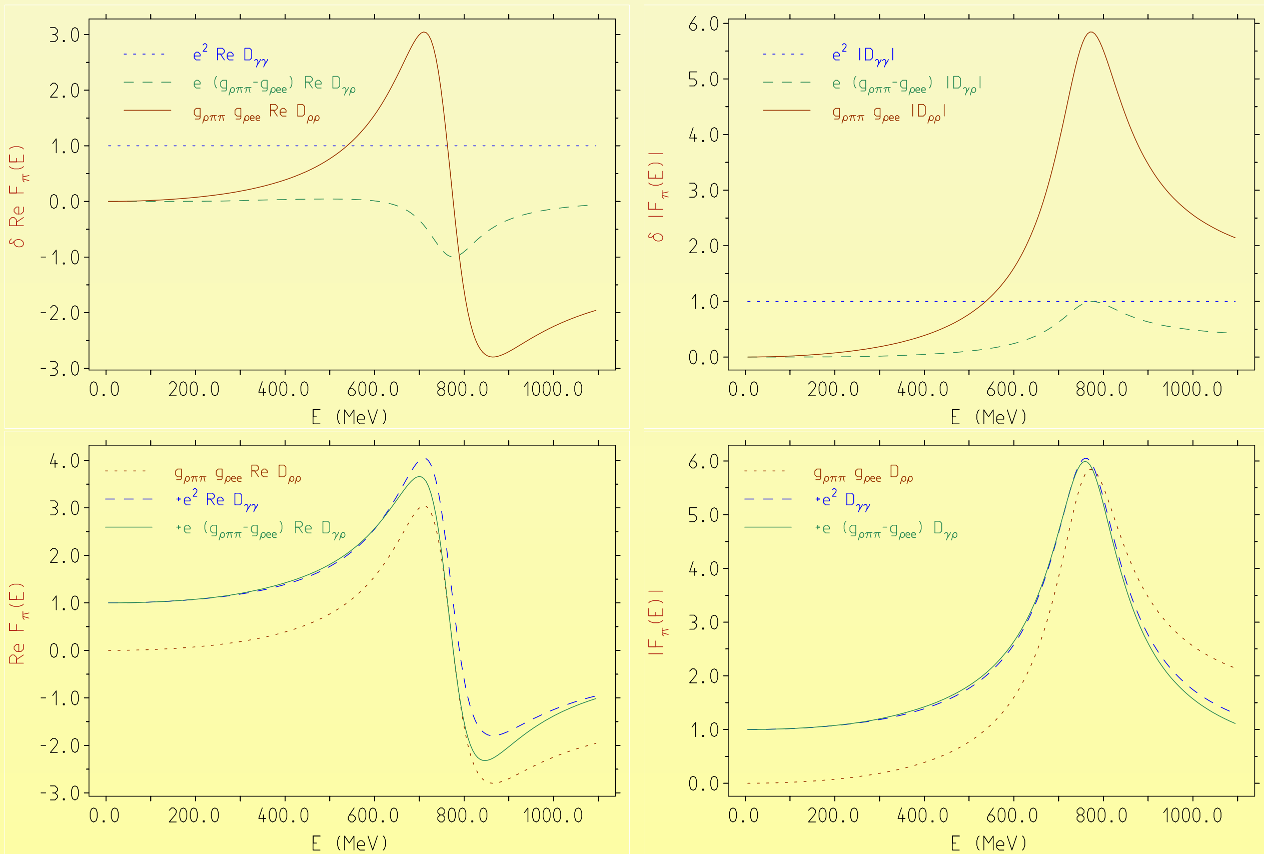
Note: in $e^+e^- \to \pi^+\pi^-$ the fields ρ^0 and γ are not external i.e. in the path integral representation they only show up as integration variables and the value of the integral is independent of the choice of integration variables. As the field redefinition is a regular transformation, we have the choice:

- interpolating fields of mass eigenstates, have induced direct ρee coupling uniquely determined by the leptonic width of the ρ^0. While the mass matrix is diagonal, coupling scheme looks “off-diagonal” (not normal)

- in original coordinates (quasi bare fields) the ρee coupling is absent. In a way the formulation is diagonal in the coupling but not diagonal in masses (ev. photon mass counterterm to be adjusted). In order to find out what are the masses diagonalization is needed anyway.

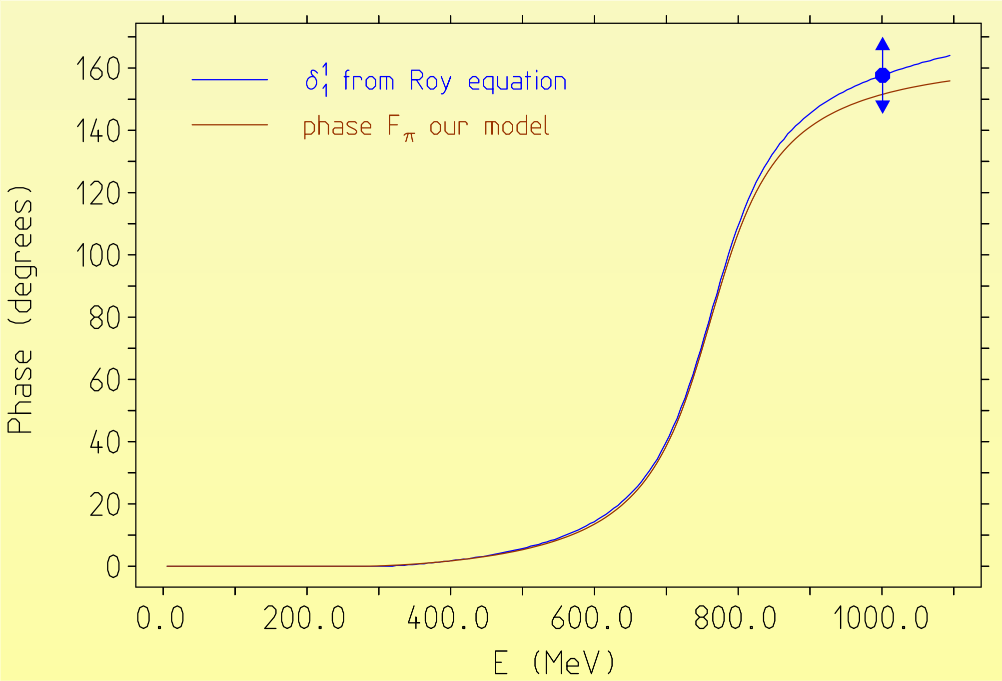
The interference of terms in $F_\pi^{(e)}$

Real parts and moduli of the 3 individual and added terms normalized to the sQED term are displayed:



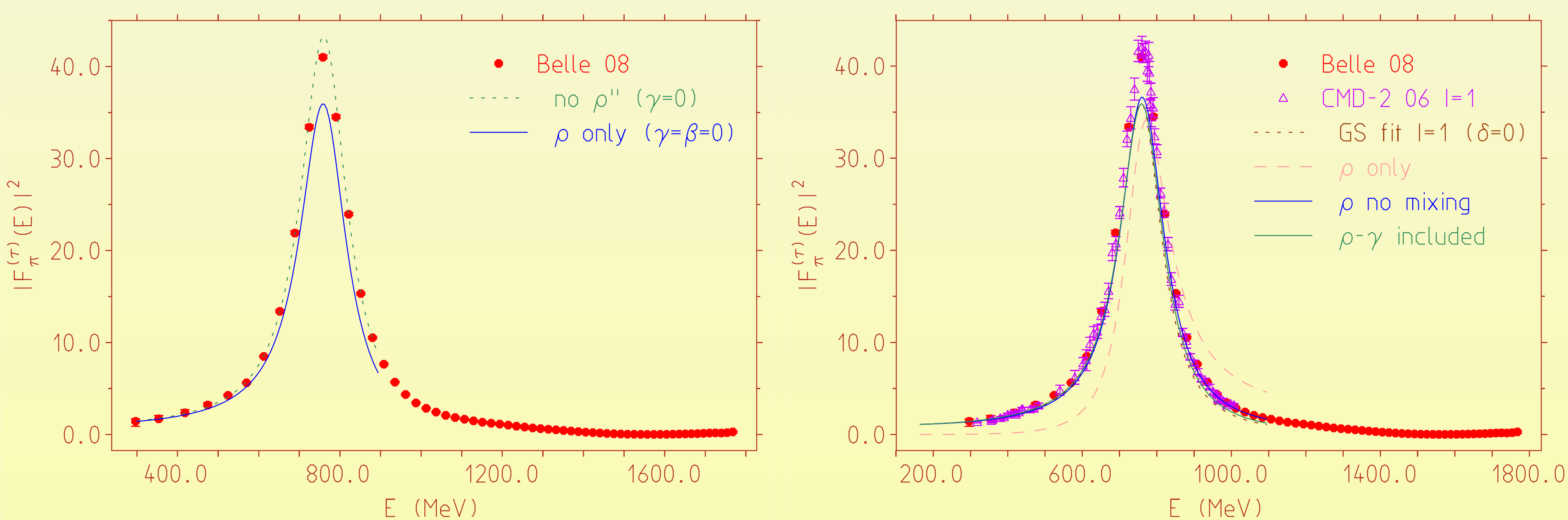
Comparison of $\pi\pi$ rescattering with Colangelo-Leutwyler's first principles approach

One of the key ingredients in this approach is the strong interaction phase shift $\delta_1^1(s)$ of $\pi\pi$ (re)scattering in the final state. We compare the phase of $F_\pi(s)$ in our model with the one obtained by solving the Roy equation with $\pi\pi$-scattering data as input. We notice that the agreement is surprisingly good up to about 1 GeV. It is not difficult to replace our phase by the more precise exact one.



Actually, the normalization just below the K^+K^- threshold in the Roy Equation approach is to be matched from $\sigma(e^+e^- \to \pi^+\pi^-)$ at about 1 GeV (vertical arrow). In our approach it can be adjusted changing $g_{\rho\pi\pi}$.

Relation to data:

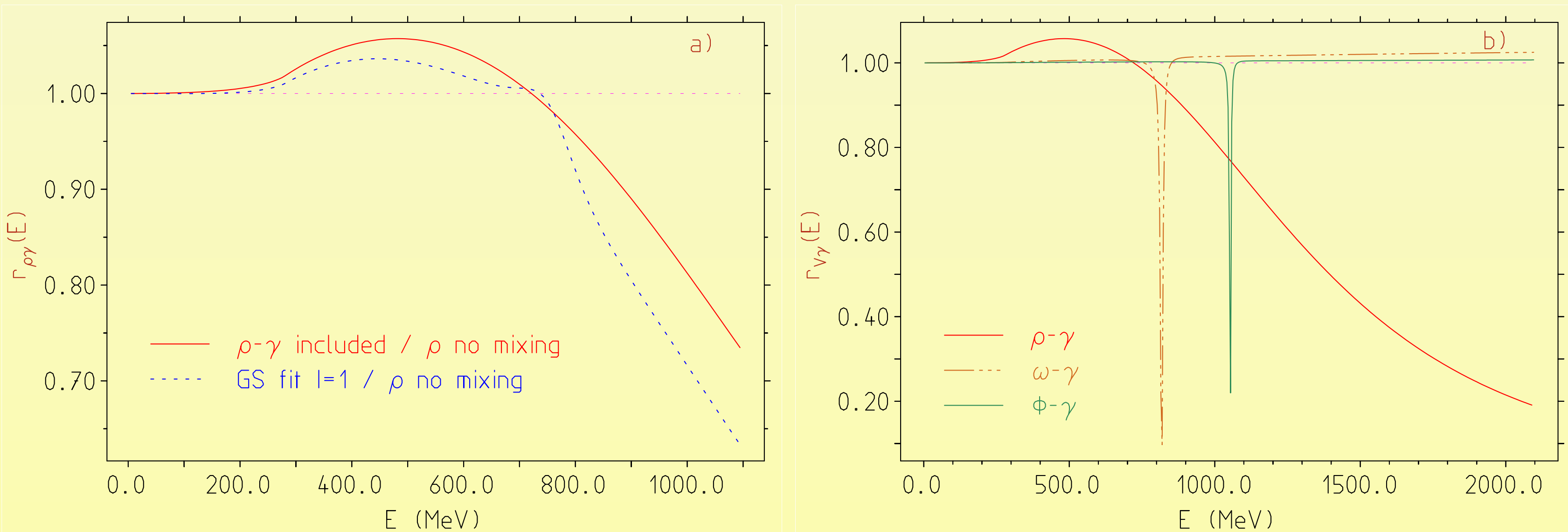


Left: GS fits of the Belle data and the effects of including higher states ρ' and ρ'' at fixed M_ρ and Γ_ρ. Right: Effect of $\gamma - \rho$ mixing in our simple EFT model

Parameters: $M_\rho = 775.5$ MeV, $\Gamma_\rho = 143.85$ MeV,
$\mathcal{B}[(\rho \to ee)/(\rho \to \pi\pi)] = 4.67 \times 10^{-5}$, $e = 0.302822$, $g_{\rho\pi\pi} = 5.92$, $g_{\rho ee} = 0.01826$.

Detailed comparison, in terms of the ratio:

$$r_{\rho\gamma}(s) \equiv \frac{|F_\pi(s)|^2}{|F_\pi(s)|^2_{D_{\gamma\rho}=0}}$$



a) Ratio of $|F_\pi(E)|^2$ with mixing vs. no mixing. Same ratio for GS fit with PDG parameters. b) The same mechanism scaled up by the branching fraction $\Gamma_V/\Gamma(V \to \pi\pi)$ for $V = \omega$ and ϕ. In the $\pi\pi$ channel the effects for resonances $V \neq \rho$ are tiny if not very close to resonance.

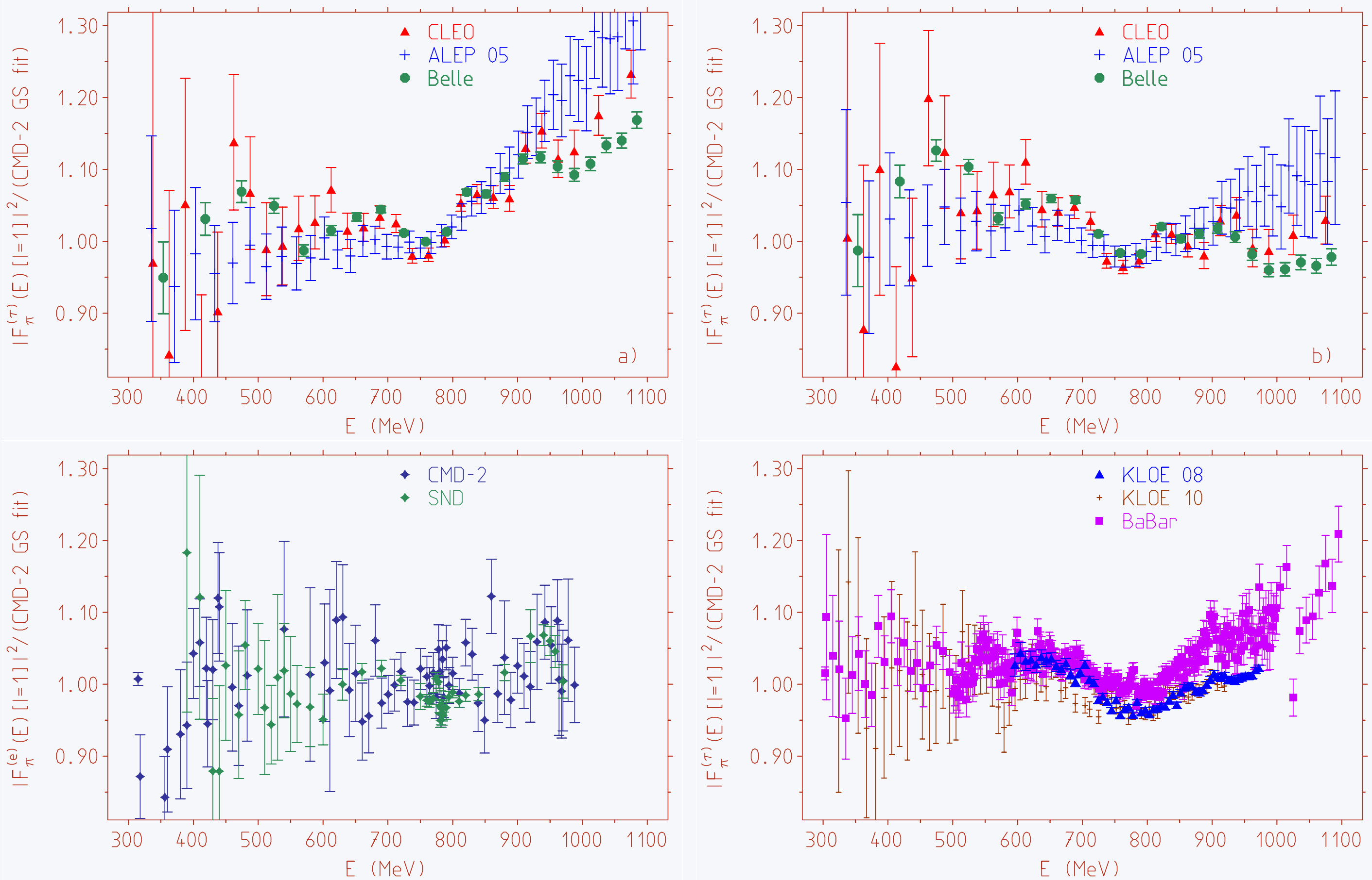
If mixing not included in $F_0(s)$ $\Rightarrow$ total correction formula on spectral functions

$$\boxed{v_0(s) = r_{\rho\gamma}(s)\, R_{\rm IB}(s)\, v_-(s)}$$

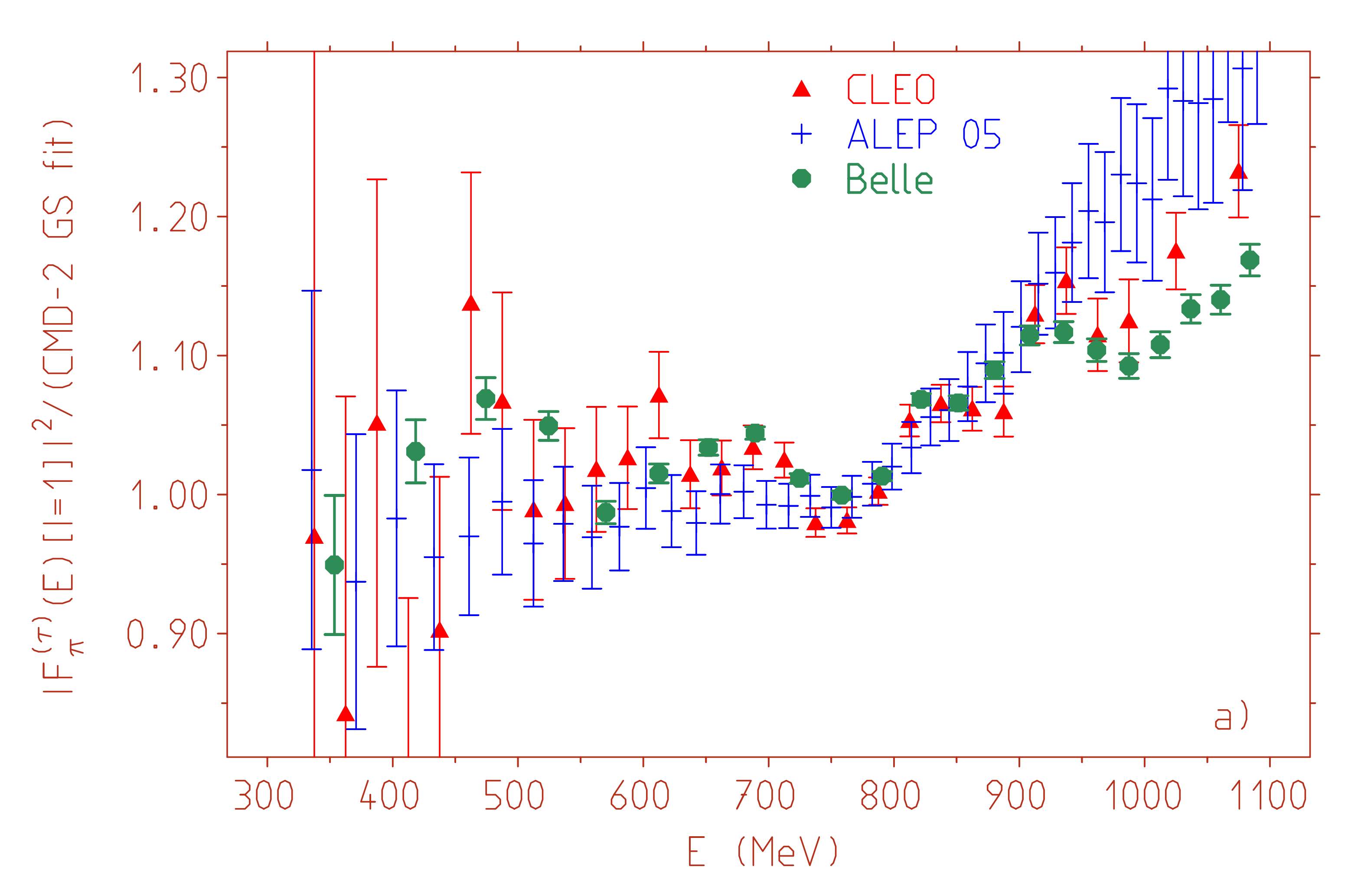
$$R_{\rm IB}(s) = \frac{1}{G_{\rm EM}(s)} \frac{\beta_0^3(s)}{\beta_-^3(s)} \left| \frac{F_0(s)}{F_-(s)} \right|^2$$

- $G_{\rm EM}(s)$ electromagnetic radiative corrections
- $\beta_0^3(s)/\beta_-^3(s)$ phase space modification by $m_{\pi^0} \neq m_{\pi^\pm}$
- $|F_0(s)/F_-(s)|^2$ incl. shifts in masses, widths etc

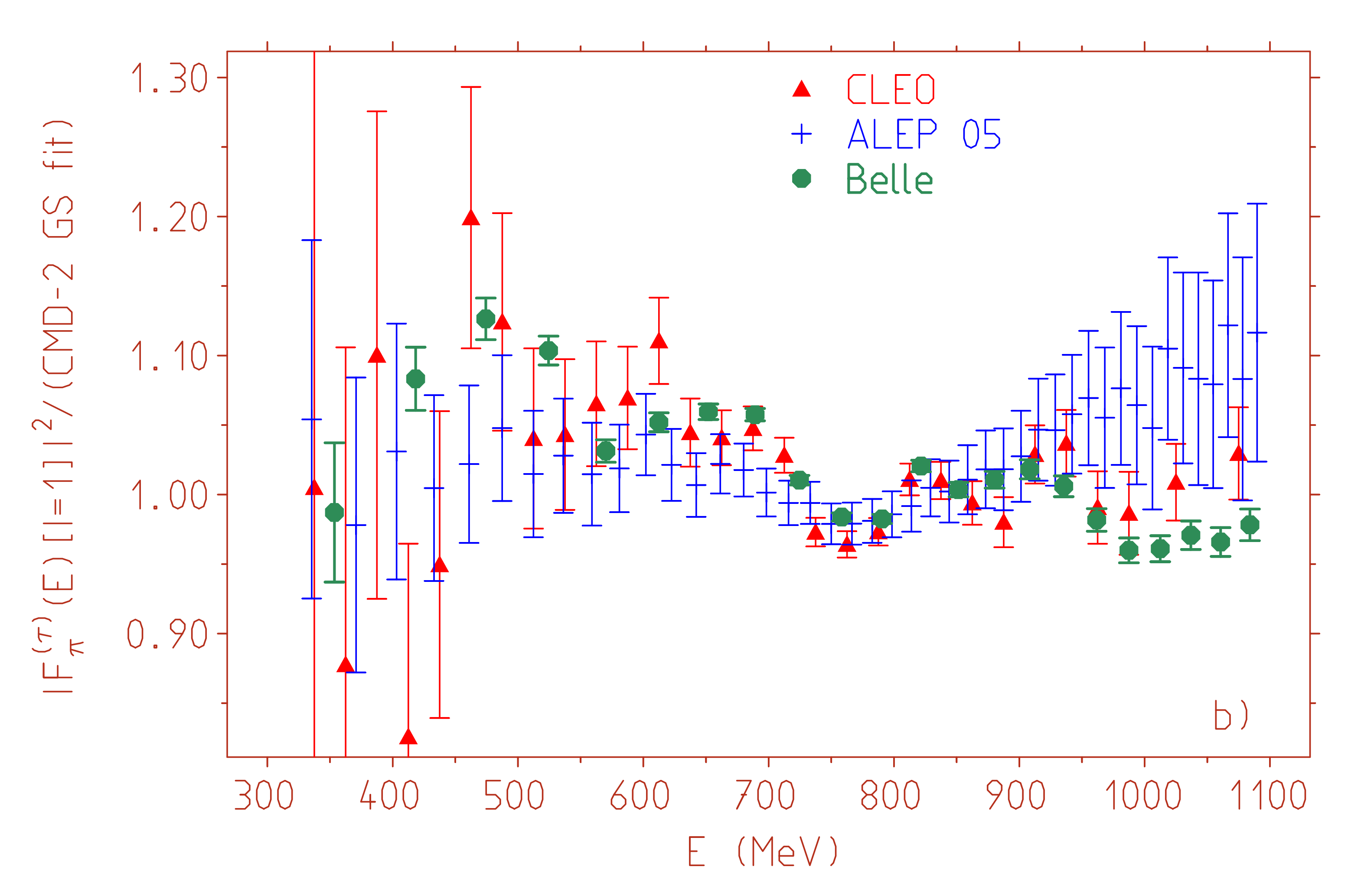
Final state radiation correction FSR(s) and vacuum polarization effects $(\alpha/\alpha(s))^2$ and I=0 component ($\rho - \omega$) we have been subtracted from all e^+e^--data.



$|F_\pi(E)|^2$ in units of e^+e^- I=1 (CMD-2 GS fit): a) τ data uncorrected for $\rho-\gamma$ mixing, and b) after correcting for mixing. Lower panel: e^+e^- energy scan data [left] and e^+e^- radiative return data [right]



$|F_\pi(E)|^2$ in units of e^+e^- I=1 (CMD-2 GS fit)



$|F_\pi(E)|^2$ in units of e^+e^- I=1 (CMD-2 GS fit)

❑ Applications: a_μ and $B^{\rm CVC}_{\pi\pi^0} = \Gamma(\tau \to \nu_\tau \pi\pi^0)/\Gamma_\tau$

❶ How does the new correction affect the evaluation of the hadronic contribution to a_μ ? To lowest order in terms of e^+e^--data, represented by $R(s)$, we have

$$a_\mu^{\rm had,LO}(\pi\pi) = \frac{\alpha^2}{3\pi^2} \int_{4m_\pi^2}^{\infty} {\rm d}s\, R^{(0)}_{\pi\pi}(s) \frac{K(s)}{s} \;,$$

with the well-known kernel $K(s)$ and

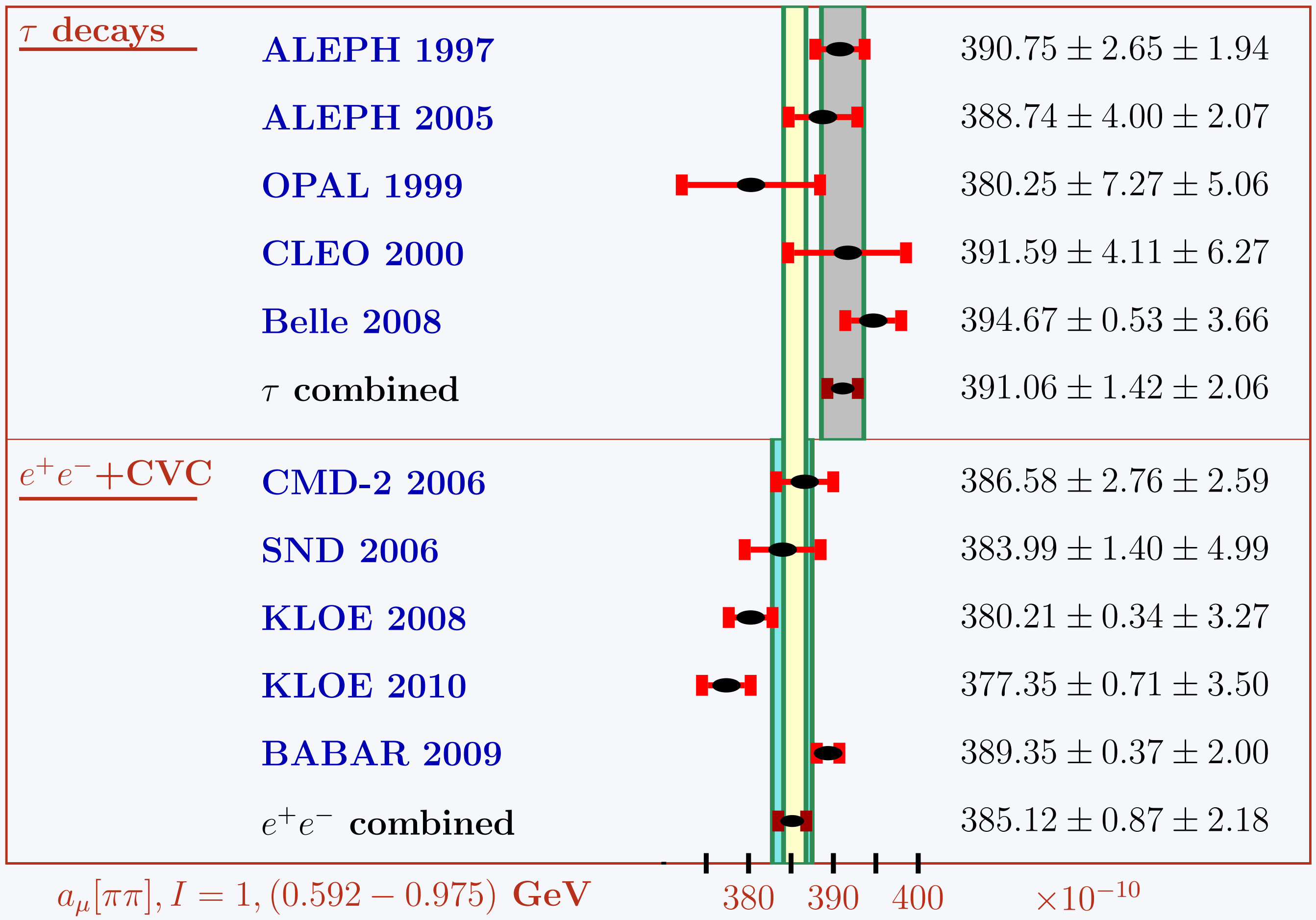
$$R^{(0)}_{\pi\pi}(s) = (3s\sigma_{\pi\pi})/(4\pi\alpha^2(s)) = 3v_0(s) \;.$$

Note that the $\rho-\gamma$ interference is included in the measured e^+e^--data, and so is its contribution to $a_\mu^{\rm had}$. In fact $a_\mu^{\rm had}$ is intrinsic an e^+e^--based “observable” (neutral current channel).

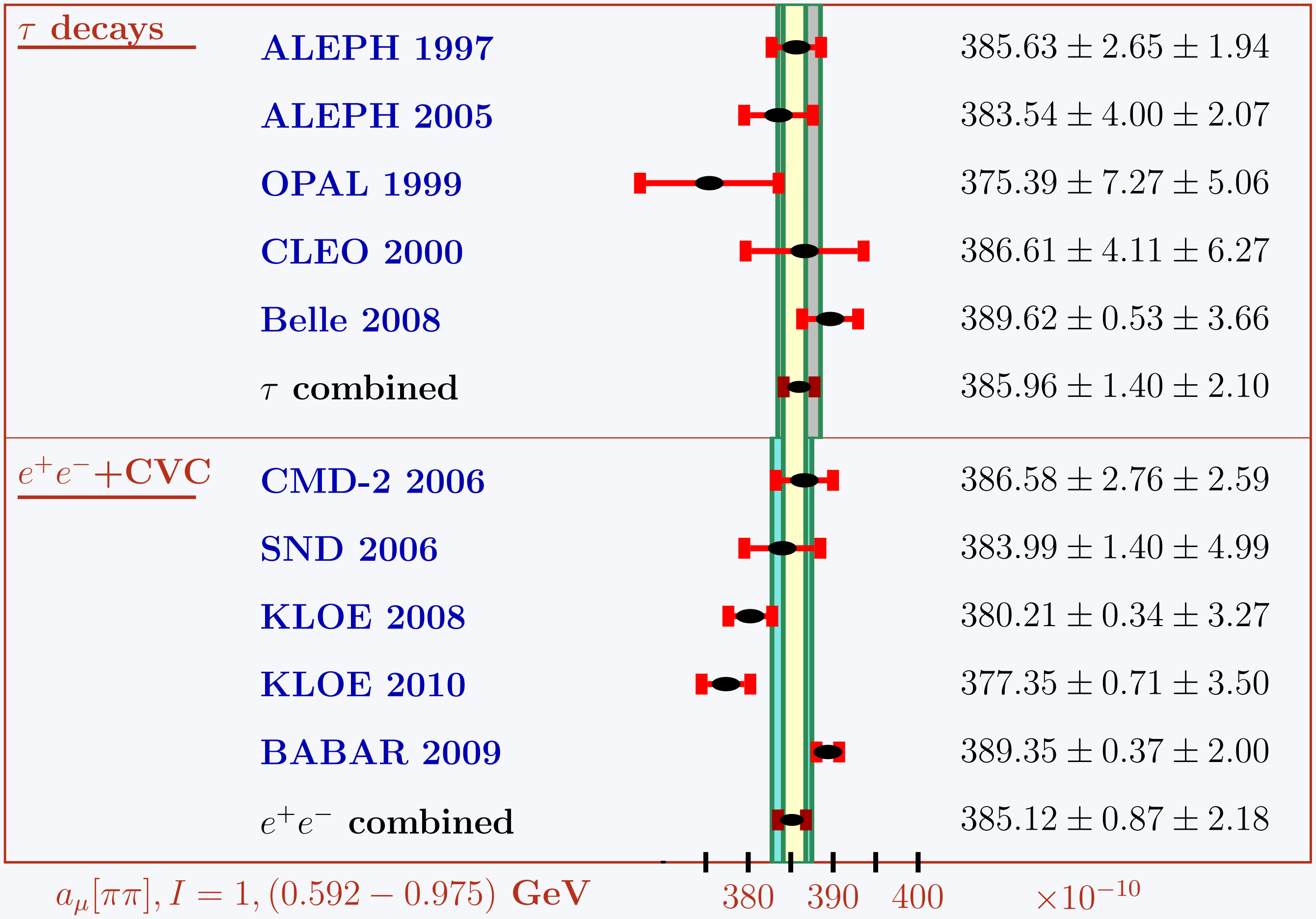
How to utilize τ data: subtract CVC violating corrections

- traditionally $v_-(s) \to v_0(s) = R_{\rm IB}(s)\, v_-(s)$
- our correction $v_-(s) \to v_0(s) = \boxed{r_{\rho\gamma}(s)}\; R_{\rm IB}(s)\, v_-(s)$

Result for the I=1 part of $a_\mu^{\rm had}[\pi\pi]$: $\boxed{\delta a_\mu^{\rm had}[\rho\gamma] \simeq (-5.1 \pm 0.5) \times 10^{-10}}$



I=1 part of $a_\mu^{\rm had}[\pi\pi]$

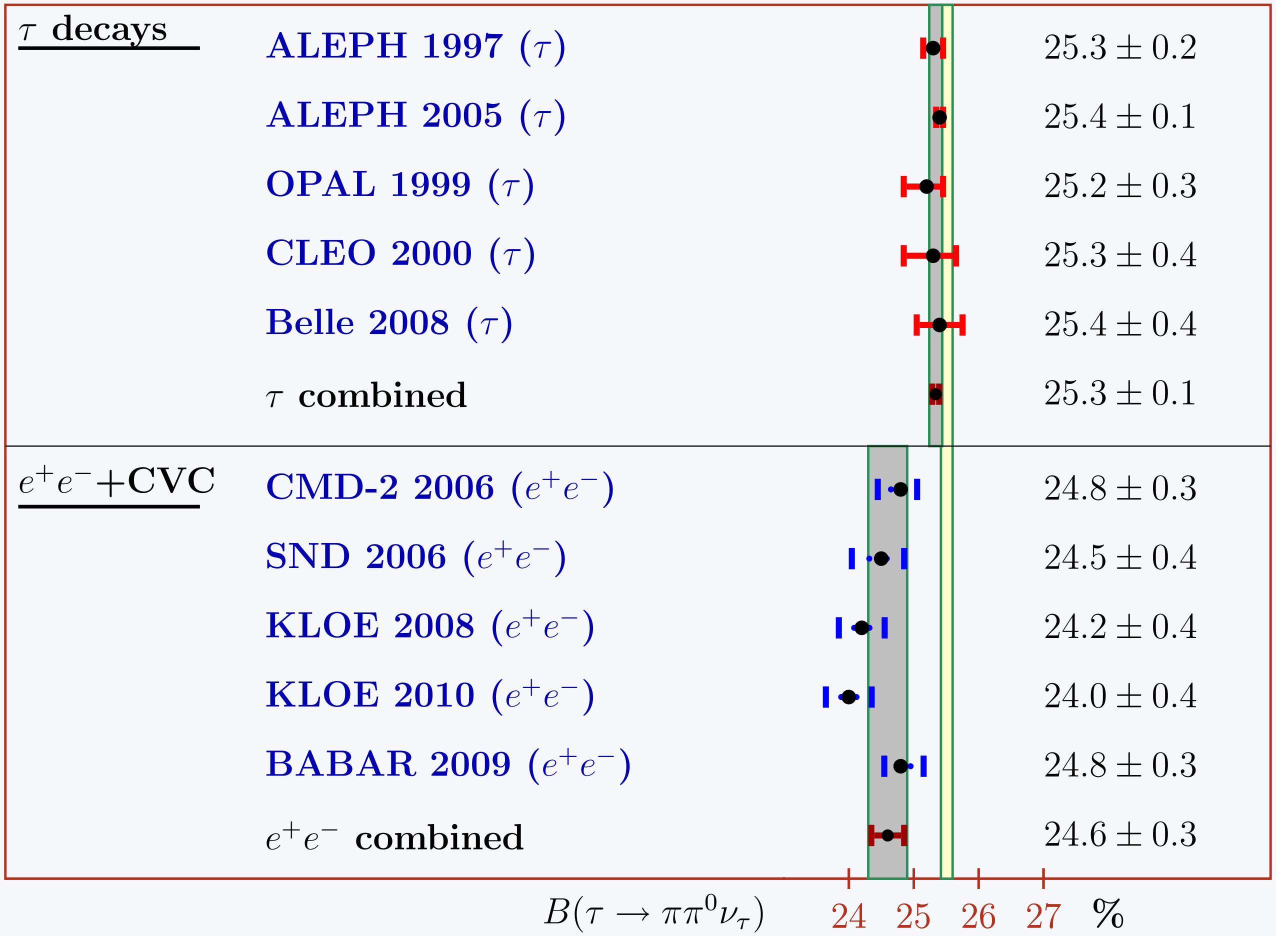


I=1 part of $a_\mu^{\rm had}[\pi\pi]$

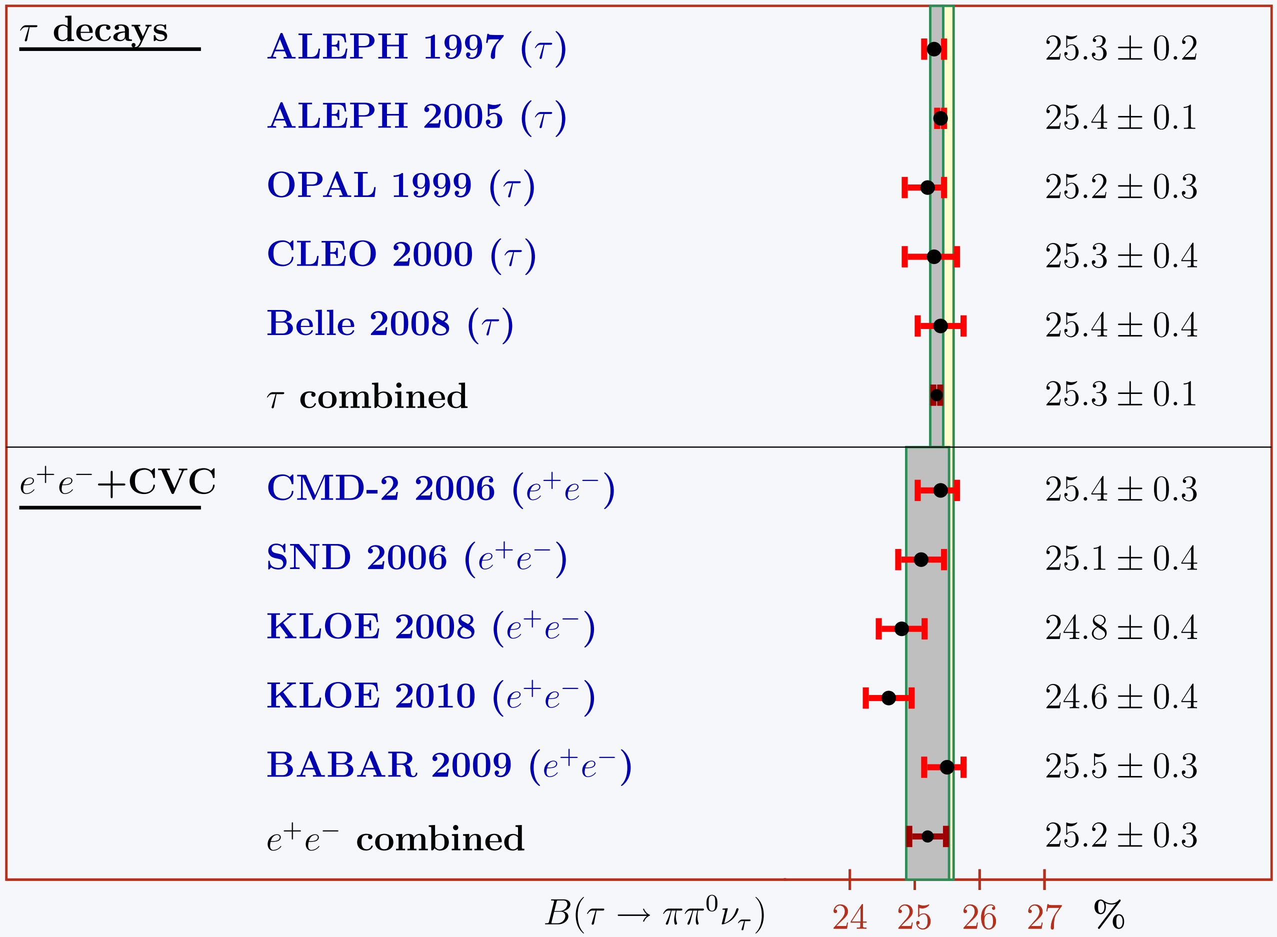
❷ The $\tau \to \pi^0 \pi \nu_\tau$ branching fraction $B_{\pi\pi^0} = \Gamma(\tau \to \nu_\tau \pi \pi^0)/\Gamma_\tau$ is another important quantity which can be directly measured. This "τ-observable" can be evaluated in terms of the I=1 part of the $e^+e^- \to \pi^+\pi^-$ cross section, after taking into account the IB correction $v_0(s) \to v_-(s) = v_0(s)/R_{\mathrm{IB}}(s)$ $/r_{\rho\gamma}(s)$,

$$B^{\mathrm{CVC}}_{\pi\pi^0} = \frac{2S_{\mathrm{EW}} B_e |V_{ud}|^2}{m_\tau^2} \int_{4m_\pi^2}^{m_\tau^2} \mathrm{d}s\, R^{(0)}_{\pi^+\pi^-}(s) \left(1 - \frac{2}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) \frac{1}{r_{\rho\gamma}(s)\, R_{\mathrm{IB}}(s)} \,,$$

where here we also have to "undo" the $\rho - \gamma$ mixing which is absent in the charged isovector channel. The shift is $\delta B^{\mathrm{CVC}}_{\pi\pi^0}[\rho\gamma] = +0.62 \pm 0.06~\%$



Branching fractions $B(\tau \to \pi\pi^0\nu_\tau)$



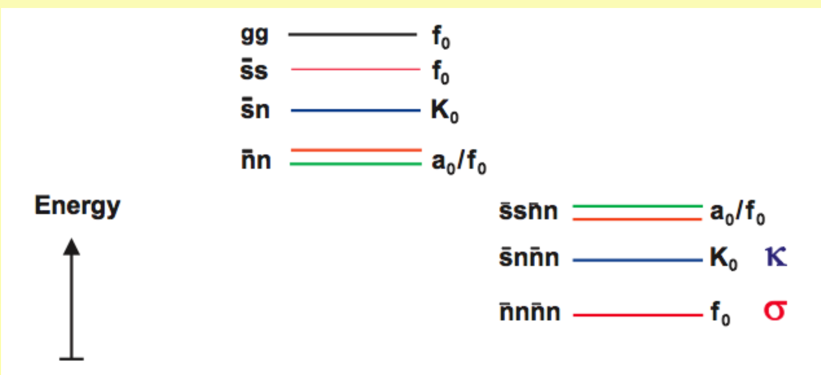
Branching fractions $B(\tau \to \pi\pi^0\nu_\tau)$

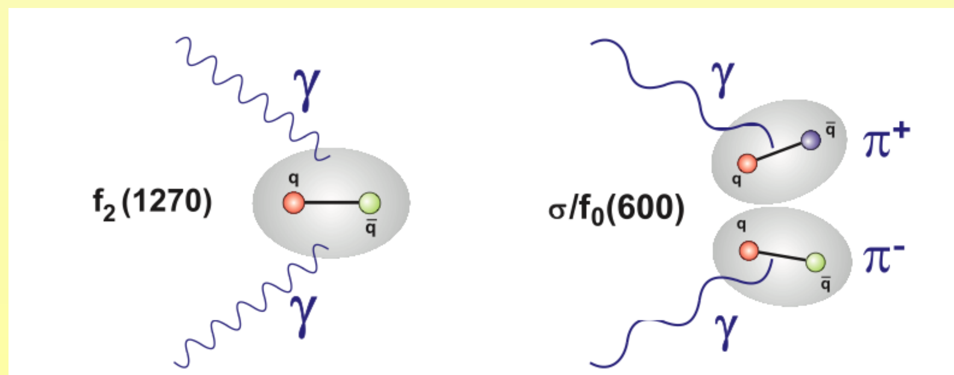
Is our model viable?

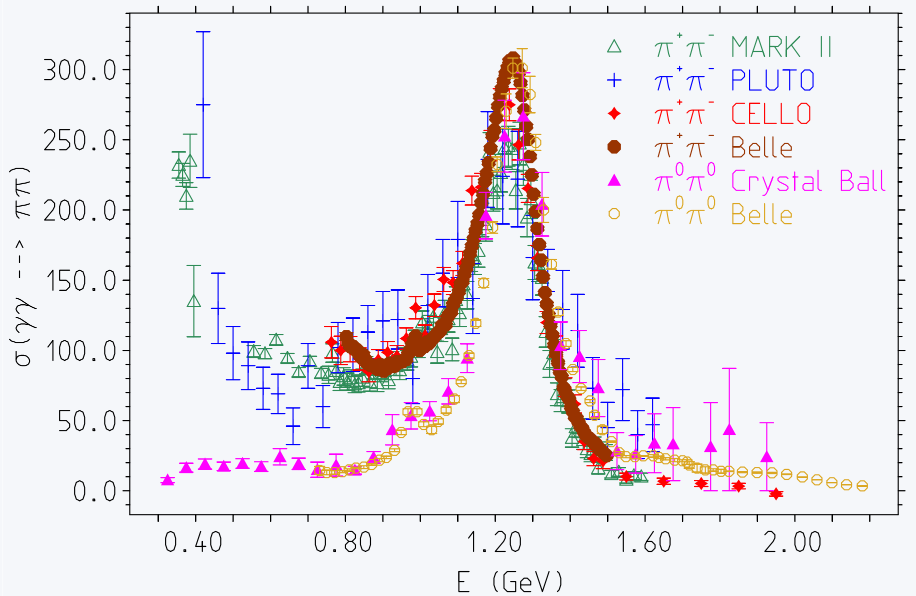
Look at $\gamma\gamma \to \pi\pi$

Laboratory to study scalar exchanges: $a_0, f_0', f_0, \cdots$ in $\gamma\gamma \to \pi\pi$ and more!

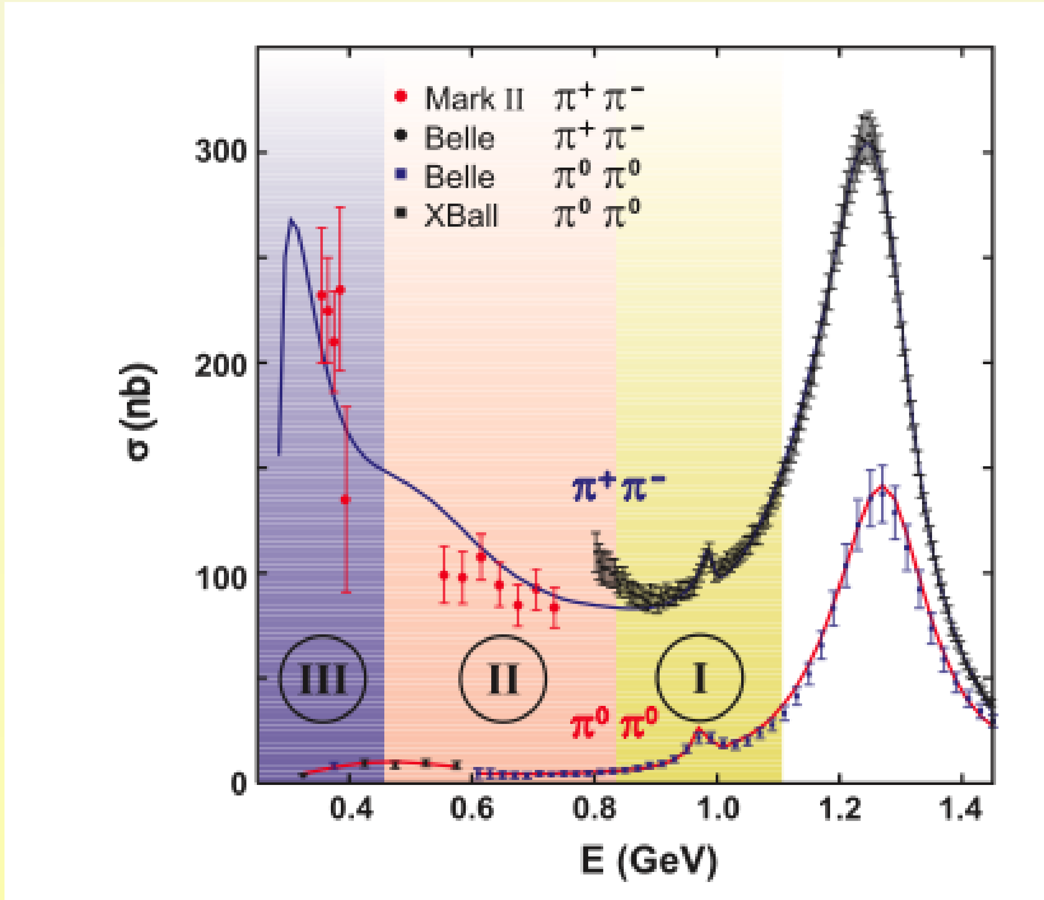
Mesons: $M(q\bar{q})$, $M(qq\bar{q}\bar{q})$, glueballs mixing
Experimental: Crystal Ball, Mark II, Belle!
Theory: Mennessier, Pennington et al., Mousallam et al., Achasov et al., ...



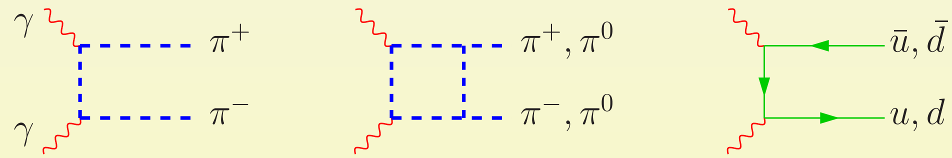




How photons couple to pions? This is obviously probed in reactions like $\gamma\gamma \to \pi^+\pi^-, \pi^0\pi^0$. Data infer that below about 1 GeV photons couple to pions as point-like objects (i.e. to the charged ones overwhelmingly), at higher energies the photons see the quarks exclusively and form the prominent tensor resonance $f_2(1270)$. The $\pi^0\pi^0$ cross section in this figure is enhanced by the isospin symmetry factor 2, by which it is reduced in reality.



Normalization as in reality: $\sigma(\pi^0\pi^0)/\sigma(\pi^+\pi^-) = \frac{1}{2}$ at the $f_2(1270)$ peak
(art work by Mike Pennington)



Di-pion production in $\gamma\gamma$ fusion. At low energy we have direct $\pi^+\pi^-$ production and by strong rescattering $\pi^+\pi^- \to \pi^0\pi^0$, however with very much suppressed rate. As energy goes up, above about 1 GeV, the strong $q\bar{q}$ resonance $f_2(1270$ shows up at equal strength at isospin ratio $\sigma(\pi^0\pi^0)/\sigma(\pi^+\pi^-) = \frac{1}{2}$. This demonstrates convincingly that we may safely work with point-like pions below 1 GeV.

Strong tensor meson resonance in $\pi\pi$ channel $f_2(1270)$ with photons directly probe the quarks!

- Photons seem to see pions below 1 GeV
- Photons definitely look at the quarks in $f_2(1270)$ resonance region
- We apply the sQED model up to 0.975 GeV (relevant for a_μ). This should be pretty save (still we assume a 10% model uncertainty)

- Switching off the electromagnetic interaction of pions, is definitely not a realistic approximation in trying to describe what data we see in $e^+e^- \to \pi^+\pi^-$

Summary and Conclusions

❖ We understand the EFT “VMD+sQED” as the tail of the more appropriate resonance Lagrangian approach (Ecker et al. 1989) or alternatives (like HLS). Applied to low energy $\pi\pi$ production yields

- proper ρ propagator self-energy effects for GS form factor ($\rho \to \pi\pi$)
- pion-loop effect in $\rho - \gamma$ mixing contributes sizable interference
- proper energy dependence off the resonances; decoupling of heavier states, high energy behavior in accord with QCD

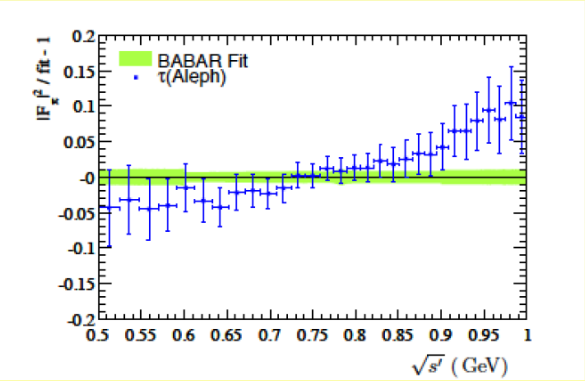
Note: so far PDG parameters masses, widths, branching fractions etc. of resonances like ρ^0 all extracted from data assuming GS like form factors (model dependent!)

❒ Note: ratio $F_0(s)/F_-(s)$ could be measured within lattice QCD, without reference to sQED or other hadronic models. Do it!

Pattern:

- moderate positive interference (up to +5%) below ρ, substantial negative interference (-10% and more) above the ρ (must vanish at $s = 0$ and $s = M_\rho^2$)
- remarkable agreement with pattern of e^+e^- vs τ discrepancy
- shift of the τ data to lie perfectly within the ballpark of the e^+e^- data

Best “proof”:



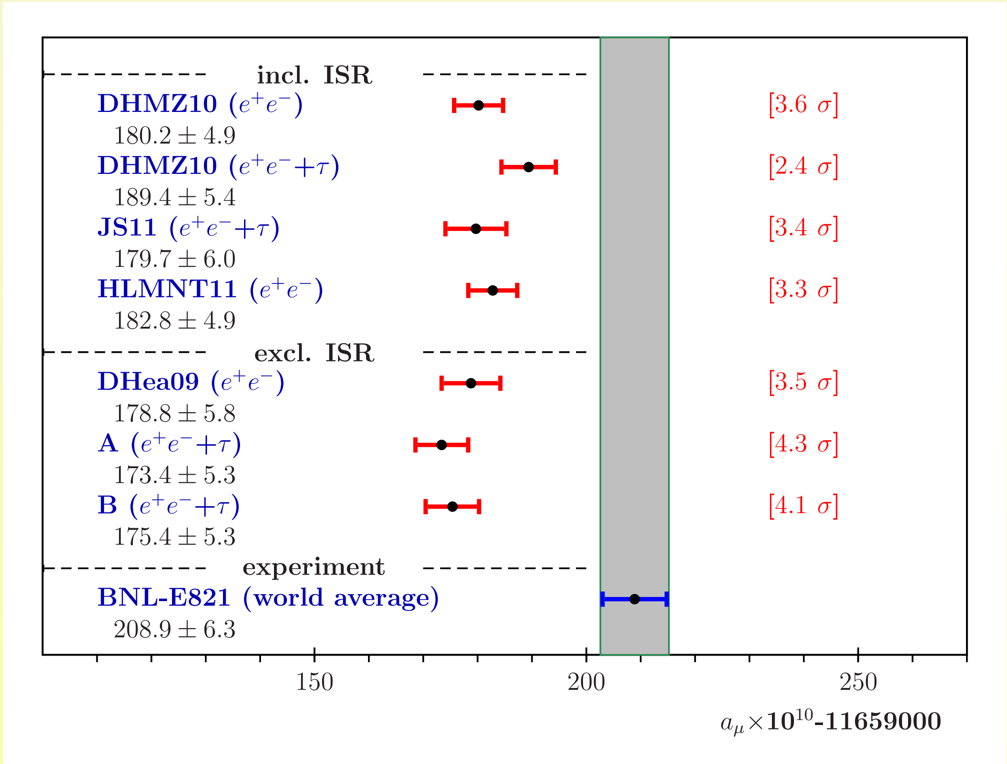
Lesson: effective field theory the basic tool (not ad hoc pheno. ansätze)

- $\rho - \gamma$ correction function $r_{\rho\gamma}(s)$ entirely fixed from neutral channel

- τ data provide independent information

- Including $\omega, \phi, \rho', \rho'', \cdots$ requires to go to appropriate Resonance Lagrangian extension

M. Benayoun et al. HLS effective resonance Lagrangian model global fits



Data below $E_0 = 1.05$ GeV (just above the ϕ) constrain effective Lagrangian couplings, using 45 different data sets (6 annihilation channels and 10 partial width decays).

- Effective theory predicts cross sections:

$$\pi^+\pi^-,\ \pi^0\gamma,\ \eta\gamma,\ \eta'\gamma,\ \pi^0\pi^+\pi^-,\ K^+K^-,\ K^0\bar{K}^0 \quad \text{(83.4\%)},$$

- Missing part:

$$4\pi, 5\pi, 6\pi, \eta\pi\pi, \omega\pi \text{ and regime } E > E_0$$

  evaluated using data directly and pQCD for perturbative region and tail

- Including self-energy effects is mandatory ($\gamma\rho$-mixing, $\rho\omega$-mixing ..., decays with proper phase space, energy dependent width etc)

- Method works in reducing uncertainties by using indirect constraints

- Able to reveal inconsistencies in data. In our case in region [1.00,1.05] GeV tension between KK and 3π data sets. All data: Solution A 71.2% CL; excluding 3π above 1 GeV: Solution B 97.0% CL. Conflict in data?, model?

- Singling out effective resonance Lagrangian by global fit is expected to help in improving EFT calculations of hadronic light-by-light scattering (such concept so far missing)

What does it mean for the muon $g-2$?

- it looks we have fairly reliable model to include τ data to improve a_μ^{had}
- there is no τ vs. e^+e^- alternative of a_μ^{had}

For the lowest order hadronic vacuum polarization (VP) contribution to a_μ we find

$$\boxed{a_\mu^{\mathrm{had,LO}}[e] = 690.75(4.72) \times 10^{-10} \quad (e)}$$

$$\Downarrow$$

$$\boxed{a_\mu^{\mathrm{had,LO}}[e,\tau] = 690.96(4.65) \times 10^{-10} \quad (e+\tau)}$$

$$\boxed{a_\mu^{\mathrm{the}} = 116591797(60) \times 10^{-11}} \qquad \boxed{a_\mu^{\mathrm{exp}} = 116592089(54)(33) \times 10^{-11}}$$

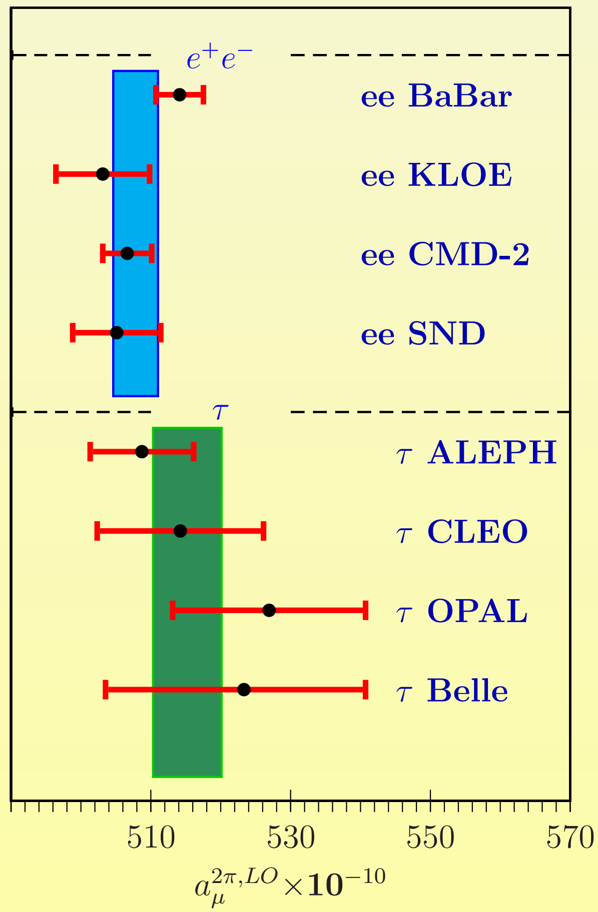
$$a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{the}} = (293 \pm 87) \times 10^{-11} \quad \Longrightarrow \quad \boxed{3.4\,\sigma}$$

Compare: Höcker 2010 (theory-driven analysis)

$$a_\mu^{\mathrm{had,LO}}[e] \quad = (692.3 \pm 1.4 \pm 3.1 \pm 2.4 \pm 0.2 \pm 0.3) \times 10^{-10}$$ (e^+e^- based),

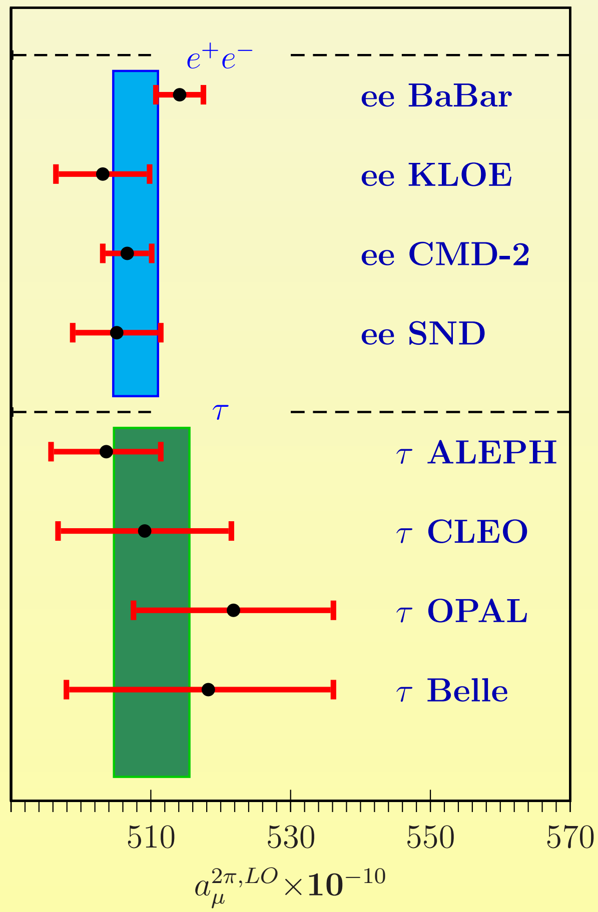
$$a_\mu^{\mathrm{had,LO}}[e,\tau] = (701.5 \pm 3.5 \pm 1.9 \pm 2.4 \pm 0.2 \pm 0.3) \times 10^{-10}$$ (e^+e^-+τ based),

New BaBar ISR 2π result: arXiv:1205.2228v1 10 May 2012



$\gamma - \rho$ correction: not applied

New BaBar ISR 2π result: arXiv:1205.2228v1 10 May 2012



$\gamma - \rho$ correction: applied

$$a_\mu^{2\pi,\mathrm{LO}}[2m_\pi - 1.8\ \text{GeV}]: \quad \begin{array}{ll} (514.1 \pm 3.8) \times 10^{-10} & \text{[BaBar]} \\ (507.8 \pm 3.2) \times 10^{-10} & \text{[ee all]} \\ (508.7 \pm 2.5) \times 10^{-10} & [\text{ee} + \tau\ \text{JS}] \end{array}$$

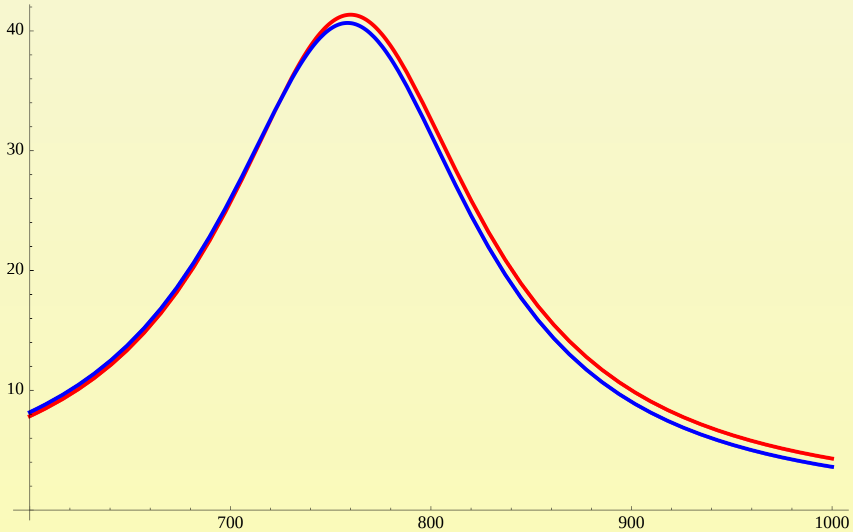
$$a_\mu^{\mathrm{the}}: \quad \begin{array}{ll} 116591865(54) \times 10^{-11} & \text{[BaBar]} \\ 116591802(50) \times 10^{-11} & \text{[ee all]} \\ 116591811(46) \times 10^{-11} & [\text{ee} + \tau\ \text{JS}] \end{array}$$

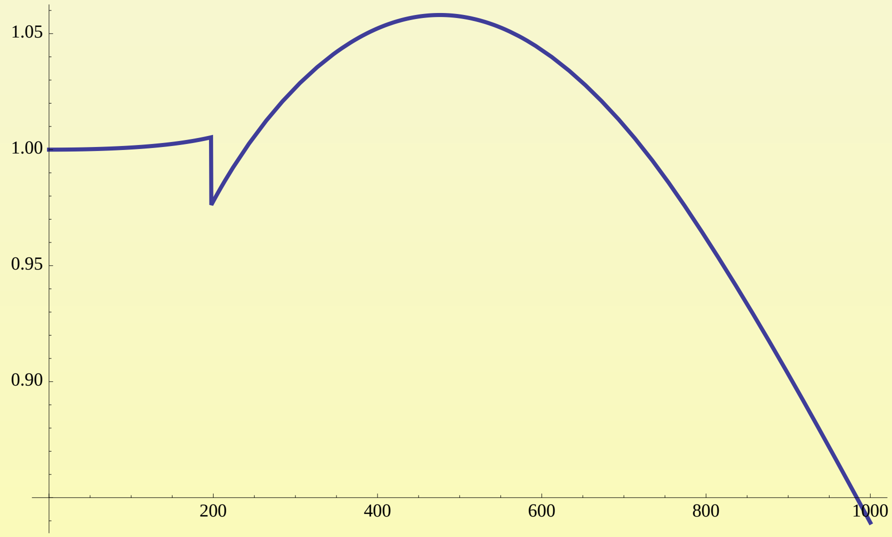
$$\delta a_\mu = a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{the}}: \quad \begin{array}{lll} (224 \pm 83) \times 10^{-11}, & 2.7\,\sigma & \text{[BaBar]} \\ (287 \pm 80) \times 10^{-11}, & 3.6\,\sigma, & \text{[ee all]} \\ (278 \pm 78) \times 10^{-11}, & 3.6\,\sigma, & [\text{ee} + \tau\ \text{JS}] \end{array}$$

- τ-data give 100% consistent results

- no way to understand without (effective) Lagrangian field theory concept taken serious

❐ then its an easy exercise





Robert Saffron's first attempt (within 3 days) to $\rho - \gamma$ mixing (based on my QCD lectures at Katowice
(see: `http://www-com.physik.hu-berlin.de/~fjeger/books.html`)).