Lectures on modern methods for scattering amplitudes

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Outline of lectures

Lectures 1 & 2

- tree-level scattering amplitudes
  (and *loop integrands*)
- analytic structure; on-shell methods
- to supersymmetric amplitudes and back
- symmetries
- IR divergences at loop level;
- massive IR regulator for planar amplitudes

Goals:

understand ideas for tree amplitudes and loop integrands

analytic structure+ symmetries

enable to do calculations for instructive examples

guide to the literature for technical details

origin of IR divergences and ways to regularize them

Lecture 3

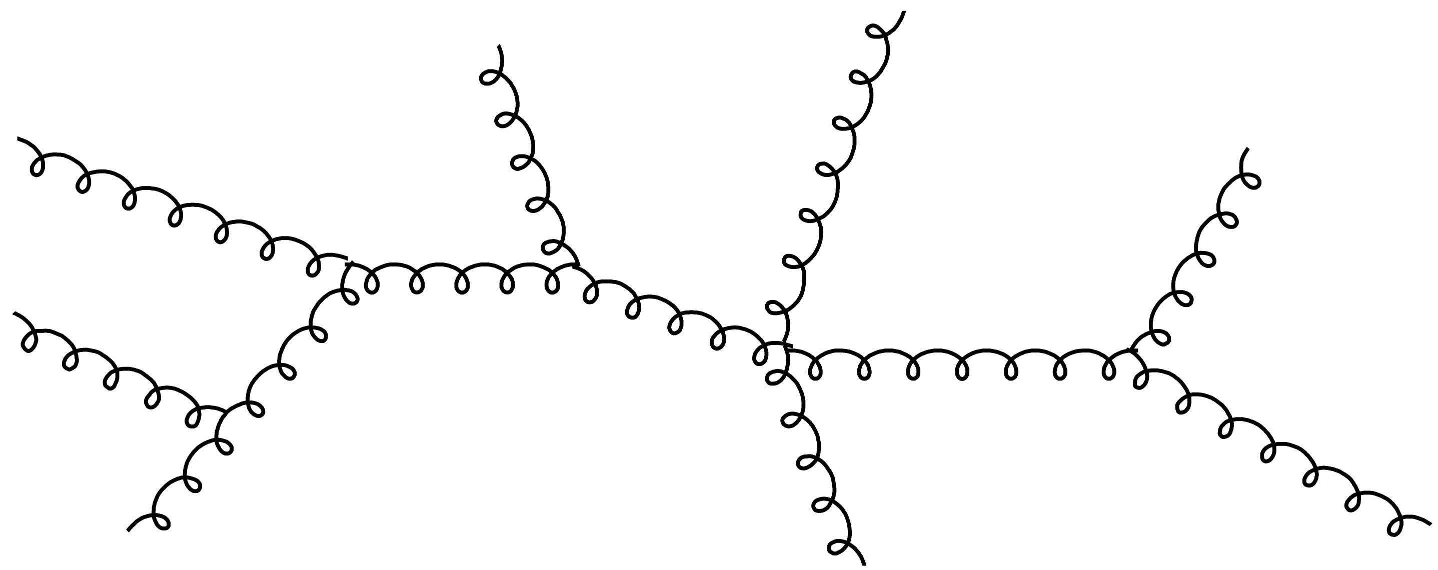
- velocity dependent cusp anomalous dimension
- relation to massive scattering amplitudes
- systematic calculation at loop level

recent development related to scattering amplitudes

Examples of loop-level results

Motivation

consider tree-level gluon amplitudes



number of external gluons	4	5	6	7	8	9	10
number of diagrams	4	25	220	2485	34300	559405	10525900

- Can we compute amplitudes for an arbitrary number of gluons?
- What are the symmetries of the amplitudes?

Some introductory literature:

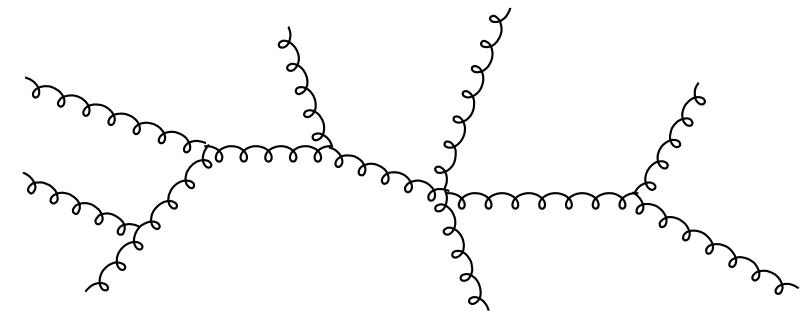
- ``Calculating scattering amplitudes efficiently'',
many detailed examples [Dixon, hep-ph/9601359]

- BCFW recursion relations [Britto, Cachazo, Feng, Witten, hep-th/0501052]

- ``Simplifying Multi-Jet QCD Computation'' [Peskin, arXiv:1101.2414 [hep-ph]
very recent, contains many examples for collider physics

Color decomposition

Amplitudes depend on



- on-shell momenta p_i^μ

$$\sum_{i=1}^{n} p_i^\mu = 0 \qquad p_i^2 = 0$$

- helicities $h_i = \pm 1$
- color a_i

Color decomposition

$$A^{\rm tree}(\{p_i, h_i, a_i\}) = \sum_{\sigma} {\rm Tr}\,(t^{a_{\sigma(1)}} \ldots t^{a_{\sigma(n)}}) A^{\rm tree}(\sigma(1), \ldots \sigma(n))$$

non-cyclic permutations — partial amplitudes

can use simplified color-ordered Feynman rules

at loop level: also double & multiple traces

U(1) decoupling identities

On-shell kinematics and helicity

- on-shell momenta p_i^μ $\quad p_i^2 = 0 \quad \sum_{i=1}^{n} p_i^\mu = 0$

$$p^{\alpha\dot\alpha} = \sigma_\mu^{\alpha\dot\alpha} p^\mu$$

$$p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha} \qquad \langle i\, j\rangle = \lambda_i^\alpha \lambda_{j\,\alpha} \qquad [ij] = \tilde\lambda_{i\,\dot\alpha} \tilde\lambda_j^{\dot\alpha}$$

- polarization vectors

$$\epsilon_{+,i}^{\alpha\dot\alpha} = -\sqrt{2}\,\frac{\tilde\lambda_i^{\dot\alpha}\,\mu_i^\alpha}{\langle\lambda_i\,\mu_i\rangle} \qquad \epsilon_{-,i}^{\alpha\dot\alpha} = \sqrt{2}\,\frac{\lambda_i^\alpha\,\tilde\mu_i^{\dot\alpha}}{[\lambda_i\,\mu_i]}$$

Helicity classification

- supersymmetry $\quad A(+,+,\ldots,+) = 0\,, \qquad A(-,+,\ldots,+) = 0$

(except for n=3)

maximally helicity-violating (MHV) $\quad A(-,+,\ldots,+,-,+,\ldots,+)$

more negative helicities: non-MHV:
e.g. next-to-MHV (NHMV), NNMHV, etc...

Some examples

MHV

$$A(-,+,\ldots,+,-_k,+,\ldots,+) = \frac{\langle 1\, k\rangle^4}{\langle 1\, 2\rangle \ldots \langle n\, 1\rangle}\, \delta^{(4)}(\sum_i p_i)$$

[Parke,Taylor, 1986]

[Berends,Giele, 1988]

non-MHV more complicated (at first sight)

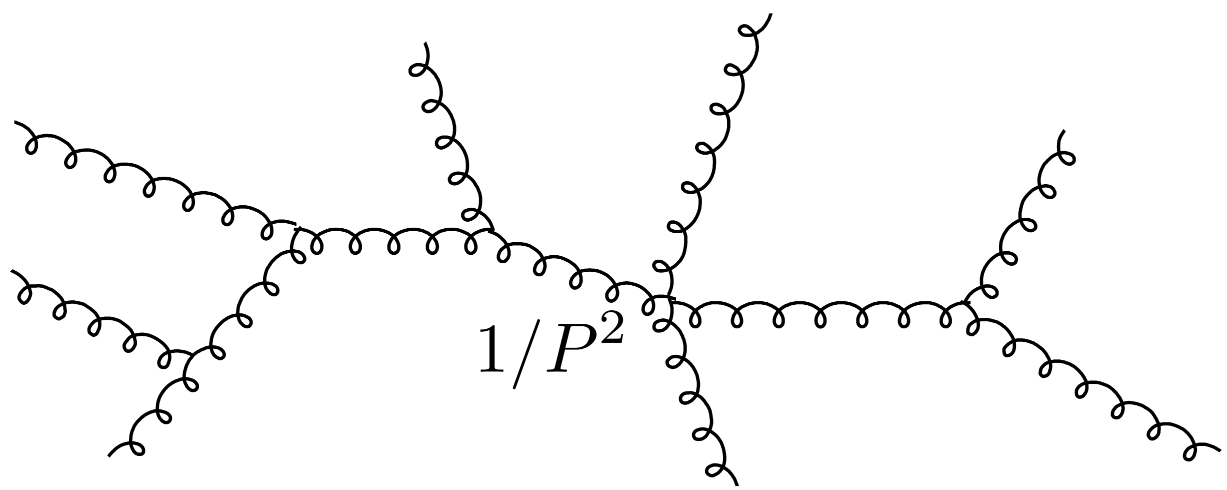
e.g. next-to-MHV (NMHV) 7-point

[Bern, Del Duca, Dixon, Kosower, 2004]

$$\begin{aligned}
&A(1^-,2^-,3^+,4^-,5^+,6^+,7^+) = \\
&\frac{\langle 1\ 2\rangle^3[3\ 5]^4}{t_3^{[3]}[3\ 4][4\ 5]\langle 6\ 7\rangle\langle 7\ 1\rangle\langle 2|3+4|5]\langle 6|4+5|3]} \\
&+ \frac{\langle 2\ 4\rangle^4\langle 1|7+6|5]^3}{t_2^{[3]}t_6^{[3]}\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 6\ 7\rangle\langle 7\ 1\rangle\langle 2|3+4|5]\langle 6|(7+1)(2+3)|4\rangle} \\
&+ \frac{\langle 1\ 2\rangle^3\langle 4|5+6|3]^4}{t_4^{[3]}t_7^{[3]}\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 7\ 1\rangle\langle 6|4+5|3]\langle 7|1+2|3]\langle 4|(5+6)(7+1)|2\rangle} \\
&+ \frac{\langle 4|1+2|3]^4}{t_1^{[3]}[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 4|3+2|1]\langle 7|1+2|3]} \\
&+ \frac{\langle 2\ 4\rangle^4\langle 4|5+6|7]^3}{\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 6\rangle[7\ 1]\langle 4|3+2|1]\langle 4|(5+6)(7+1)|2\rangle\langle 6|(7+1)(2+3)|4\rangle}.
\end{aligned}$$

Factorization formula

- multi-particle poles: internal propagator goes on-shell $P^2 \to 0$



amplitude behaves as

$$A \overset{P^2 \to 0}{\sim} \sum_{\pm} A^{(L)}(P^{\pm}) \frac{1}{P^2} A^{(R)}(-P^{\mp})$$

sum over helicities
(in general sum over all possible intermediate states)

On-shell recursion relation 1/2

- idea: amplitudes as functions in complex plane
- analytically continue to complex momenta

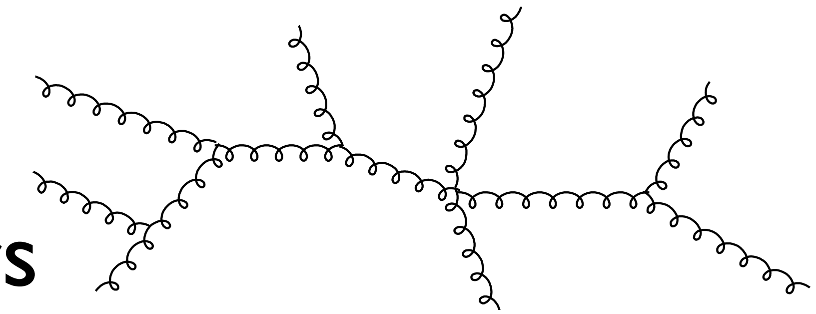
[Britto, Cachazo, Feng + Witten (BCFW), 2004]

$$A \longrightarrow A(z) \qquad \begin{aligned} \lambda_1 &\to \lambda_1 + z\lambda_n \\ \tilde{\lambda}_n &\to \tilde{\lambda}_n - z\tilde{\lambda}_1 \end{aligned}$$

Note: momentum conservation: $\lambda_n\tilde{\lambda}_n + \lambda_1\tilde{\lambda}_1$ unchanged
and on-shell conditions are preserved

called $[h_n h_1\rangle$ shift

- $A(z)$ has poles in the complex plane
- position of poles determined by propagators



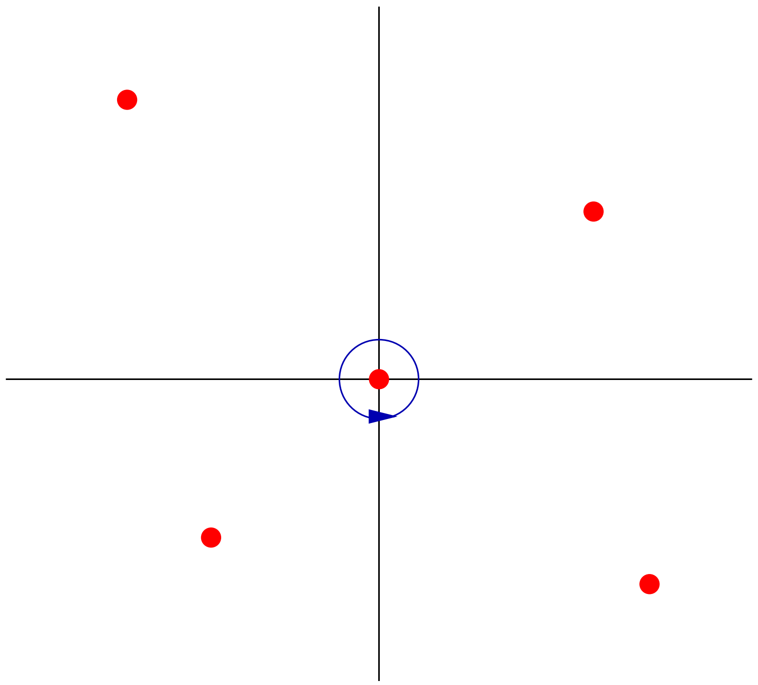
$$\frac{1}{(p_1 + \ldots p_j)^2} \longrightarrow \frac{1}{(p_1 + \ldots + p_n + z\lambda_n\tilde{\lambda}_1)^2}$$

Note: only consecutive momenta (color-ordering)
simple poles

On-shell recursion relation 2/2

Complex analysis:

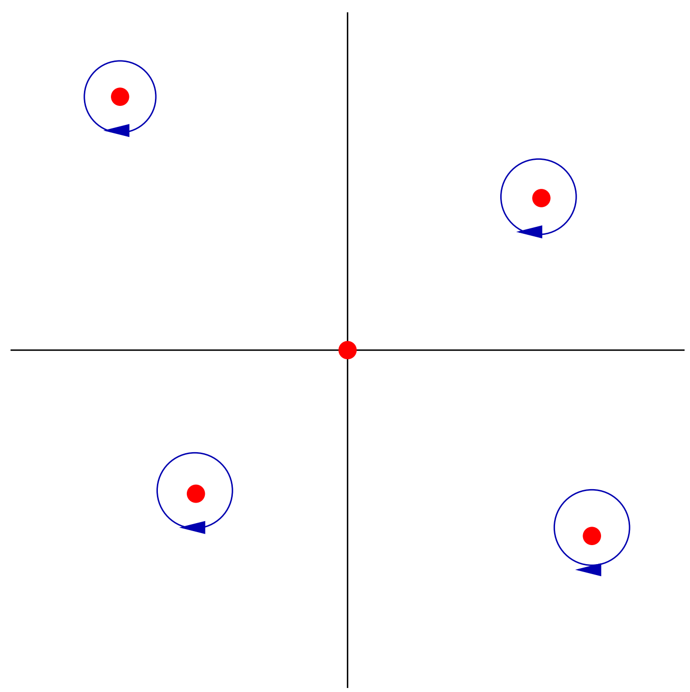
$$A_n = \oint dz \frac{A_n(z)}{z}$$



- large z behavior $A(z) \stackrel{z\to\infty}{\longrightarrow} 0$ (prove later)

$$A_n = -\sum_i \oint_{z_i} \frac{A_n(z)}{z}$$

$$= -\sum_i \mathrm{Res}_{z_i=z}\left(\frac{A_n(z)}{z}\right)$$



- factorization property

$$A_n = \sum_i \sum_{\pm} A^{(L)}(z_i;\pm)\frac{1}{P_i^2}A^{(R)}(z_i;\mp)$$

behavior at large z

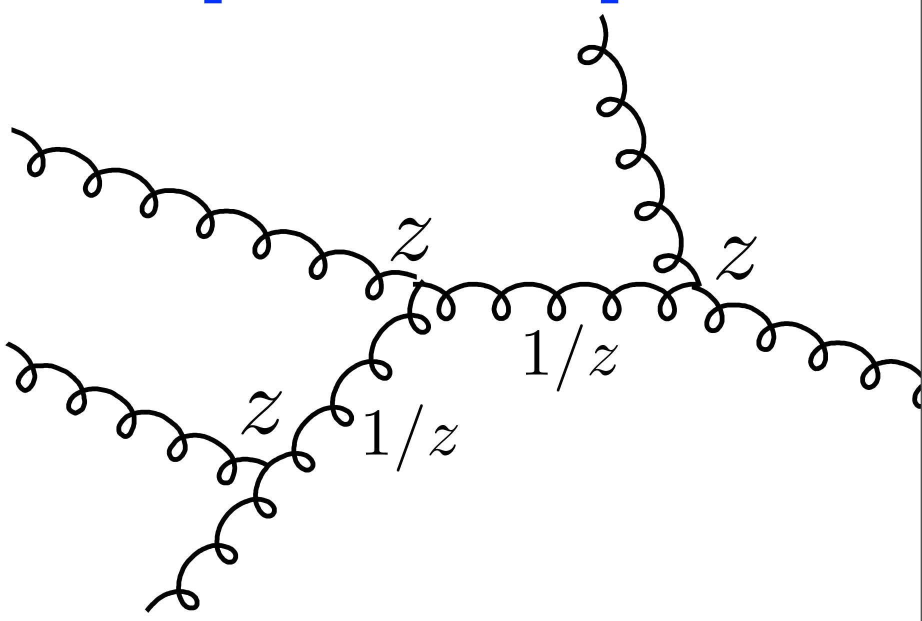
- from analyzing Feynman diagrams

[BCFW, 2005]

3-particle vertex: $\sim k^\mu \sim z$ at worst

internal propagators: $\sim z^{-1}$

=> Feynman diagrams go at worst as $\sim z$



polarization vectors of shifted particles:

$[-+\rangle$ shift: extra z^{-2} from polarization vectors

$$A(z) \stackrel{z\to\infty}{\longrightarrow} z^{-1}$$

- more general analysis (also in higher dimensions):

[Arkani-Hamed, Kaplan, 2008]

$[++\rangle$ $[--\rangle$ shifts also allowed

important for supersymmetric generalization

On-shell 3-point kinematics

$$p_1^2 = 0\,,\; p_2^2 = 0\,,\; p_3^2 = 0\,,\; p_1^\mu \sim p_2^\mu \sim p_3^\mu$$

- for real momenta implies $p_i^\mu = 0$
- complex momenta: two solutions

MHV

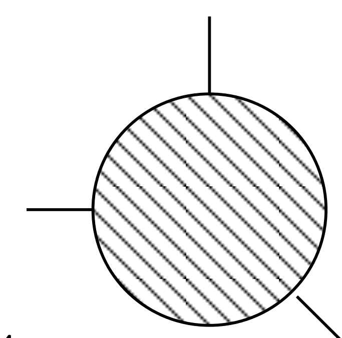
$$\tilde\lambda_1^{\dot\alpha} \sim \tilde\lambda_2^{\dot\alpha} \sim \tilde\lambda_3^{\dot\alpha}$$

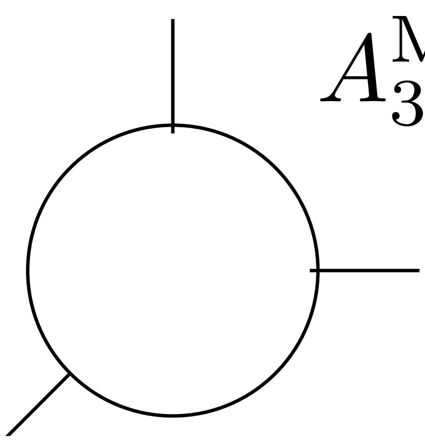
$$[\tilde\lambda_i, \tilde\lambda_j] = 0\,,\; \langle \lambda_i \lambda_j \rangle \neq 0$$

MHV-bar

$$\lambda_1^{\dot\alpha} \sim \lambda_2^{\dot\alpha} \sim \lambda_3^{\dot\alpha}$$

$$\langle \lambda_i, \lambda_j \rangle = 0\,,\; [\tilde\lambda_i \tilde\lambda_j] \neq 0$$





$$A_3^{\mathrm{MHV}} = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}$$

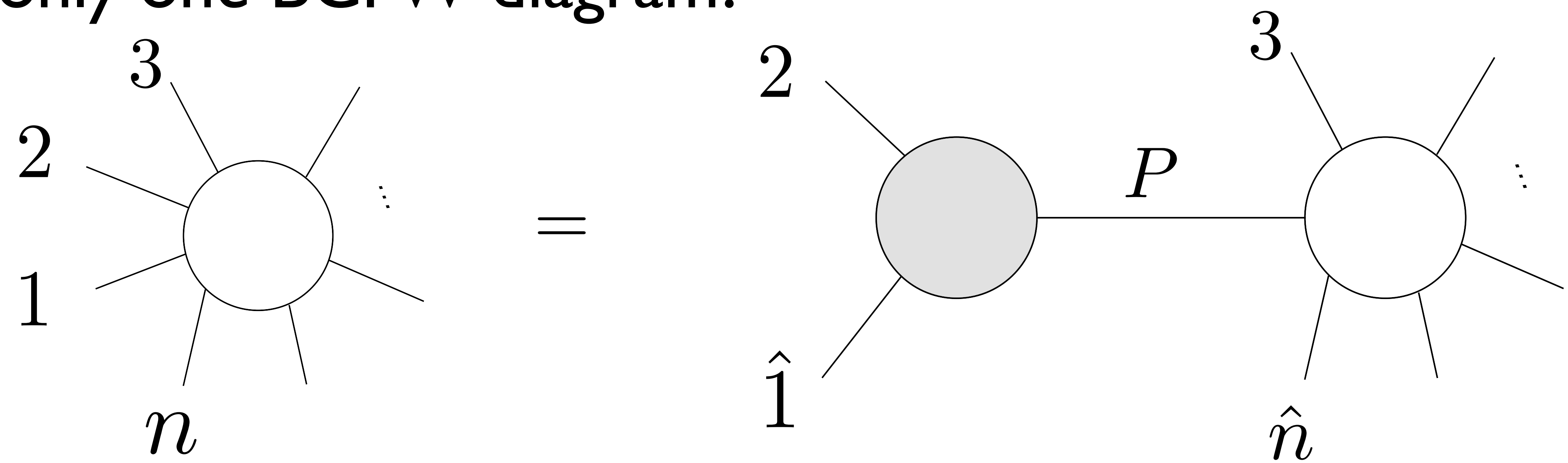
$$A_3^{\overline{\mathrm{MHV}}} = \frac{[ij]^4}{[12][23][31]}$$

Example: MHV n-point formula

- use BCFW to prove Parke-Taylor formula

$$A_n^{\mathrm{MHV}} = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$

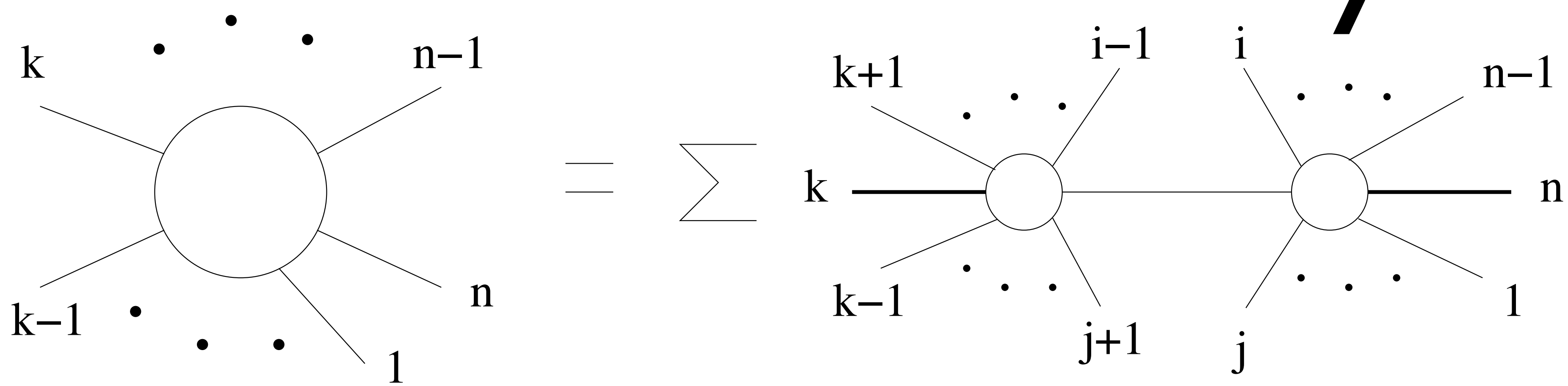
- only one BCFW diagram!



$$A_n^{\mathrm{MHV}}(1,2,\ldots,n) = A_3^{\overline{\mathrm{MHV}}}(\hat{1},2,\hat{P})\,\frac{1}{P^2}\,A_{n-1}^{\mathrm{MHV}}(-\hat{P},3,\ldots,\hat{n})$$

very instructive exercise!

BCFW summary



- n-point amplitudes obtained recursively from lower-point ones
- all ingredients are on-shell
- BCFW recursion and on-shell three-point amplitudes determine all amplitudes

- closed-form expressions known for special classes e.g. split-helicity $A(-,\ldots,-,+,\ldots+)$

  supersymmetry: extend this to any helicities
- note: different shifts lead to different (but equivalent) representations

BCFW for planar loop integrands

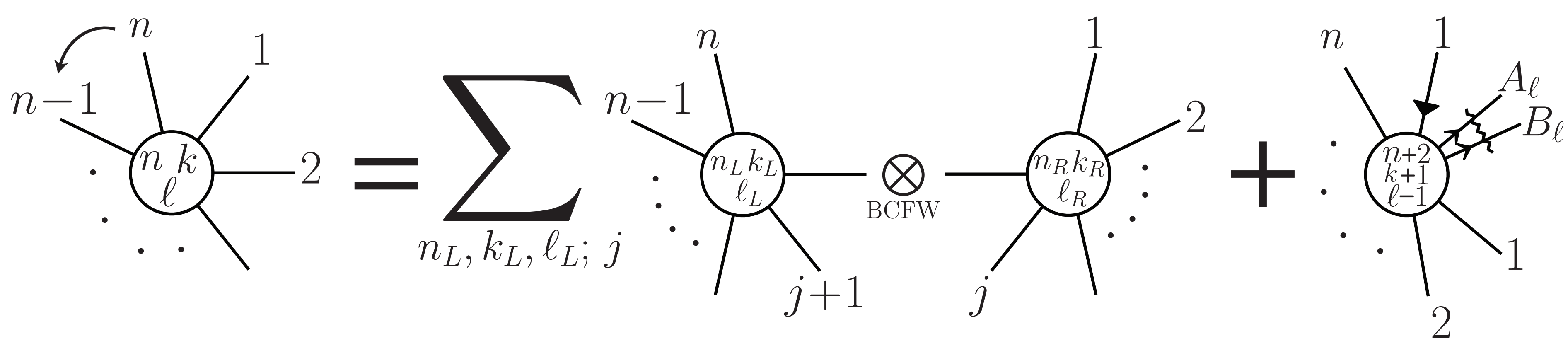
[Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka , 2010]
[related work by Boels, 2010]

- idea: loop integrand is more or less a tree (with loops...)
- planar: dual coordinates to define loop variables $x_i - x_{i+1} = p_i$

$$A_n^{(L)}(x_1, \ldots x_n) = \int d^4y_1 \ldots \int d^4y_L \, I^{(L)}(x_1, \ldots, x_n; y_1, \ldots, y_L)$$

Apply BCFW shift:

- extra term in recursion from loop propagator going on-shell
- forward limit of N=4 SYM trees well defined [Caron-Huot, 2010]



- same idea for D-dimensional integrand, but hard in practice

or: extend to massive case to deal with IR divergences

[Alday, JMH, Plefka, Schuster, 2008]

Symmetries (first part)

conformal symmetry follows from Lagrangian in position space => Fourier transform

[Witten, 2003]

single-particle generators:

$$p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha} \qquad d_i = \lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} + \tilde\lambda_{i\,\dot\alpha} \frac{\partial}{\partial \tilde\lambda_{i\,\dot\alpha}} + 1$$

$$k_{i\,\alpha\dot\alpha} = \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \tilde\lambda_i^{\dot\alpha}} \qquad m_i^{\alpha\beta} = \lambda_i^\alpha \frac{\partial}{\partial \lambda_{i\beta}} - \lambda_i^\beta \frac{\partial}{\partial \lambda_{i\alpha}} \qquad m_i^{\dot\alpha\dot\beta}$$

realization on scattering amplitude $p = \sum_{i=1}^n p_i$ etc.

Exercise 1: verify conformal algebra

Exercise 2: show symmetry of MHV amplitudes

e.g. $$k \left[\delta^{(4)}(p) \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} \right] = 0$$

Note: for collinear momenta modification of generators required. Important at loop level

[Beisert et al.,2009]

End of lecture 1

Thank you!

Scattering amplitudes in N=4 SYM

- contain all gluon amplitudes, and amplitudes of massless QCD
- N=4 SYM: max. supersymmetry, AdS/CFT, expected integrability

On-shell particle content in single supermultiplet:

$$\Phi(p,\eta) = G^{+}(p) + \eta^{A}\Gamma_{A}(p) + 1/2\eta^{A}\eta^{B}S_{AB} + \ldots + 1/4!\epsilon_{ABCD}\eta^{A}\eta^{B}\eta^{C}\eta^{D}G^{-}(p)$$

Grassmann variable η^A for bookkeeping, $A = 1,2,3,4$

- superamplitudes $\mathcal{A}_n = < \Phi_1 \ldots \Phi_n >$
- supersymmetry $p = \sum_{i=1}^{n} \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}}\,, \qquad q = \sum_{i=1}^{n} \lambda_i^{\alpha} \eta_i^{A}\,, \qquad \bar{q} = \sum_{i=1}^{n} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \eta_i^{A}}$

implies $\mathcal{A}_n = \delta^{(4)}(p)\delta^{(8)}(q)\frac{\mathcal{P}_n}{\langle 1\,2\rangle \ldots \langle n\,1\rangle}$

$\mathcal{P}_n$ is a polynomial in the $\{\eta_i\}$, MHV: η^0, NMHV: η^4, NNMHV: η^8 etc.

$\delta(\eta^1) = \eta^1$

- MHV case:

$$\mathcal{P}_n^{\rm MHV} = 1\,, \qquad \mathcal{A}_n^{\rm MHV} = \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\langle 1\,2\rangle \ldots \langle n\,1\rangle}$$

[Nair, 1988]

On-shell susy simplifications

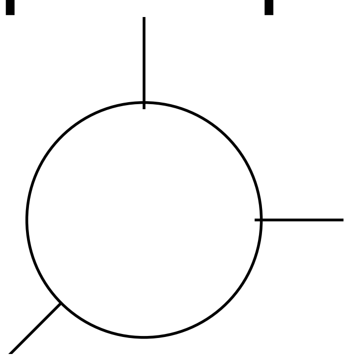
- Sums over intermediate states/helicities become trivial Grassmann integrals

$$\sum_{\text{states}} A_L \frac{1}{P^2} A_R \quad \Rightarrow \quad \int d^4\eta \mathcal{A}_L \frac{1}{P^2} \mathcal{A}_R$$

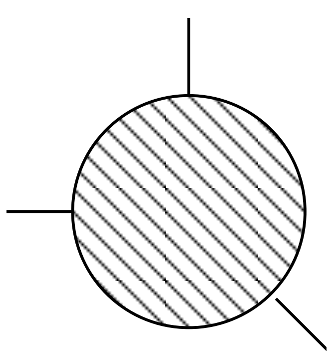
- Grassmann integrals carried out using susy delta functions

=> also very important at loop level

- simpler 3-particle building blocks



$$\mathcal{A}_3^{\mathrm{MHV}} = \frac{\delta^{(8)}(\lambda_1^\alpha \eta_1^A + \lambda_2^\alpha \eta_2^A + \lambda_3^\alpha \eta_3^A)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} = \frac{\delta^{(8)}(q^{\alpha A})}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$



$$\mathcal{A}_3^{\overline{\mathrm{MHV}}} = \frac{\delta^{(4)}(\eta_1^A[23] + \eta_2^A[31] + \eta_3^A[12])}{[12][23][31]}$$

Supersymmetric BCFW

- Supersymmetric shift

$$\hat{\tilde{\lambda}}_n^{\dot{\alpha}} = \tilde{\lambda}_n^{\dot{\alpha}} + z\tilde{\lambda}_1^{\dot{\alpha}} \qquad \hat{\lambda}_1^{\alpha} = \lambda_1^{\alpha} - z\lambda_n^{\alpha}$$

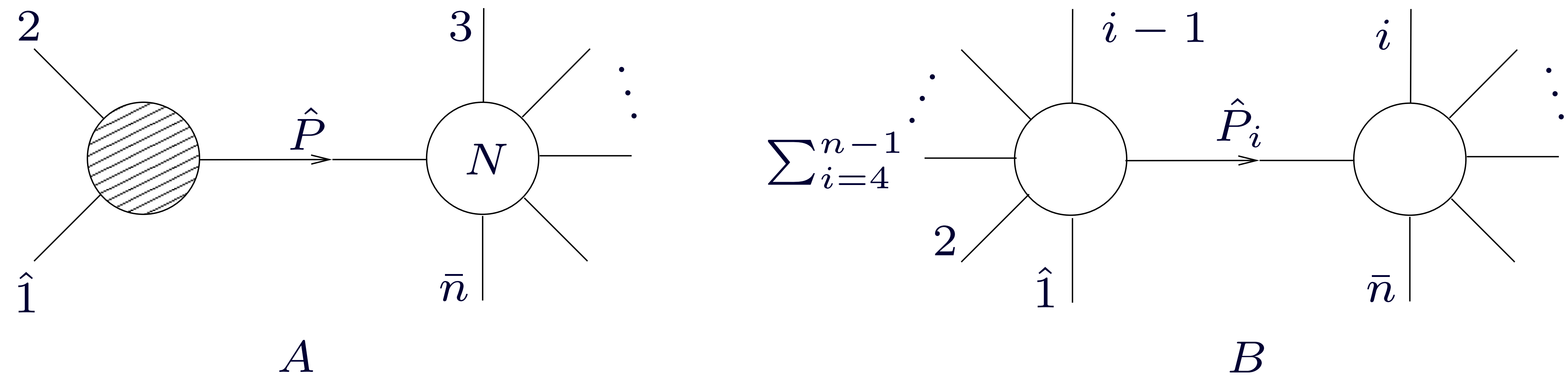
$$\hat{\eta}_n^A = \eta_n^A + z\eta_1^A$$

- large z behavior related to bosonic case by susy
  => all BCFW shifts possible [Arkani-Hamed, Cachazo, Kaplan, 2008]

bosonic:
$$A = \sum_{P_i} \sum_h A_L^h(z_{P_i}) \frac{1}{P_i^2} A_R^{-h}(z_{P_i})$$

supersymmetric:
$$\mathcal{A} = \sum_{P_i} \int d^4\eta_{P_i} \mathcal{A}_L(z_{P_i}) \frac{1}{P_i^2} \mathcal{A}_R(z_{P_i})$$

Example: NHMV tree in N=4 SYM



- MHV superamplitude $\mathcal{A}_n^{\mathrm{MHV}} = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}$
- inhomogeneous term $B = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle} \sum_{i=4}^{n-1} R_{n;2\,i}$
- ansatz $\mathcal{A}_n^{\mathrm{NMHV}} = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle} \sum_{2\leq s<t\leq n-1} R_{n;st}$

prove by induction!

Example: NHMV tree in N=4 SYM

- final result $$\mathcal{A}_n^{\rm NMHV} = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle} \sum_{2\le s<t\le n-1} R_{n;st}$$

$$R_{n;i,j} = \frac{< i\ i-1 >< j\ j-1 > \delta^{(4)}(\Xi_{n;ij})}{x_{ij}^2 < n|x_{ni}x_{ij}|j >< n|x_{ni}x_{ij}|j-1 >< n|x_{nj}x_{ji}|i >< n|x_{nj}x_{ji}|i-1 >}$$

$$\Xi_{n;ij} = < n|x_{ni}x_{ij}|\theta_{jn} > + < n|x_{nj}x_{ji}|\theta_{in} >$$

- with dual coordinates $\lambda_i^\alpha \tilde{\lambda}_i^{\dot\alpha} = x_i^{\alpha\dot\alpha} - x_{i+1}^{\alpha\dot\alpha}\,, \qquad \lambda_i^\alpha \eta_i^A = \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A}$

- Note: different shifts lead to different representations; interpretation as residue theorems [Arkani-Hamed]

- solution for all tree-level amplitudes in similar way

[Drummond, JMH, 2008]

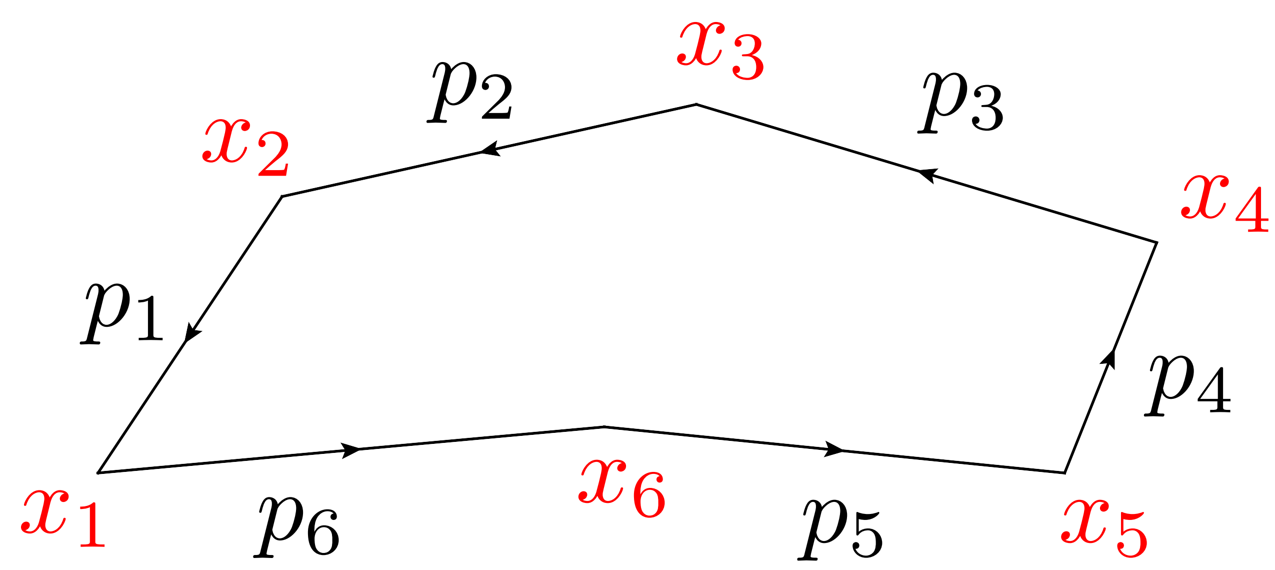
Symmetries (second part)

(super)conformal symmetry from Lagrangian [Witten, 2003]

why is the result so simple?

dual space

$$x_i^\mu - x_{i+1}^\mu = p_i^\mu$$



dual conformal symmetry = conformal symmetry in dual space

- first hints seen in loop integrals for 4pt amplitude

  [Drummond, JMH, Sokatchev, V. Smirnov, 2006]

- for generic helicity amplitude;
  extension to dual superconformal symmetry

  [Drummond, JMH, Korchemsky Sokatchev, 2008]

- natural in string theory [Alday, Maldacena, 2007]
  [Maldacena, Berkovits, 2008; Beisert et al, 2008]

Dual conformal symmetry

[Drummond, JMH, Korchemsky Sokatchev, 2008]

Explicit form of infinitesimal generators

- dual space $x_i^{\alpha\dot\alpha} - x_{i+1}^{\alpha\dot\alpha} = p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha}$
- bosonic part of generator

$$K_{\alpha\dot\alpha} = \sum_i \left[x_{i\alpha}{}^{\dot\beta} x_{i\dot\alpha}{}^{\beta} \frac{\partial}{\partial x_i^{\beta\dot\beta}} + x_{i\dot\alpha}{}^{\beta} \lambda_{i\alpha} \frac{\partial}{\partial \lambda_i^{\beta}} + x_{i+1\,\alpha}{}^{\dot\beta} \tilde\lambda_{i\dot\alpha} \frac{\partial}{\partial \tilde\lambda_i^{\dot\beta}} \right]$$

- dual superspace $\theta_{i\,\alpha}^{A} - \theta_{i+1\,\alpha}^{A} = q_{i\,\alpha}^{A} = \eta_i^{A} \lambda_{i\,\alpha}$
- final expression

$$K^{\alpha\dot\alpha} = \sum_i \left[x_i^{\alpha\dot\beta} x_i^{\dot\alpha\beta} \frac{\partial}{\partial x_i^{\beta\dot\beta}} + x_i^{\dot\alpha\beta} \lambda_i^{\alpha} \frac{\partial}{\partial \lambda_i^{\beta}} + x_{i+1}^{\alpha\dot\beta} \tilde\lambda_i^{\dot\alpha} \frac{\partial}{\partial \tilde\lambda_i^{\dot\beta}} + x_i^{\dot\alpha\beta} \theta_i^{\alpha B} \frac{\partial}{\partial \theta_i^{\beta B}} + \tilde\lambda_i^{\dot\alpha} \theta_{i+1}^{\alpha B} \frac{\partial}{\partial \eta_i^{B}} \right]$$

- some dual conformal covariants $x_{qr}^2 \xrightarrow{\text{inversion}} \dfrac{x_{qr}^2}{x_q^2 x_r^2}$

e.g. $x_{qr}^2 \quad \langle q-1\, q\rangle \quad \langle p|x_{pq}x_{q\,r}|r\rangle \quad \langle r|x_{rt}x_{ts}|\theta_{sr}\rangle$

=> easy to verify symmetry!

Yangian symmetry

superconformal psu(2,2|4) symmetry $[J_a, J_b\} = f_{ab}{}^c J_c\,, \qquad J_a = \sum_{i=1}^{n} J_{ia}$

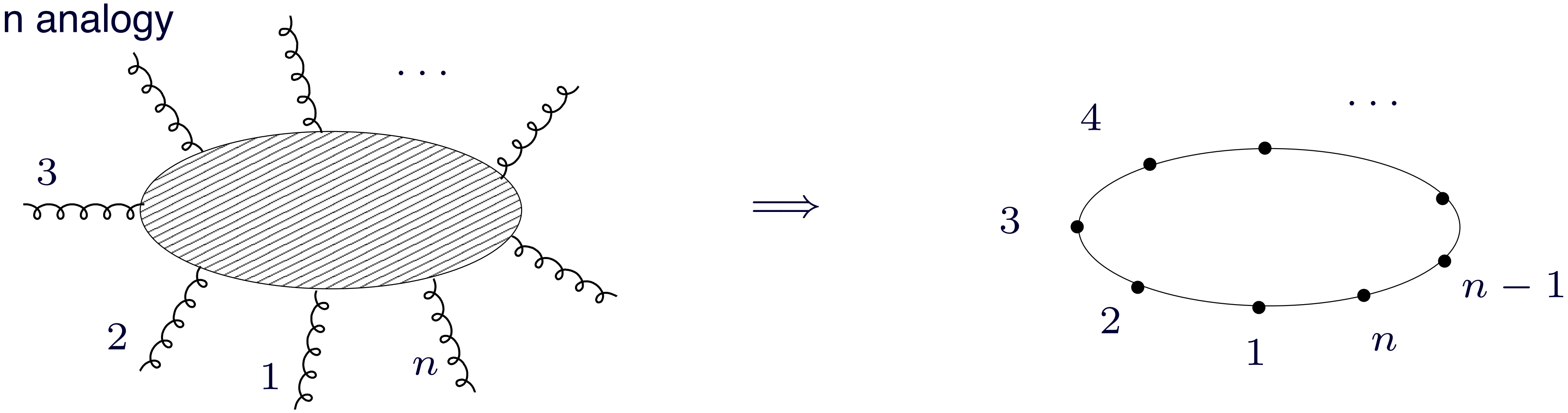
dual (super)conformal symmetry

closure of algebra? Yangian of psu(2,2|4) [Drummond, JMH, Plefka, 2009]

level-one Yangian generators $Q_a = f_a{}^{cb} \sum_{1 \le i < j \le n} J_{ib} J_{jc}$

- Yangian algebra $[J_a^{(1)}, J_b^{(0)}\} = f_{ab}{}^c J_c^{(1)}$ and higher levels
- Serre relations

spin chain analogy



all tree amplitudes have Yangian symmetry

$$J_a^{(0)} \mathcal{A}^{\rm tree} = 0$$
$$J_a^{(1)} \mathcal{A}^{\rm tree} = 0$$

generators very simple in (super)twistor space

$$J^{(0)\,\bar A}{}_{\bar B} = \sum_i \mathcal{Z}_i^{\bar A} \frac{\partial}{\partial \mathcal{Z}_i^{\bar B}} \qquad J^{(1)\,\bar A}{}_{\bar B} = -\sum_{i>j} \left[\mathcal{Z}_i^{\bar A} \mathcal{Z}_j^{\bar C} \frac{\partial}{\partial \mathcal{Z}_i^{\bar C}} \frac{\partial}{\partial \mathcal{Z}_j^{\bar B}} - (i \leftrightarrow j) \right]$$

How to extract component amplitudes

[Drummond, JMH, 2008]

[Dixon, JMH, Plefka, Schuster, 2010]

- use N=4 SYM result to obtain gluon and fermion amplitudes of massless QCD
- reminder: $\Phi(p,\eta) = G^{+}(p) + \eta^{A}\Gamma_{A}(p) + \ldots + 1/4!\epsilon_{ABCD}\eta^{A}\eta^{B}\eta^{C}\eta^{D}G^{-}(p)$

negative helicity gluon state at point i $\iff$ $1/4!\epsilon_{ABCD}\eta_i^A\eta_i^B\eta_i^C\eta_i^D = (\eta_i)^4$

positve helicity gluon state at point j $\iff$ $\eta_j^A = 0$

- e.g. $\mathcal{A}_n = (\eta_i)^4(\eta_j)^4 A_n^{\mathrm{MHV}}(i^-,j^-) + \ldots$

equivalently $A_n^{\mathrm{MHV}}(i^-,j^-) = \int d^4\eta_i \int d^4\eta_j\, \mathcal{A}_n$

- Grassmann delta functions

=> extraction of components is simple linear algebra

- similarly: up to (at least) 3 quark/antiquark pairs

Numerical applications

- fast evaluation of tree amplitudes important
  in numerical implementation of generalized unitarity;
  and for real emission

  many existing programs: MadGraph, CompHEP, AMEGIC++, COMIX, ALPHA, HELAC, O`MEGA/WHIZARD, ...

  based on Feynman diagrams, Berends-Giele (off-shell) recursions, ...

- analytic tree amplitudes derived from N=4 SYM [Dixon, JMH, Plefka, Schuster, 2010]
are beingvused at one-loop in Blackhat
[Bern, Diana, Dixon, Febres Cordero, Hoeche, Ita, Kosower, Maitre, Ozeren]

- analytic NMHV formulas faster;
  NNMHV at large n Berends-Giele faster

[Badger, Biedermann, Hackl, Plefka, Schuster, Uwer, 2012]

Extra slides

[Badger, Biedermann, Hackl, Plefka, Schuster, Uwer, 2012]
[GGT: Dixon, JMH, Plefka, Schuster, 2010]

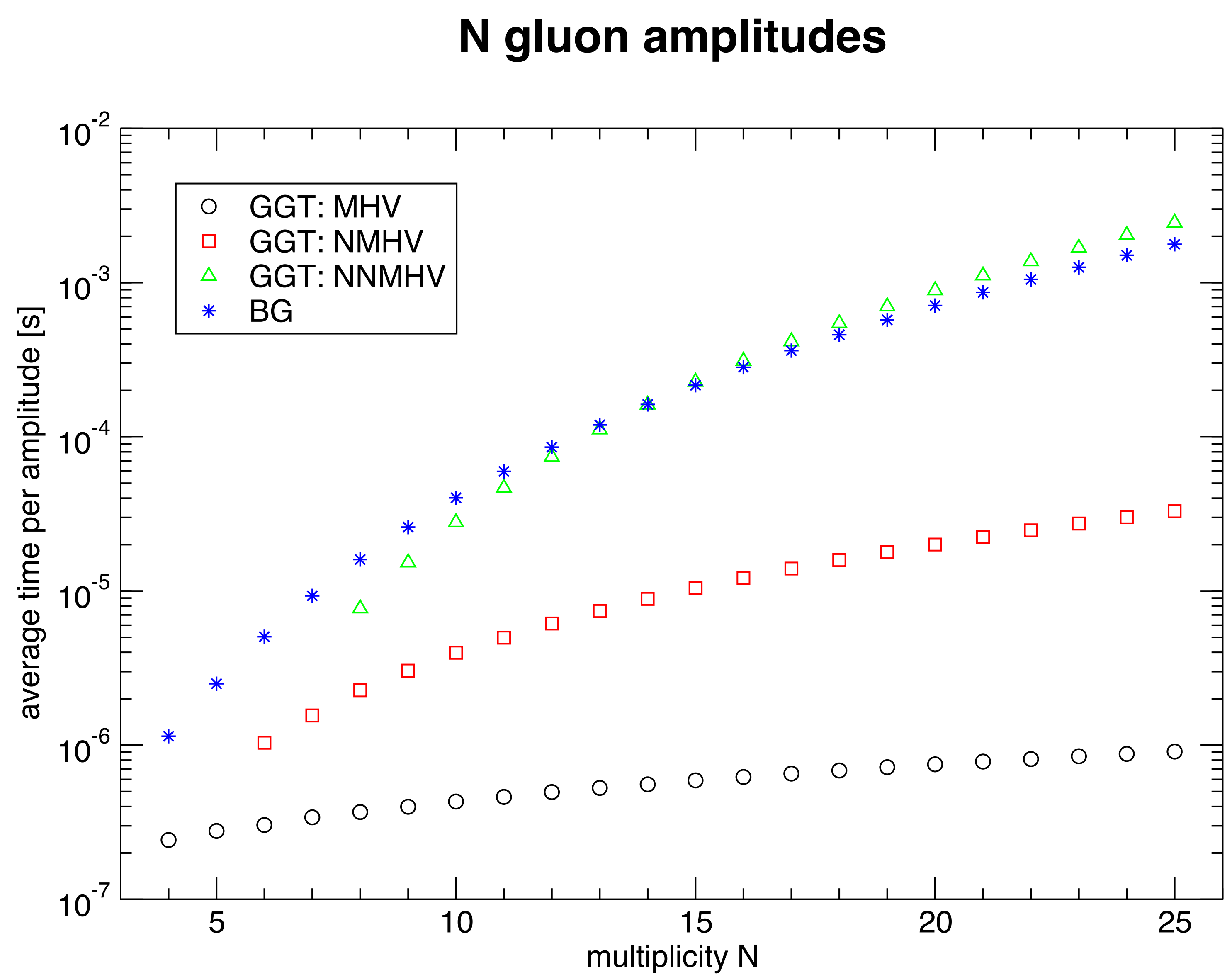


Figure 1: Average time required per phase space point for the evaluation of pure gluon amplitudes as function of the parton multiplicity.

Extra slides

[Badger, Biedermann, Hackl, Plefka, Schuster, Uwer, 2012]
[GGT: Dixon, JMH, Plefka, Schuster, 2010]

(N-2) gluon 2 quark amplitudes

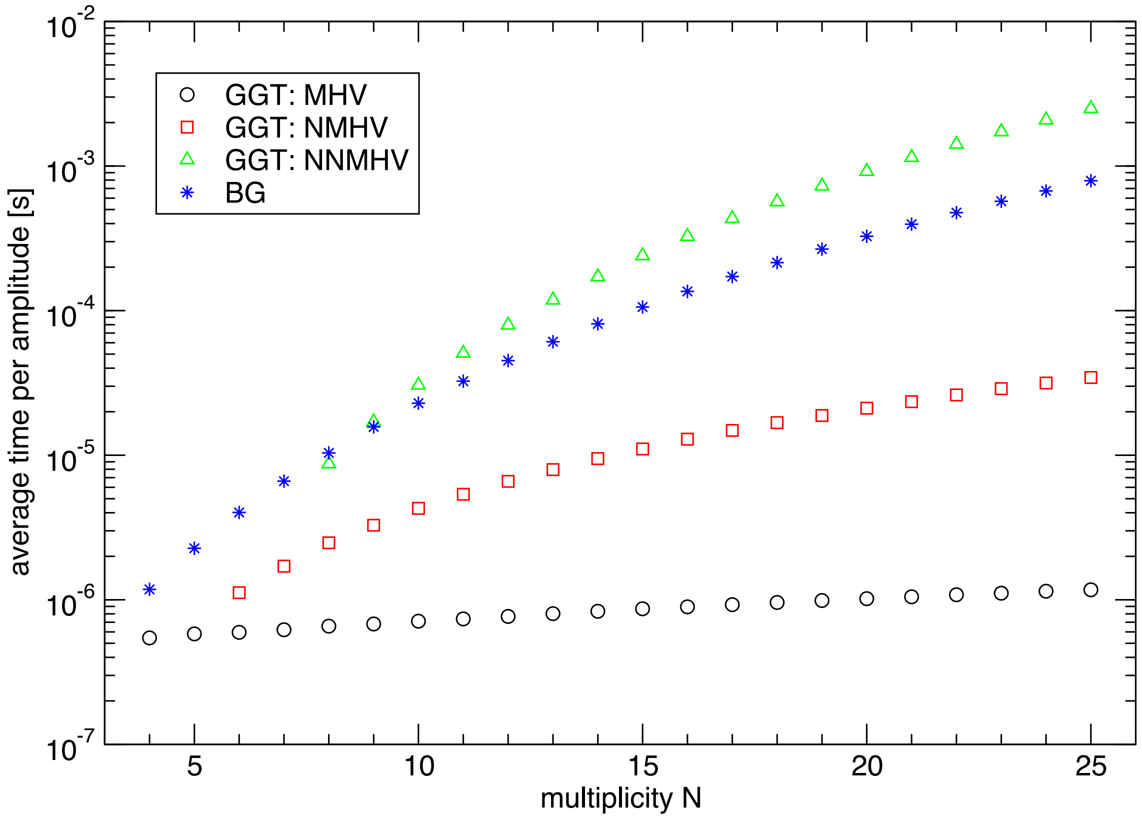


Figure 2: Evaluation time per phase space point for amplitudes with a quark–anti-quark pair and $N-2$ gluons.

(N-6) gluon 6 quark amplitudes

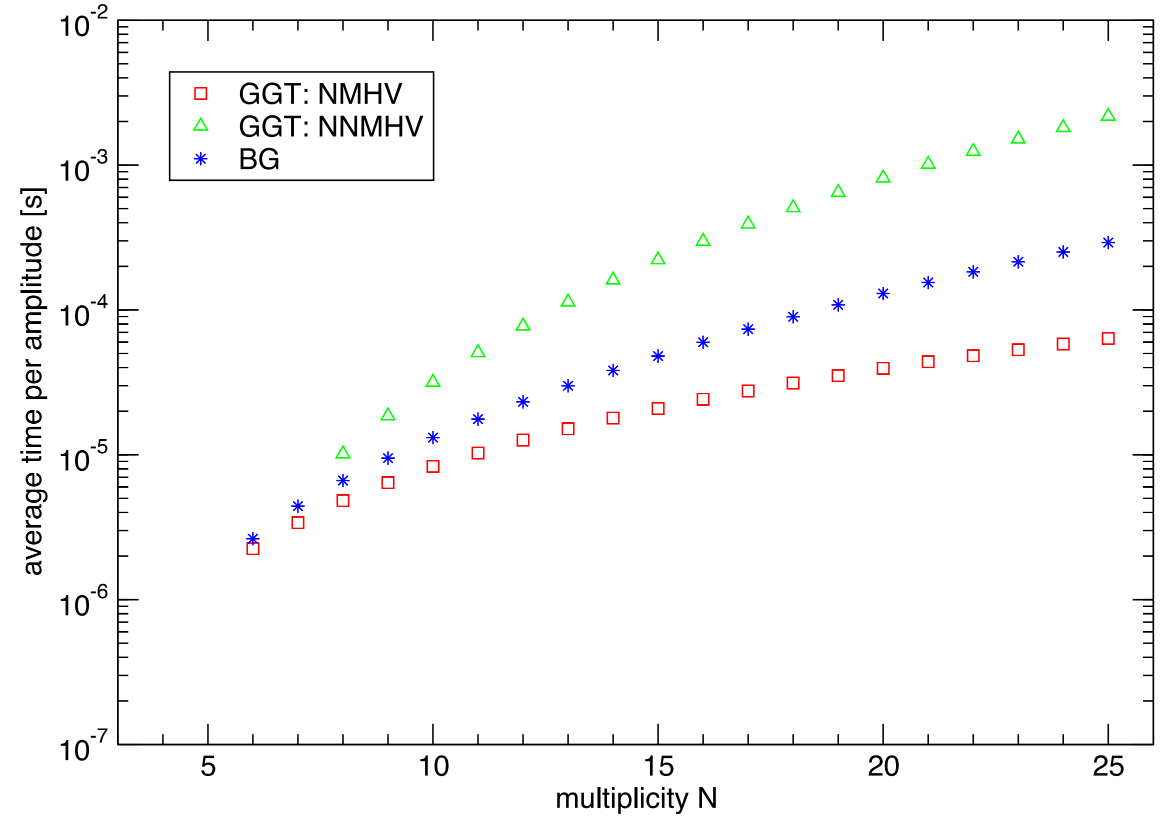


Figure 4: Evaluation time per phase space point for amplitudes with three quark–anti-quark pairs (different flavors) and $N-6$ gluons.

Extra slides

[Badger, Biedermann, Hackl, Plefka, Schuster, Uwer, 2012]
[GGT: Dixon, JMH, Plefka, Schuster, 2010]

Numerical precision of quark amplitudes

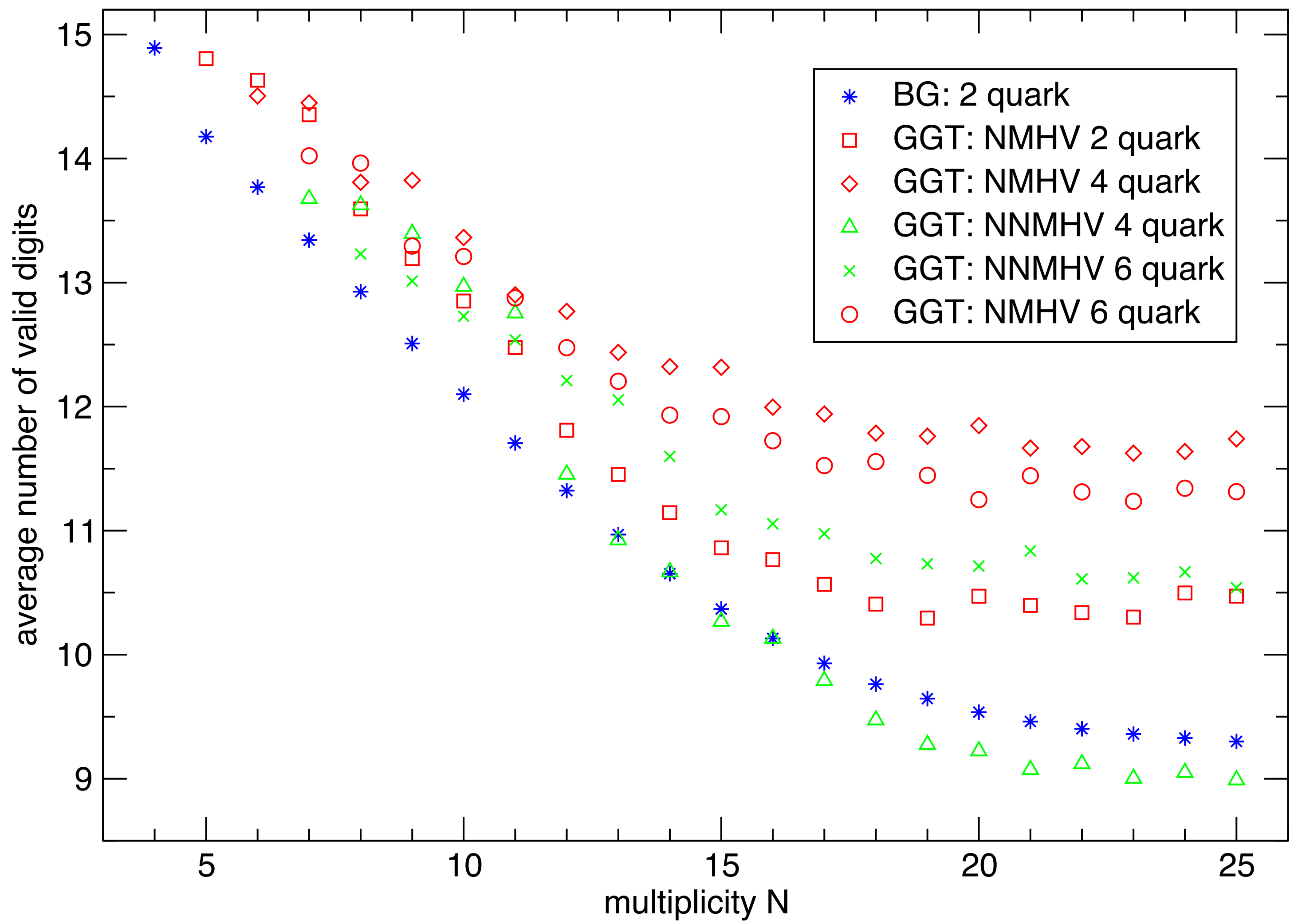


Figure 9: Average accuracy for amplitudes involving quarks. Phase-space generation by sequential splitting.

A first look at loop integrals and divergences

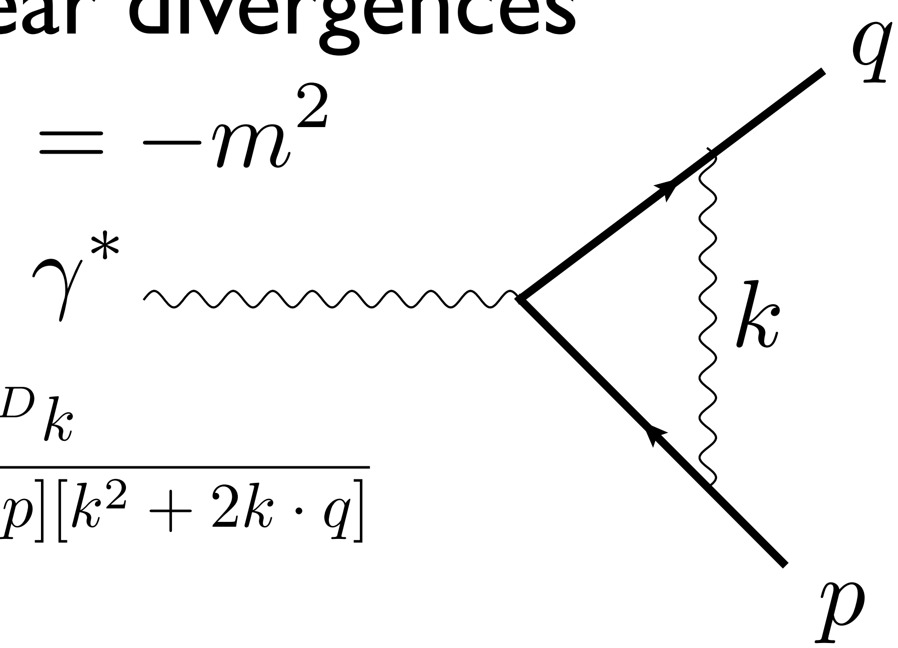
• ultraviolet (UV) divergences:
regularization (e.g. dimensional regularization);
renormalization of parameters of Lagrangian

• well understood, renormalization group (RG),
discussed in all textbooks

are there other types of divergences?

Infrared divergences

- on-shell amplitudes can have soft/collinear divergences
- example: massive form factor $p^2 = q^2 = -m^2$



sample integral:

$$I = \int \frac{d^D k}{k^2[(k+p)^2+m^2][(k+q)^2+m^2]} = \int \frac{d^D k}{k^2[k^2+2k\cdot p][k^2+2k\cdot q]}$$

dimensional regulator: $D = 4 - 2\epsilon$

logarithmic divergence for small k $\int_0^\Lambda dr\, r^{-1-2\epsilon} = -\frac{1}{2\epsilon}\Lambda^{-2\epsilon}$ $\epsilon < 0$

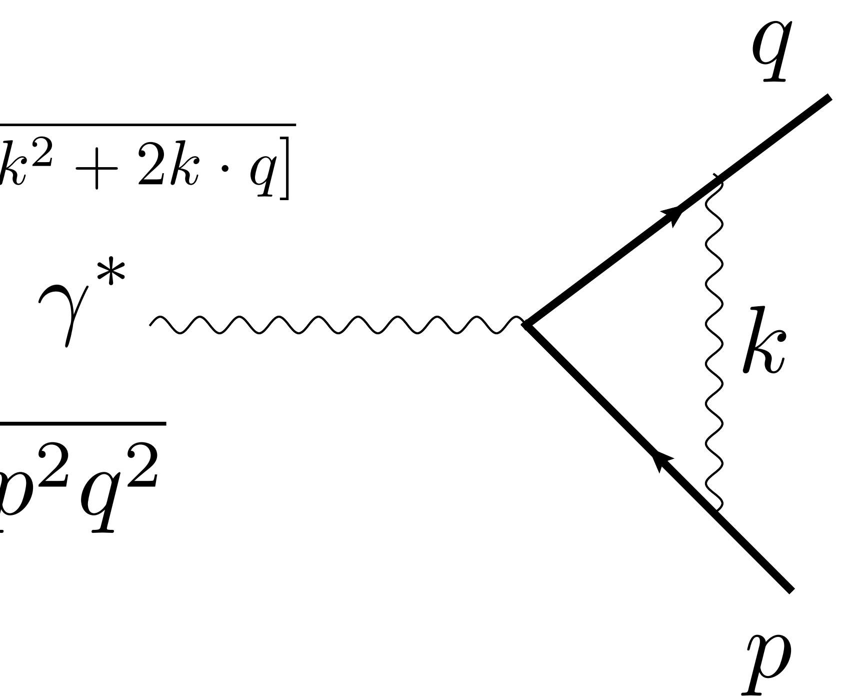
``soft region''

- for massless quarks: $p^2 = q^2 = 0$ extra divergence from collinear regions $k^\mu \sim p^\mu$ or $k^\mu \sim q^\mu$

$$I \sim \Gamma \frac{1}{\epsilon^2} + \ldots$$

An IR/UV connection

$$I = \int \frac{d^D k}{k^2[(k+p)^2+m^2][(k+q)^2+m^2]} = \int \frac{d^D k}{k^2[k^2+2k\cdot p][k^2+2k\cdot q]}$$



- compute UV divergence:

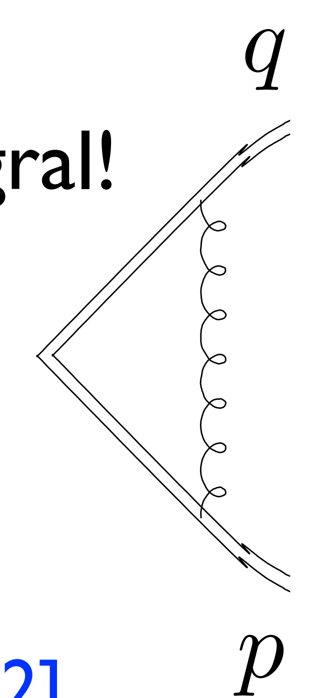
$$I \sim \frac{1}{\epsilon}\,\phi\cot\phi \qquad\qquad \cos\phi = p\cdot q/\sqrt{p^2q^2}$$

- consider eikonal limit $k^\mu \to 0$; formally we have

$$I \to \int \frac{d^D k}{k^2[2k\cdot p][2k\cdot q]}$$

integral is formally zero, UV and IR divergence cancellation

above formula is momentum space version of Wilson loop integral!



$$\langle W\rangle \sim \frac{1}{\epsilon_{\rm UV}}\Gamma_{\rm cusp}(\phi)$$

IR divergences of massive scattering governed by cusp anomalous dimension [Korchemksy, Radyushkin, 1986; 1992]

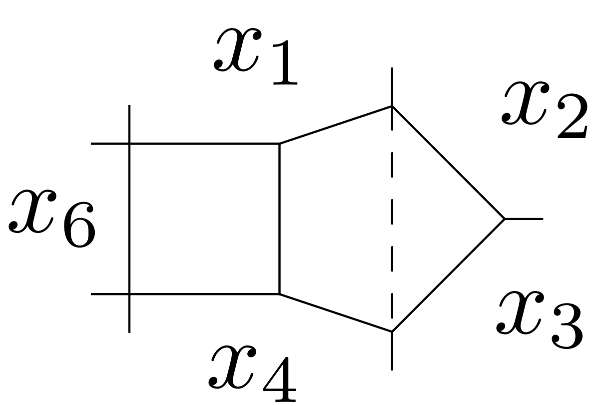
Comments on loop integrals & IR divergences

or: what is a good basis for loop integral basis?

- at two loops: integrals with irreducible numerators; good choice?
- issue clearest in N=4 super Yang-Mills: UV finite theory

Key lesson: use numerators that soften IR divergences

example: [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka , 2010]



integration points: $x_a \, , x_b$

numerator: $\sim \mathrm{tr}(x_{12}\tilde{x}_{2b}x_{b3}\tilde{x}_{34}x_{46}\tilde{x}_{61})$

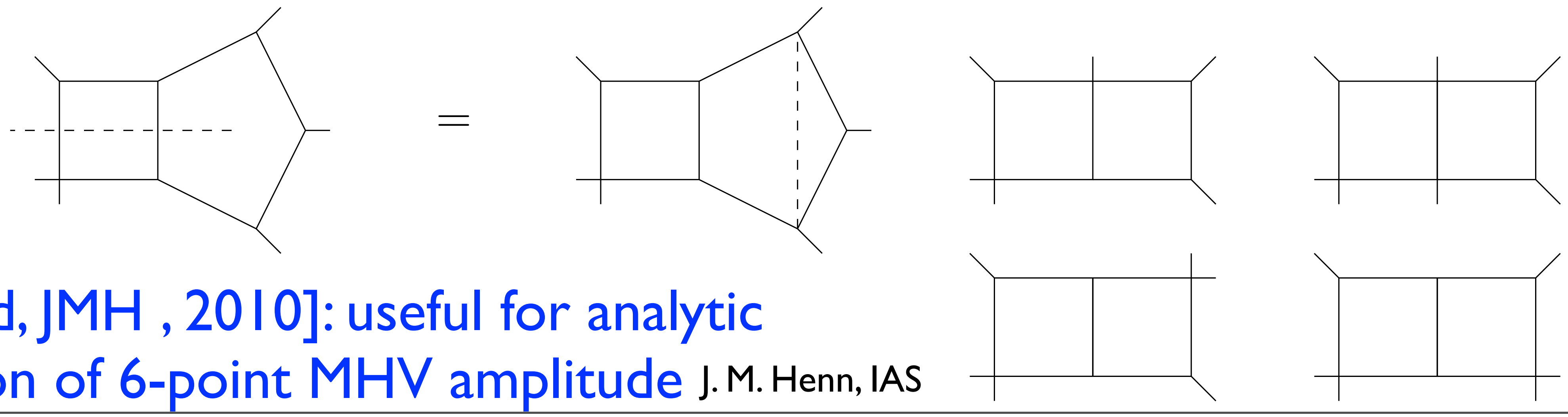
$$x_{ij} = x_i - x_j$$

vanishes in collinear regions!

$$x_b = z x_1 + (1-z) x_2$$
$$x_b = z x_3 + (1-z) x_4$$

Integral is finite!

useful variables: momentum twistors [Hodges , 2010]



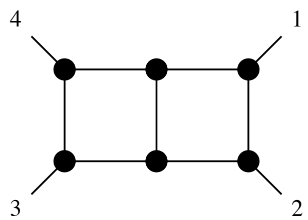
[Drummond, JMH , 2010]: useful for analytic evaluation of 6-point MHV amplitude J. M. Henn, IAS

Simple representation of loop integrands

[Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka , 2010]

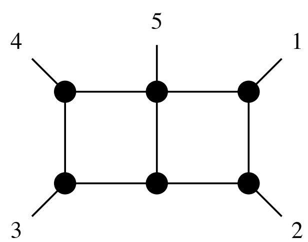
6.1 All 2-loop MHV Amplitudes

The two-loop amplitude for 4- and 5-particles is given by, respectively,

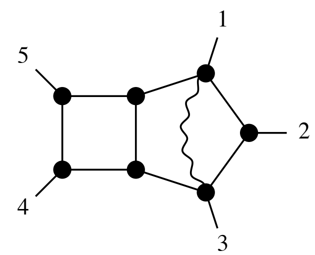


$$\langle 2341\rangle\langle 3412\rangle\langle 4123\rangle \quad + \begin{matrix}\text{cyclic}\\ \text{(no repeat)}\end{matrix} \tag{46}$$

and

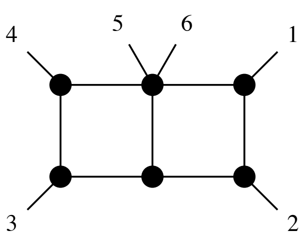


$\langle 2345\rangle\langle 5123\rangle\langle 3412\rangle$ +

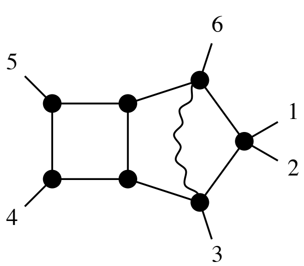


$$\langle 3451\rangle\langle 4513\rangle \times\langle AB|(512)\bigcap(234)\rangle \quad + \begin{matrix}\text{cyclic}\\ \text{(no repeat)}\end{matrix} \tag{47}$$

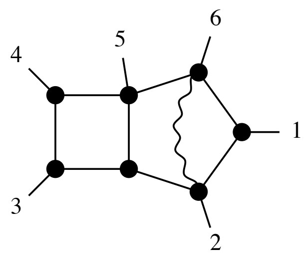
while the 6-particle amplitude is



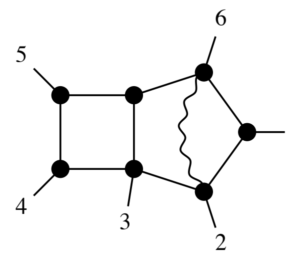
$\langle 2345\rangle\langle 6123\rangle\langle 3412\rangle$ +



$\langle 3456\rangle\langle 4563\rangle \times\langle AB|(561)\bigcap(234)\rangle$ +

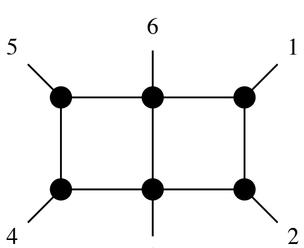


$\langle 2345\rangle\langle 3462\rangle \times\langle AB|(561)\bigcap(123)\rangle$ +

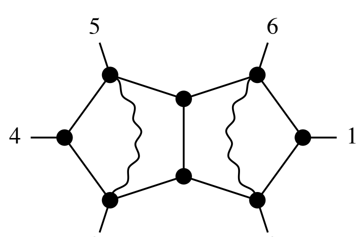


$\langle 3456\rangle\langle 4562\rangle \times\langle AB|(561)\bigcap(123)\rangle$

+



$\langle 3456\rangle\langle 6123\rangle\langle 4512\rangle$ +



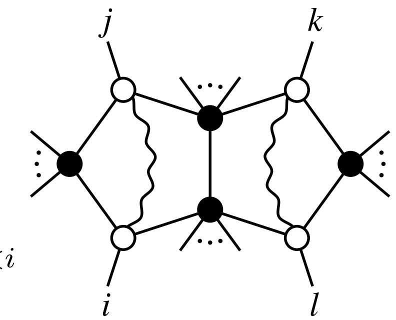
$\langle 6235\rangle \times\langle AB|(234)\bigcap(456)\rangle \times\langle CD|(561)\bigcap(123)\rangle$

+ cyclic (no repeat)

soft limits manifest:

[Drummond, JMH, 2010]

$$\mathcal{A}^{2-\text{loop}}_{\text{MHV}} = \frac{1}{2}\sum_{i<j<k<l<i}$$

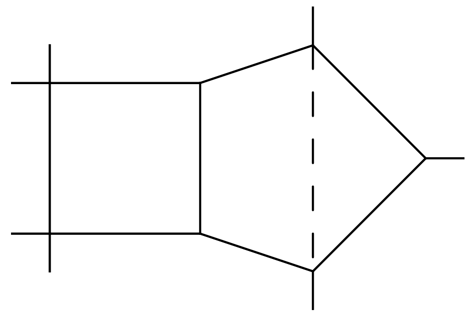


integrals satisfy 2nd-order differential equations!

[Drummond, JMH, Trnka, 2010]

Non-trivial two-loop examples

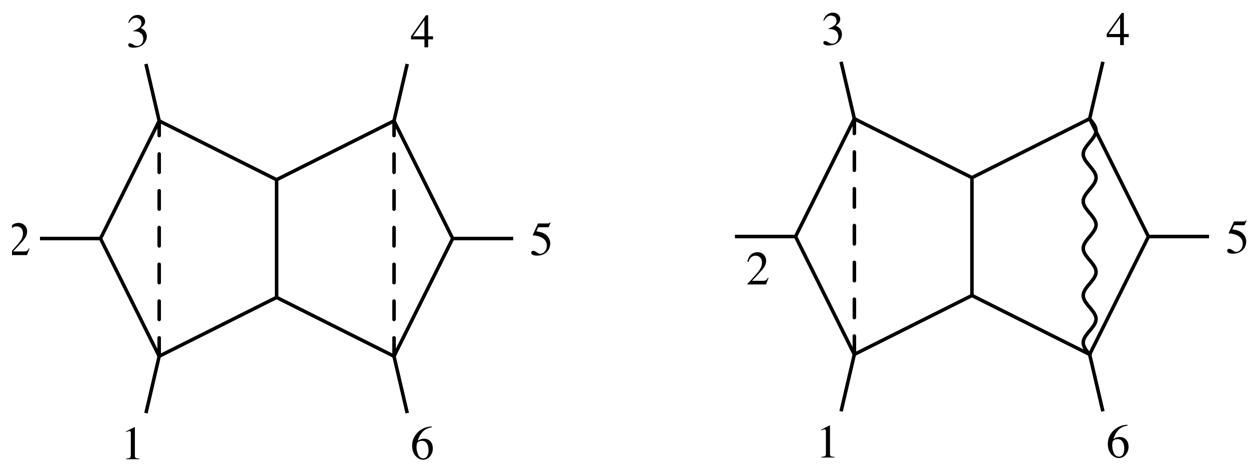
- seven-point penta-box integral



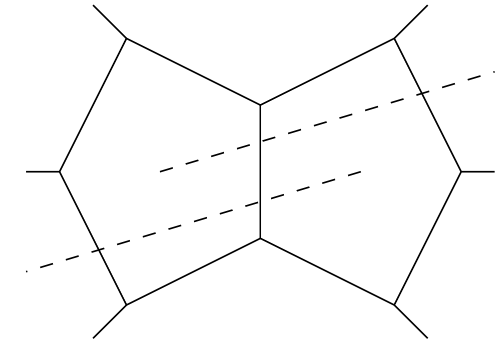
computed analytically using differential equations
in terms of 2d harmonic polylogarithms [Drummond, JMH, Trnka, 2010]

- six-point double pentagon integrals [Dixon, Drummond, JMH, 2011]

also known analytically ; most difficult piece of
2-loop MHV and NMHV amplitudes in N=4 super Yang-Mills



most difficult integral in
conventional representation:



not needed!

Integrals finite, AND (relatively) easy to evaluate!

End of lecture 2

Thank you!