Decoupling

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Introduction

QCD where all 6 flavors is rarely used. If characteristic $p_i \ll M_Q$, it is better to use a low-energy effective theory without Q. Its Lagrangian has the QCD form plus $1/M_Q^n$ corrections. Coefficients in the effective Lagrangian are tuned to reproduce scattering amplitudes of the full theory expanded in p_i/M_Q up to some order. Operators in the full QCD are expanded in $1/M_Q$ via appropriate operators in the effective theory.

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- QED with e and $\mu \Rightarrow$ QED with e
- QCD with q_i and $Q \Rightarrow$ QCD with q_i

History

2 loops	W. Bernreuther, W. Wetzel, Nucl. Phys. **B 197** (1982) 228; Erratum: **B 513** (1998) 758 S. A. Larin, T. van Ritbergen, J. A. M. Vermaseren, Nucl. Phys. **B 438** (1995) 278 [hep-ph/9411260]
3 loops	K. G. Chetyrkin, B. A. Kniehl, M. Steinhauser, Nucl. Phys. **B 510** (1998) 61 [hep-ph/9708255]
4 loops	K. G. Chetyrkin, J. H. Kühn, C. Sturm, Nucl. Phys. **B 744** (2006) 121 [hep-ph/0512060] Y. Schröder, M. Steinhauser, JHEP **01** (2006) 051 [hep-ph/0512058]

Full QED and effective theory

$$L = \bar{\psi}_0 i \not{D}_0 \psi_0 + \bar{\Psi}_0 \left(\not{D}_0 - M_0 \right) \Psi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2a_0} \left(\partial_\mu A_0^\mu \right)^2$$

$$L' = \bar{\psi}'_0 i \not{D}'_0 \psi'_0 - \frac{1}{4} F'_{0\mu\nu} F_0'^{\mu\nu} - \frac{1}{2a'_0} \left(\partial_\mu A_0'^\mu \right)^2 + \frac{1}{M_0^2} \sum_i C_i^0 O_i^0 + \cdots$$

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$$L' = \bar{\psi}_0' i \not{D}_0' \psi_0' - \frac{1}{4} F_{0\mu\nu}' F_0'^{\mu\nu} - \frac{1}{2a_0'} \left(\partial_\mu A_0'^\mu \right)^2 + \frac{1}{M_0^2} \sum_i C_i^0 O_i^0 + \cdots$$

Fields

$$A_0 = \zeta_{A0}^{-1/2} A_0' + \frac{1}{M_0^2} \sum_i C_{Ai}^0 O_{Ai}^0 + \cdots$$

$$\psi_0 = \zeta_{\psi 0}^{-1/2} \psi_0' + \frac{1}{M_0^2} \sum_i C_{\psi i}^0 O_{\psi i}^0 + \cdots$$

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Parameters

$$e_0 = \zeta_{\alpha_0}^{-1/2} e'_0 \qquad a_0 = \zeta_{A0}^{-1} a'_0$$

Renormalization

Photon propagator

$$\Pi_{\mu\nu}(p) = \left(p^2 g_{\mu\nu} - p_\mu p_\nu\right) \Pi(p^2)$$

$$D_{\mu\nu}(p) = \frac{1}{p^2\left[1 - \Pi(p^2)\right]} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + a_0 \frac{p_\mu p_\nu}{p^2}$$

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Electron propagator

$$\Sigma(p) = \not{p}\,\Sigma_V(p^2)$$
$$S(p) = \frac{1}{\not{p}\left[1 - \Sigma_V(p^2)\right]}$$

$\overline{\text{MS}}$ renormalization

$$A_0 = Z_A^{1/2}(\alpha(\mu))\, A(\mu) \qquad \psi_0 = Z_\psi^{1/2}(\alpha(\mu), a(\mu))\, \psi(\mu)$$

$$e_0 = Z_\alpha^{1/2}(\alpha(\mu))\, e(\mu) \qquad a_0 = Z_A(\alpha(\mu))\, a(\mu)$$

$$Z_i(\alpha) = 1 + \frac{z_1}{\varepsilon}\frac{\alpha}{4\pi} + \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon}\right)\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$

$$\frac{\alpha(\mu)}{4\pi} = \mu^{-2\varepsilon}\frac{e^2(\mu)}{(4\pi)^{d/2}} e^{-\gamma\varepsilon}$$

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$$\Gamma^\mu(p,p') = Z_\Gamma \Gamma_r^\mu(p,p';\mu)$$

$$e_0 \Gamma^\mu Z_\psi Z_A^{1/2} = e\Gamma_r^\mu Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2} = \text{finite}$$

$$Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2} = \text{finite} = 1 \qquad Z_\alpha = [Z_\Gamma Z_\psi]^{-2} Z_A^{-1}$$

Ward identity

$$\Gamma^\mu(p,p')\, q_\mu = S^{-1}(p') - S^{-1}(p) \qquad Z_\Gamma = Z_\psi^{-1}$$

On-shell renormalization

$$A_0 = \left[Z_A^{\text{os}}(e_0)\right]^{1/2} A_{\text{os}} \qquad \psi_0 = \left[Z_\psi^{\text{os}}(e_0, a_0)\right]^{1/2} \psi_{\text{os}}$$

$$e_0 = \left[Z_\alpha^{\text{os}}(e_0)\right]^{1/2} e_{\text{os}} \qquad a_0 = Z_A^{\text{os}}(e_0)\, a_{\text{os}}$$

$$p^2 \to 0 : \qquad D_\perp^{\text{os}}(p^2) \to D_\perp^0(p^2) \qquad S_{\text{os}}(p) \to S_0(p)$$

$$Z_A^{\text{os}}(e_0) = \frac{1}{1 - \Pi(0)} \qquad Z_\psi^{\text{os}}(e_0, a_0) = \frac{1}{1 - \Sigma_V(0)}$$

On-shell renormalization

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At $q \to 0$, p on mass shell, physical polarizations

$$e_{\rm os}\gamma^\mu = e_0 \Gamma^\mu Z_\psi^{\rm os} \left[Z_A^{\rm os}(e_0)\right]^{1/2}$$

$$\Gamma^\mu = Z_\Gamma^{\rm os}\gamma^\mu$$

$$Z_\alpha^{\rm os} = \left[Z_\Gamma^{\rm os} Z_\psi^{\rm os}\right]^{-2} \left[Z_A^{\rm os}\right]^{-1}$$

Ward identity

$$\Gamma^\mu = \frac{\partial S^{-1}(p)}{\partial p_\mu} = \frac{\partial}{\partial p_\mu}\left[\frac{Z_\psi^{\rm os}}{\not p}\right]^{-1} = \left[Z_\psi^{\rm os}\right]^{-1}\gamma^\mu \qquad Z_\Gamma^{\rm os} = \left[Z_\psi^{\rm os}\right]^{-1}$$

$\overline{\text{MS}}$ renormalized fields and parameters

$$A(\mu) = \zeta_A^{-1/2}(\mu) A'(\mu) \qquad \psi(\mu) = \zeta_\psi^{-1/2}(\mu) \psi'(\mu)$$

$$e(\mu) = \zeta_\alpha^{-1/2}(\mu) e'(\mu) \qquad a(\mu) = \zeta_A^{-1}(\mu) a'(\mu)$$

where

$$\zeta_A(\mu) = \frac{Z_A(\alpha(\mu))}{Z'_A(\alpha'(\mu))} \zeta_A^0$$

$$\zeta_\psi(\mu) = \frac{Z_\psi(\alpha(\mu), a(\mu))}{Z'_\psi(\alpha'(\mu), a'(\mu))} \zeta_\psi^0$$

$$\zeta_\alpha(\mu) = \frac{Z_\alpha(\alpha(\mu))}{Z'_\alpha(\alpha'(\mu))} \zeta_\alpha^0$$

RG equations

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$$\frac{d\ \log \zeta_A(\mu)}{d\ \log \mu} = \gamma_A(\alpha(\mu)) - \gamma_A'(\alpha'(\mu))$$

$$\frac{d\ \log \zeta_\psi(\mu)}{d\ \log \mu} = \gamma_\psi(\alpha(\mu), a(\mu)) - \gamma_\psi'(\alpha'(\mu), a'(\mu))$$

$$\frac{d\ \log \zeta_\alpha(\mu)}{d\ \log \mu} = 2\left[\beta(\alpha(\mu)) - \beta'(\alpha'(\mu))\right]$$

where

$$\frac{d\ \log Z_A(\alpha(\mu))}{d\ \log \mu} = \gamma_A(\alpha(\mu)) \qquad \frac{d\ \log a(\mu)}{d\ \log \mu} = -\gamma_A(\alpha(\mu))$$

$$\frac{d\ \log Z_\psi(\alpha(\mu), a(\mu))}{d\ \log \mu} = \gamma_\psi(\alpha(\mu), a(\mu))$$

$$\frac{d\ \log Z_\alpha(\alpha(\mu))}{d\ \log \mu} = 2\beta(\alpha(\mu)) \qquad \frac{d\ \log \alpha(\mu)}{d\ \log \mu} = -2\varepsilon - 2\beta(\alpha(\mu))$$

Photon field

$$D_\perp^{\mathrm{os}}(p^2) = D_\perp'^{\mathrm{os}}(p^2)\left[1 + \mathcal{O}(p^2)\right]$$

$$A_{\mathrm{os}} = A'_{\mathrm{os}} + \mathcal{O}\left(\frac{1}{M^2}\right)$$

$$\zeta_A^0 = \frac{Z_A'^{\mathrm{os}}(e'_0)}{Z_A^{\mathrm{os}}(e_0)}$$

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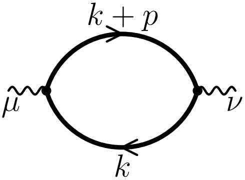
$$Z_A^{\text{os}}(e_0) = \frac{1}{1 - \Pi(0)}$$

$$Z_A'^{\text{os}}(e'_0) = \frac{1}{1 - \Pi'(0)} = 1 \qquad \Pi'(0) = 1$$

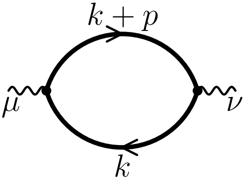
$$\zeta_A^0 = 1 - \Pi(0)$$

1 loop



$$= i\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right)\Pi(p^2)$$

1 loop



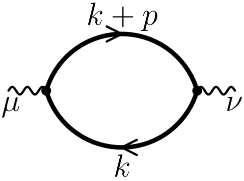
$$= i\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right)\Pi(p^2)$$

$$\Pi(p^2) = \frac{4ie_0^2}{(d-1)p^2}\int \frac{d^dk}{(2\pi)^d}\frac{N}{D_1 D_2}$$

$$D_1 = M_0^2 - (k+p)^2 \qquad D_2 = M_0^2 - k^2$$

$$N = \frac{1}{4}\,\mathrm{Tr}\,\gamma_\mu(\not{k} + \not{p} + M_0)\gamma^\mu(\not{k} + M_0)$$

1 loop



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$$\Pi(0) = -\frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)$$

Photon field decoupling

$$\zeta_A^0 = 1 + \frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon) + \cdots$$

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$$= 1 + \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon} Z_\alpha(\alpha(\mu)) Z_m^{-2\varepsilon}(\alpha(\mu)) \left(\frac{\mu}{M(\mu)}\right)^{2\varepsilon} \Gamma(1+\varepsilon) e^{\gamma\varepsilon} + \cdots$$

where

$$M_0 = Z_m(\alpha(\mu))\, M(\mu)$$

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where

$$M_0 = Z_m(\alpha(\mu))\, M(\mu)$$

$$Z_A^{(\prime)}(\alpha) = 1 - \frac{4}{3} n_f \frac{\alpha}{4\pi\varepsilon}$$

$$\zeta_A(\mu) = 1 + \frac{4}{3} L \frac{\alpha(\mu)}{4\pi} + \cdots \qquad L = 2\log\frac{\mu}{M(\mu)}$$

It is convenient to do choose $\mu = \bar{M}$

$$M(\bar{M}) = \bar{M}$$

Electron charge

Scattering: on-shell electron (physical polarization), $q \to 0$

$$e_0 \Gamma^\mu Z_\psi^{\text{os}} \left[Z_A^{\text{os}}\right]^{1/2} = e_0' \Gamma'^\mu Z_\psi'^{\text{os}} \left[Z_A'^{\text{os}}\right]^{1/2}$$

$$\Gamma^\mu = Z_\Gamma^{\text{os}} \gamma^\mu \qquad \Gamma'^\mu = Z_\Gamma'^{\text{os}} \gamma^\mu$$

$$\zeta_\alpha^0 = \frac{\left[Z_\Gamma^{\text{os}} Z_\psi^{\text{os}}\right]^2 Z_A^{\text{os}}}{\left[Z_\Gamma'^{\text{os}} Z_\psi'^{\text{os}}\right]^2 Z_A'^{\text{os}}} = \frac{Z_\alpha'^{\text{os}}}{Z_\alpha^{\text{os}}}$$

$$Z_\Gamma'^{\text{os}} = Z_\psi'^{\text{os}} = Z_A'^{\text{os}} = 1 \qquad Z_\Gamma^{\text{os}} Z_\psi^{\text{os}} = 1$$

$$\zeta_\alpha^0 = \left[\zeta_A^0\right]^{-1}$$

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$$e_0 \Gamma^\mu Z_\psi^{\text{os}} \left[Z_A^{\text{os}}\right]^{1/2} = e'_0 \Gamma'^\mu Z_\psi'^{\text{os}} \left[Z_A'^{\text{os}}\right]^{1/2}$$

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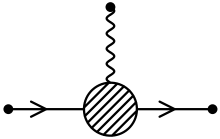
$$\zeta_\alpha^0 = \left[\zeta_A^0\right]^{-1}$$

The on-shell charge (measured at large distances) is the same in both theories

$$\alpha_{\text{os}} = \alpha'_{\text{os}}$$

Electron charge

Green function $\bar{\psi}_0$, ψ_0, A_0



$$e_0 \Gamma S S D = \left[\zeta_\psi^0\right]^{-1} \left[\zeta_A^0\right]^{-1/2} e_0' \Gamma' S' S' D'$$

$$e_0 \Gamma^\mu = \zeta_\psi^0 \left[\zeta_A^0\right]^{1/2} e_0' \Gamma'^\mu$$

$$\Gamma^\mu = \left[\zeta_\Gamma^0\right]^{-1} \Gamma'^\mu \qquad \zeta_\Gamma^0 = \frac{Z_\Gamma'^{\text{os}}}{Z_\Gamma^{\text{os}}}$$

$$\zeta_\alpha^0 = \left[\zeta_\Gamma^0 \zeta_\psi^0\right]^{-2} \left[\zeta_A^0\right]^{-1}$$

Electron charge

$\overline{\text{MS}}$ decoupling

$$\alpha(\mu) = \zeta_\alpha^{-1}(\mu)\alpha'(\mu) \qquad \zeta_\alpha(\mu) = \frac{Z_\alpha(\alpha(\mu))}{Z'_\alpha(\alpha'(\mu))}\zeta_\alpha^0$$

$$Z_\alpha^{(\prime)} = \left[Z_A^{(\prime)}\right]^{-1} \qquad \zeta_\alpha(\mu) = \zeta_A^{-1}(\mu)$$

Electron charge

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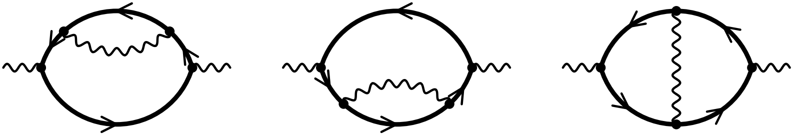
$$Z_\alpha^{(\prime)} = \left[Z_A^{(\prime)}\right]^{-1} \qquad \zeta_\alpha(\mu) = \zeta_A^{-1}(\mu)$$

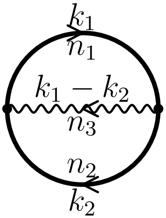
$$\zeta_A(\mu) = 1 - \frac{4}{3}L\frac{\alpha(\mu)}{4\pi} + \cdots$$

2 loops



$$\Pi(0) = -\frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon) - \frac{2}{3}\frac{(d-4)(5d^2-33d+34)}{d(d-5)}\left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\right)^2 + \cdots$$

A. Vladimirov (1980)



$$\int \frac{d^d k_1 \, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} = -\pi^d M^{2(d-n_1-n_2-n_3)} V(n_1,n_2,n_3)$$

$$D_1 = M^2 - k_1^2 \quad D_2 = M^2 - k_2^2 \quad D_3 = -(k_1-k_2)^2$$

$$V(n_1,n_2,n_3) = \frac{\Gamma\left(\frac{d}{2} - n_3\right)\Gamma\left(n_1 + n_3 - \frac{d}{2}\right)\Gamma\left(n_2 + n_3 - \frac{d}{2}\right)\Gamma(n_1+n_2+n_3-d)}{\Gamma\left(\frac{d}{2}\right)\Gamma(n_1)\Gamma(n_2)\Gamma(n_1+n_2+2n_3-d)}$$

Photon field decoupling: 2 loops

$$
\begin{aligned}
\zeta_A^0 &= 1 + \frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon) \\
&\quad + \frac{2}{3}\frac{(d-4)(5d^2-33d+34)}{d(d-5)}\left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\right)^2 + \cdots \\
&= 1 + \frac{4}{3}e^{L\varepsilon}\frac{\alpha(\mu)}{4\pi\varepsilon}Z_\alpha(\alpha(\mu))Z_m^{-2\varepsilon}(\alpha(\mu)) \\
&\quad - \varepsilon\left(6 - \frac{13}{3}\varepsilon + \cdots\right)e^{2L\varepsilon}\left(\frac{\alpha(\mu)}{4\pi\varepsilon}\right)^2 + \cdots
\end{aligned}
$$

Photon field decoupling: 2 loops

$$\zeta_A^0 = 1 + \frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)$$

$$+ \frac{2}{3}\frac{(d-4)(5d^2-33d+34)}{d(d-5)}\left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\right)^2 + \cdots$$

$$= 1 + \frac{4}{3}e^{L\varepsilon}\frac{\alpha(\mu)}{4\pi\varepsilon}Z_\alpha(\alpha(\mu))Z_m^{-2\varepsilon}(\alpha(\mu))$$

$$-\varepsilon\left(6 - \frac{13}{3}\varepsilon + \cdots\right)e^{2L\varepsilon}\left(\frac{\alpha(\mu)}{4\pi\varepsilon}\right)^2 + \cdots$$

$$Z_A^{(\prime)}(\alpha) = 1 - \frac{4}{3}n_f\frac{\alpha}{4\pi\varepsilon} - 2\varepsilon n_f\left(\frac{\alpha}{4\pi\varepsilon}\right)^2 + \cdots$$

$$Z_\alpha = Z_A^{-1} = 1 + 2\cdot\frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon} + \cdots \qquad Z_m = 1 - 3\frac{\alpha(\mu)}{4\pi\varepsilon} + \cdots$$

2-loop renormalized decoupling

$$\zeta_\alpha^{-1}(\mu) = \zeta_A(\mu) = 1 + \frac{4}{3} L \frac{\alpha(\mu)}{4\pi} + \left(-4L + \frac{13}{3}\right) \left(\frac{\alpha(\mu)}{4\pi}\right)^2 + \cdots$$

$$L = 2 \log \frac{\mu}{M(\mu)}.$$

2-loop renormalized decoupling

$$\zeta_\alpha^{-1}(\mu) = \zeta_A(\mu) = 1 + \frac{4}{3} L \frac{\alpha(\mu)}{4\pi} + \left(-4L + \frac{13}{3}\right) \left(\frac{\alpha(\mu)}{4\pi}\right)^2 + \cdots$$

$L = 2 \log \dfrac{\mu}{M(\mu)}$. At $\mu = \bar{M}$ $(M(\bar{M}) = \bar{M})$

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$$\mu = M_{\text{os}}$$

$$\frac{M(\mu)}{M_{\text{os}}} = 1 - 6 \left(\log \frac{\mu}{M_{\text{os}}} + \frac{2}{3}\right) \frac{\alpha}{4\pi} + \cdots \qquad L = 8 \frac{\alpha}{4\pi}$$

$$\zeta_\alpha(M_{\text{os}}) = 1 - 15 \left(\frac{\alpha(M_{\text{os}})}{4\pi}\right)^2 + \cdots$$

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For any $\mu = \bar{M}(1 + \mathcal{O}(\alpha))$, $\zeta_\alpha = 1 + \mathcal{O}(\alpha^2)$

Electron field

The electron propagators in the two theories are related by

$$\not{p}S(p) = \left[\zeta_\psi^0\right]^{-1} \not{p}S'(p) + \mathcal{O}\left(\frac{p^2}{M^2}\right)$$

Matching at $p \to 0$ — power-suppressed terms play no role.

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$$S(p) = \frac{Z_\psi^{\text{os}}}{\not{p}} \qquad Z_\psi^{\text{os}} = \frac{1}{1 - \Sigma_V(0)}$$

where $\Sigma(p) = \Sigma_V(p^2)\not{p}$. Only diagrams with muon loops contribute to $\Sigma_V(0)$.

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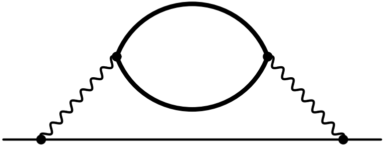
2 loops

The first diagram contributing to $\Sigma_V(0)$ appears at 2 loops



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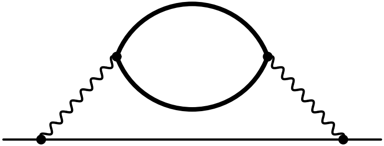

$$-i\not{p}\Sigma_V(p^2) = \int \frac{d^dk}{(2\pi)^d} ie_0\gamma^\mu i\frac{\not{k}+\not{p}}{(k+p)^2} ie_0\gamma^\nu \left(\frac{-i}{k^2}\right)^2$$
$$i(k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi(k^2)$$

2 loops

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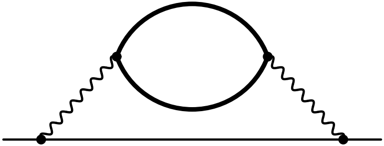

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$$\Sigma_V(0) = -ie_0^2 \frac{(d-1)(d-4)}{d} \int \frac{d^dk}{(2\pi)^d} \frac{\Pi(k^2)}{(-k^2)^2}$$

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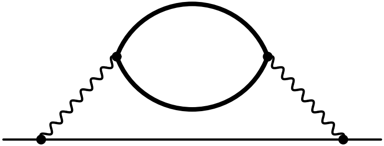

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$$\zeta_\psi^0 = 1 - \frac{2(d-1)(d-4)(d-6)}{d(d-2)(d-5)(d-7)} \left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\right)^2 + \cdots$$

Renormalized decoupling coefficient

$$\zeta_\psi(\mu) = \frac{Z_\psi(\alpha(\mu), a(\mu))}{Z'_\psi(\alpha'(\mu), a'(\mu))} \zeta_\psi^0$$

$$\zeta_\psi^0 = 1 - \varepsilon \left(1 - \frac{5}{6}\varepsilon + \cdots\right) \left(\frac{\alpha}{4\pi\varepsilon}\right)^2 + \cdots$$

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$$\alpha'(M) = \alpha(M)\,[1 + \mathcal{O}(\alpha^2)],\; a'(M) = a(M)\,[1 + \mathcal{O}(\alpha^2)]$$

$$\gamma_\psi^{(\prime)}(\alpha, a) = 2a\frac{\alpha}{4\pi} - (4n_f + 3)\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$

$$\frac{Z_\psi(\alpha(\bar{M}), a(\bar{M}))}{Z'_\psi(\alpha'(\bar{M}), a'(\bar{M}))} = 1 + \varepsilon \left(\frac{\alpha}{4\pi\varepsilon}\right)^2$$

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$$\zeta_\psi(\bar{M}) = 1 + \frac{5}{6}\left(\frac{\alpha(\bar{M})}{4\pi}\right)^2 + \cdots$$

Electron mass

$$\Sigma(p) = \not{p}\,\Sigma_V(p^2) + m_0\Sigma_S(p^2)$$

$$S(p) = \frac{1}{\not{p} - m_0 - \Sigma(p)} = \frac{1}{1 - \Sigma_V(p^2)}\;\frac{1}{\not{p} - \dfrac{1 + \Sigma_S(p^2)}{1 - \Sigma_V(p^2)} m_0}$$

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$$\frac{1}{1 - \Sigma_V(p^2)}\ \frac{1}{\not{p} - \dfrac{1 + \Sigma_S(p^2)}{1 - \Sigma_V(p^2)} m_0}$$

$$= \left[\zeta_\psi^0\right]^{-1} \frac{1}{1 - \Sigma_V'(p^2)}\ \frac{1}{\not{p} - \dfrac{1 + \Sigma_S'(p^2)}{1 - \Sigma_V'(p^2)} m_0'}$$

Electron mass

Linear approximation in m

$$\frac{1+\Sigma_S(0)}{1-\Sigma_V(0)}m_0 = \frac{1+\Sigma'_S(0)}{1-\Sigma'_V(0)}m'_0$$

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$$m_0 = \left[\zeta_m^0\right]^{-1} m'_0$$

$$\zeta_m^0 = \left[\zeta_q^0\right]^{-1} \frac{1+\Sigma_S(0)}{1+\Sigma'_S(0)} = \frac{1+\Sigma_S(0)}{1-\Sigma_V(0)}$$

$$\Sigma'_V(0) = \Sigma'_S(0) = 0$$

Electron mass

Linear approximation in m

$$\frac{1+\Sigma_S(0)}{1-\Sigma_V(0)} m_0 = \frac{1+\Sigma'_S(0)}{1-\Sigma'_V(0)} m'_0$$

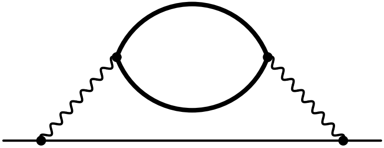
$$\begin{aligned}
m_0 &= \left[\zeta_m^0\right]^{-1} m'_0 \\
\zeta_m^0 &= \left[\zeta_q^0\right]^{-1} \frac{1+\Sigma_S(0)}{1+\Sigma'_S(0)} = \frac{1+\Sigma_S(0)}{1-\Sigma_V(0)} \\
\Sigma'_V(0) &= \Sigma'_S(0) = 0
\end{aligned}$$

On-shell masses coincide

$$m_{\text{os}} = m'_{\text{os}}$$

(less convenient for calculation)

2 loops



$$\Sigma_S(0) = -\frac{2(d-1)(d-6)}{(d-2)(d-5)(d-7)} \left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots$$

$$\zeta_m^0 = 1 - \frac{8(d-1)(d-6)}{d(d-2)(d-5)(d-7)} \left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots$$

$$= 1 + \left(2 - \frac{5}{3}\varepsilon + \frac{89}{18}\varepsilon^2 + \cdots \right) \left(\frac{\alpha}{4\pi\varepsilon} \right)^2 + \cdots$$

Renormalized decoupling coefficient

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RG equation

$$\frac{d\log\zeta_m(\mu)}{d\log\mu} + \gamma_m(\alpha(\mu)) - \gamma'_m(\alpha'(\mu)) = 0$$

QCD with a heavy flavor

$$L = \sum_{i=1}^{n_l} \bar{q}_{0i} \not{D}_0 q_{0i} + \bar{Q}_0 \left(\not{D}_0 - M_0 \right) Q_0$$
$$- \frac{1}{4} G^a_{0\mu\nu} G^{0a\mu\nu} - \frac{1}{2a_0} \left(\partial_\mu A_0^{a\mu} \right)^2 + \left(\partial_\mu \bar{c}_0^a \right) \left(D_0^\mu c_0^a \right)$$

QCD with a heavy flavor

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Low-energy effective theory

$$L' = \sum_{i=1}^{n_l} \bar{q}'_{0i} \not{D}'_0 q'_{0i} - \frac{1}{4} G'^a_{0\mu\nu} G'^{0a\mu\nu} - \frac{1}{2a'_0} \left(\partial_\mu A_0'^{a\mu} \right)^2 + \left(\partial_\mu \bar{c}_0'^a \right) \left(D_0'^\mu c_0'^a \right)$$

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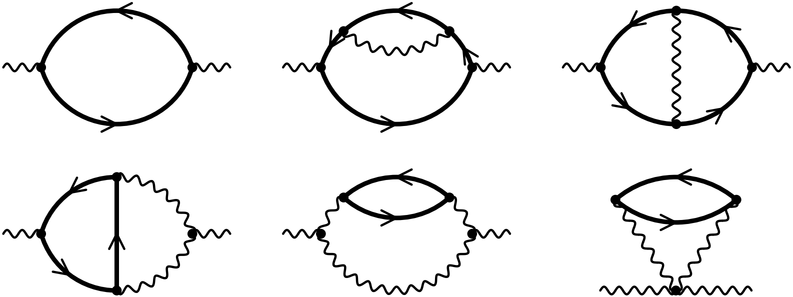
Decoupling

$$A_0 = \left[\zeta^0_A \right]^{-1/2} A'_0 \qquad q_0 = \left[\zeta^0_q \right]^{-1/2} q'_0 \qquad c_0 = \left[\zeta^0_c \right]^{-1/2} c'_0$$
$$g_0 = \left[\zeta^0_\alpha \right]^{-1/2} g'_0 \qquad a_0 = \left[\zeta^0_A \right]^{-1} a'_0$$

Gluon self energy

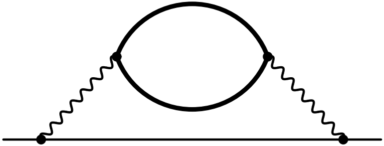


Gluon field

$$\zeta_A^0 = 1 + \frac{4}{3} T_F \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon)$$
$$+ \frac{1}{d(d-5)} \left[\frac{2}{3}(d-4)(5d^2 - 33d + 34) C_F \right.$$
$$\left. - \frac{d^5 - 20d^4 + 145d^3 - 458d^2 + 588d - 232}{(d-2)(d-7)} C_A \right]$$
$$\times T_F \left(\frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots$$

C_F term trivially follows from QED

Quark self energy



$$\Sigma_V(0) = \frac{2(d-1)(d-4)(d-6)}{d(d-2)(d-5)(d-7)} C_F T_F \left(\frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots$$

$$\Sigma_S(0) = -\frac{2(d-1)(d-6)}{(d-2)(d-5)(d-7)} C_F T_F \left(\frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots$$

Quark field and mass

$$\zeta_q^0 = 1 - \frac{2(d-1)(d-4)(d-6)}{d(d-2)(d-5)(d-7)} C_F T_F \left(\frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots$$

$$\zeta_m^0 = 1 - \frac{8(d-1)(d-6)}{d(d-2)(d-5)(d-7)} C_F T_F \left(\frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots$$

Trivially follows from QED results

Ghost field

$$G(p) = \frac{1}{p^2 - \Sigma_c(p^2)} \qquad Z_c^{\text{os}} = \frac{1}{1 - \frac{d\Sigma_c}{dp^2}(0)}$$

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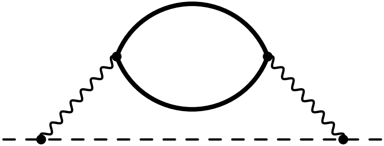


$$\frac{d\Sigma_c}{dp^2}(0) = -\frac{2(d-1)(d-6)}{d(d-2)(d-5)(d-7)} C_A T_F \left(\frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots$$

Decoupling: α_s

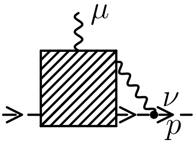
Expand vertex functions in their external momenta up to the first non-vanishing term

- Quark–gluon: γ^μ
- 3–gluon: $f^{a_1 a_2 a_3}\left(g^{\mu_1\mu_2}(k_1-k_2)^{\mu_3}+\text{cycle}\right)$, $d^{a_1 a_2 a_3}\left(g^{\mu_1\mu_2}k_3^{\mu_3}+\text{cycle}\right)$. Slavnov–Taylor identity $<T\{\partial^\mu A_\mu(x),\partial^\nu A_\nu(y),\partial^\lambda A_\lambda(z)\}>=0\Rightarrow$ $\Gamma^{a_1 a_2 a_3}_{\mu_1\mu_2\mu_3}k_1^{\mu_1}k_2^{\mu_2}k_3^{\mu_3}=0$
- Ghost–gluon: p^μ (outgoing ghost momentum)

Low–energy effective Lagrangian has the QCD form (up to power corrections)

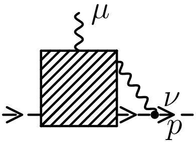
$$\zeta^0_\alpha(g_0)=\Gamma^2_{A\bar{c}c}\left[Z_c^{\text{os}}\right]^2 Z_A^{\text{os}}=\Gamma^2_{A\bar{q}q}\left[Z_q^{\text{os}}\right]^2 Z_A^{\text{os}}$$
$$=\Gamma^2_{AAA}\left[Z_A^{\text{os}}\right]^3$$

Ghost–gluon vertex



$$= A^{\mu\nu} p_\nu \qquad A^{\mu\nu}(0) = A g^{\mu\nu}$$

Ghost–gluon vertex



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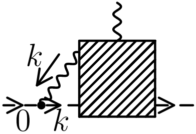
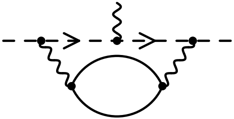
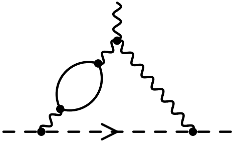


$$\sim k^\lambda$$

All corrections vanish in Landau gauge

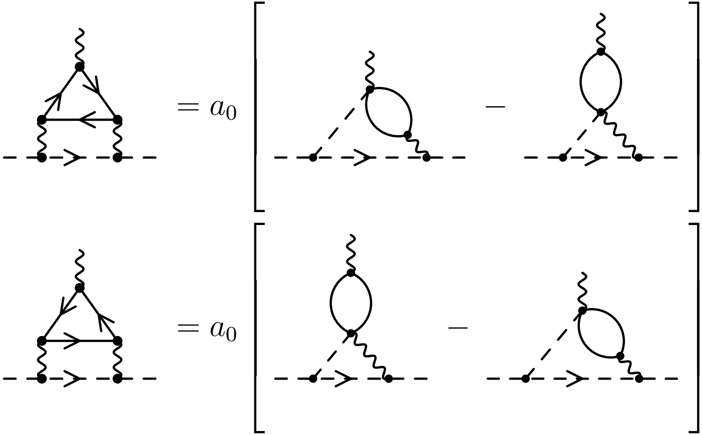


$$=$$

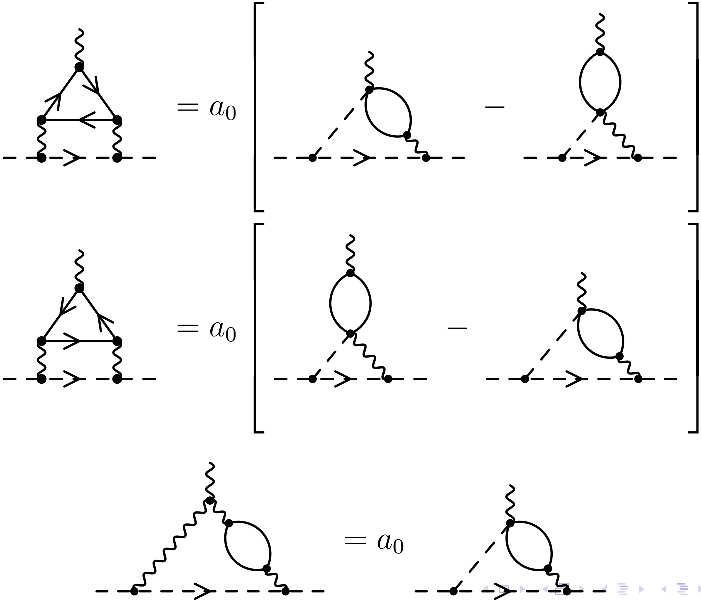


$$= 0$$

Ghost–gluon vertex



Ghost–gluon vertex



Decoupling: α_s

$$\left[\zeta_\alpha^0\right]^{-1} = 1 + \frac{4}{3} T_F \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon)$$
$$+ \frac{d-4}{d(d-5)} \left[\frac{2}{3}(5d^2 - 33d + 34) C_F - \frac{d^3 - 14d^2 + 53d - 32}{d-7} C_A \right]$$
$$\times T_F \left(\frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots$$

Decoupling: α_s

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$$\times T_F\left(\frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\right)^2 + \cdots$$

$$g_0'^2 = \zeta_\alpha^0(g_0)g_0^2$$

Step 1

Via renormalized quantities

$$\frac{g_0^2}{(4\pi)^{d/2}}\Gamma(\varepsilon) = \mu^{2\varepsilon}\frac{\alpha_s(\mu)}{4\pi\varepsilon}Z_\alpha(\alpha_s(\mu))\Gamma(1+\varepsilon)e^{\gamma\varepsilon}$$

$$Z_\alpha(\alpha) = 1 - \beta_0\frac{\alpha}{4\pi\varepsilon} + \left(\beta_0^2 - \frac{1}{2}\beta_1\varepsilon\right)\left(\frac{\alpha}{4\pi\varepsilon}\right)^2 + \cdots$$

$$M_0 = Z_m(\alpha_s(\mu))\,M(\mu)$$

$g_0'^2$ via $\alpha_s(\mu)$

Step 2

Inverting the series

$$\frac{g_0'^2}{(4\pi)^{d/2}}\Gamma(\varepsilon) = \mu'^{2\varepsilon}\frac{\alpha_s'(\mu')}{4\pi\varepsilon}Z_\alpha'(\alpha_s'(\mu'))\Gamma(1+\varepsilon)e^{\gamma\varepsilon}$$

we obtain

$$\frac{\alpha_s'(\mu')}{4\pi\varepsilon} = \frac{g_0'^2\mu'^{-2\varepsilon}}{(4\pi)^{d/2}\varepsilon}e^{-\gamma\varepsilon}\left[1 + \beta_0'\frac{g_0'^2\mu'^{-2\varepsilon}}{(4\pi)^{d/2}\varepsilon}e^{-\gamma\varepsilon}\right.$$
$$\left. + \left(\beta_0'^2 + \frac{1}{2}\beta_1'\varepsilon\right)\left(\frac{g_0'^2\mu'^{-2\varepsilon}}{(4\pi)^{d/2}\varepsilon}e^{-\gamma\varepsilon}\right)^2 + \cdots\right]$$

$\alpha_s'(\mu')$ via $\alpha_s(\mu)$

Renormalized decoupling coefficient

$$\alpha'_s(\mu) = \zeta_\alpha(\mu)\,\alpha_s(\mu)$$

$$\zeta_\alpha(\mu) = 1 - \frac{4}{3} L T_F \frac{\alpha_s(\mu)}{4\pi} + \left[\frac{16}{9} T_F L^2 + 4\left(C_F - \frac{5}{3} C_A\right) L - \left(\frac{13}{3} C_F - \frac{32}{9} C_A\right)\right] T_F \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 + \cdots$$

$$L = 2 \log \frac{\mu}{M(\mu)}$$

Renormalized decoupling coefficient

Convenient to use $\mu = \bar{M}$ $(M(\bar{M}) = \bar{M})$

$$\alpha_s'(\bar{M}) = \zeta_\alpha(\bar{M})\,\alpha_s(\bar{M})$$

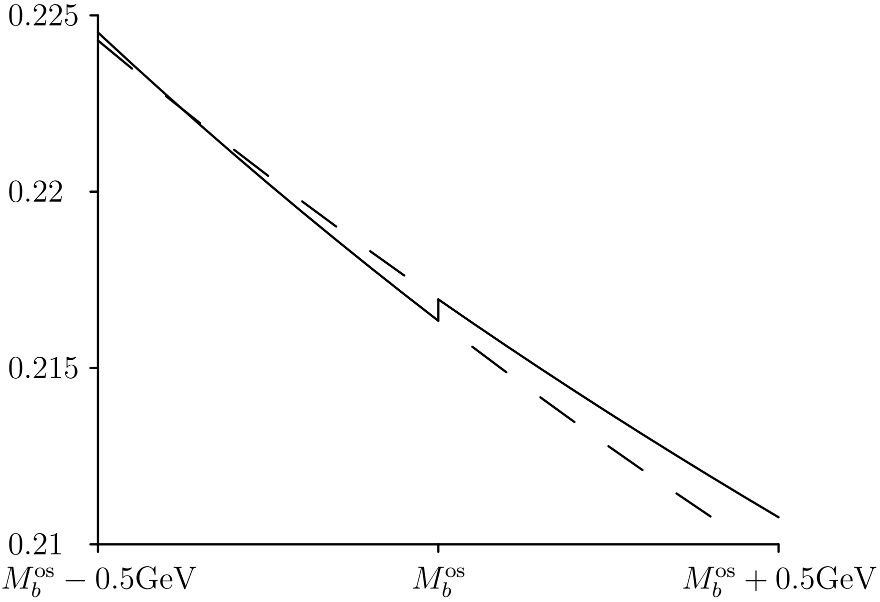
$$\zeta_\alpha(\bar{M}) = 1 - \left(\frac{13}{3}C_F - \frac{32}{9}C_A\right) T_F \left(\frac{\alpha_s(\bar{M})}{4\pi}\right)^2 + \cdots$$

C_F term from QED

For other values of μ — RG

$$\frac{d\,\log\zeta_\alpha(\mu)}{d\,\log\mu} = 2\left[\beta(\alpha(\mu)) - \beta'(\alpha'(\mu))\right]$$

Decoupling: α_s



Light-quark masses

Trivially obtained from QED results

$$m'(\bar{M}) = \zeta_m(\bar{M})\, m(\bar{M})$$
$$\zeta_m(\bar{M}) = 1 - \frac{89}{18} C_F T_F \left(\frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \cdots$$

For other values of μ, μ' — RG

Gluon field and a

$$a_0' = \zeta_A^0 a_0$$

$$\zeta_A^0 = 1 + \frac{4}{3} T_F \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \cdots$$

Gluon field and a

$$a'_0 = \zeta_A^0 a_0$$
$$\zeta_A^0 = 1 + \frac{4}{3} T_F \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \cdots$$

Re-expressing via renormalized quantities

$$a_0 = Z_A(\alpha_s(\mu), a(\mu))\, a(\mu)$$

a'_0 via $a(\mu)$, $\alpha_s(\mu)$

Gluon field and a

$$a'_0 = \zeta_A^0 a_0$$
$$\zeta_A^0 = 1 + \frac{4}{3} T_F \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \cdots$$

Re-expressing via renormalized quantities

$$a_0 = Z_A(\alpha_s(\mu), a(\mu))\, a(\mu)$$

a'_0 via $a(\mu)$, $\alpha_s(\mu)$

Solving by iterations

$$a'_0 = Z'_A(\alpha'_s(\mu'), a'(\mu'))\, a'(\mu')$$

$a'(\mu')$ via $a(\mu)$, $\alpha_s(\mu)$

Gluon field and a

Convenient to use $\mu = \bar{M}$

$$a'(\bar{M}) = \zeta_A(\bar{M})\, a(\bar{M})$$

$$\zeta_A(\bar{M}) = 1 + \frac{13}{12}(4C_F - C_A)T_F \left(\frac{\alpha_s(\bar{M})}{4\pi}\right)^2 + \cdots$$

Gauge-dependent at α_s^3
For other values of μ, μ' — RG

Quark and ghost fields

From QED

$$\zeta_q(\bar{M}) = 1 + \frac{5}{6} C_F T_F \left(\frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \cdots$$

Gauge dependent from 3 loops

Quark and ghost fields

From QED

$$\zeta_q(\bar{M}) = 1 + \frac{5}{6} C_F T_F \left(\frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \cdots$$

Gauge dependent from 3 loops

$$\zeta_c(\bar{M}) = 1 - \frac{89}{72} C_A T_F \left(\frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \cdots$$

Conclusion

- Relate $\alpha_s(m_\tau)$ to $\alpha_s(m_Z)$
- Retate $m_s(m_\tau)$ to higher μ
- Parton distribution functions (and their moments)
- Other quantities needed in a wide range of μ — QCD decoupling effects are rather large (unlike QED)

A nice and simple example of low-energy effective theories (Higgs–gluon interaction is similar).