

# Decoupling

Andrey Grozin

`A.G.Grozin@inp.nsk.su`

Budker Institute of Nuclear Physics  
Novosibirsk

# Introduction

QCD where all 6 flavors is rarely used. If characteristic  $p_i \ll M_Q$ , it is better to use a low-energy effective theory without  $Q$ . Its Lagrangian has the QCD form plus  $1/M_Q^n$  corrections. Coefficients in the effective Lagrangian are tuned to reproduce scattering amplitudes of the full theory expanded in  $p_i/M_Q$  up to some order. Operators in the full QCD are expanded in  $1/M_Q$  via appropriate operators in the effective theory.

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- ▶ QED with  $e$  and  $\mu \Rightarrow$  QED with  $e$
- ▶ QCD with  $q_i$  and  $Q \Rightarrow$  QCD with  $q_i$

# History

- 2 loops** W. Bernreuther, W. Wetzel, Nucl. Phys. **B 197** (1982) 228; Erratum: **B 513** (1998) 758  
S. A. Larin, T. van Ritbergen,  
J. A. M. Vermaseren, Nucl. Phys. **B 438** (1995)  
278 [hep-ph/9411260]
- 3 loops** K. G. Chetyrkin, B. A. Kniehl, M. Steinhauser,  
Nucl. Phys. **B 510** (1998) 61 [hep-ph/9708255]
- 4 loops** K. G. Chetyrkin, J. H. Kühn, C. Sturm, Nucl.  
Phys. **B 744** (2006) 121 [hep-ph/0512060]  
Y. Schröder, M. Steinhauser, JHEP **01** (2006)  
051 [hep-ph/0512058]

# Full QED and effective theory

$$L = \bar{\psi}_0 i \not{D}_0 \psi_0 + \bar{\Psi}_0 (\not{D}_0 - M_0) \Psi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2a_0} (\partial_\mu A_0^\mu)^2$$
$$L' = \bar{\psi}'_0 i \not{D}'_0 \psi'_0 - \frac{1}{4} F'_{0\mu\nu} F_0'^{\mu\nu} - \frac{1}{2a'_0} (\partial_\mu A_0'^\mu)^2 + \frac{1}{M_0^2} \sum_i C_i^0 O_i^0 + \dots$$

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## Fields

$$A_0 = \zeta_{A0}^{-1/2} A'_0 + \frac{1}{M_0^2} \sum_i C_{Ai}^0 O_{Ai}^0 + \dots$$
$$\psi_0 = \zeta_{\psi 0}^{-1/2} \psi'_0 + \frac{1}{M_0^2} \sum_i C_{\psi i}^0 O_{\psi i}^0 + \dots$$

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## Parameters

$$e_0 = \zeta_{\alpha 0}^{-1/2} e'_0 \quad a_0 = \zeta_{A0}^{-1} a'_0$$

# Renormalization

## Photon propagator

$$\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu) \Pi(p^2)$$

$$D_{\mu\nu}(p) = \frac{1}{p^2 [1 - \Pi(p^2)]} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + a_0 \frac{p_\mu p_\nu}{p^2}$$



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## Electron propagator

$$\Sigma(p) = \not{p} \Sigma_V(p^2)$$

$$S(p) = \frac{1}{\not{p} [1 - \Sigma_V(p^2)]}$$

# $\overline{\text{MS}}$ renormalization

$$A_0 = Z_A^{1/2}(\alpha(\mu)) A(\mu) \quad \psi_0 = Z_\psi^{1/2}(\alpha(\mu), a(\mu)) \psi(\mu)$$

$$e_0 = Z_\alpha^{1/2}(\alpha(\mu)) e(\mu) \quad a_0 = Z_A(\alpha(\mu)) a(\mu)$$

$$Z_i(\alpha) = 1 + \frac{z_1}{\varepsilon} \frac{\alpha}{4\pi} + \left( \frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon} \right) \left( \frac{\alpha}{4\pi} \right)^2 + \dots$$

$$\frac{\alpha(\mu)}{4\pi} = \mu^{-2\varepsilon} \frac{e^2(\mu)}{(4\pi)^{d/2}} e^{-\gamma\varepsilon}$$

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$Z_A$  converts  $D_\perp(p^2) \rightarrow D_\perp^r(p^2; \mu)$  and  $a_0 \rightarrow a(\mu)$

$$\Gamma^\mu(p, p') = Z_\Gamma \Gamma_r^\mu(p, p'; \mu)$$

$$e_0 \Gamma^\mu Z_\psi Z_A^{1/2} = e \Gamma_r^\mu Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2} = \text{finite}$$

$$Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2} = \text{finite} = 1 \quad Z_\alpha = [Z_\Gamma Z_\psi]^{-2} Z_A^{-1}$$

Ward identity

$$\Gamma^\mu(p, p') q_\mu = S^{-1}(p') - S^{-1}(p) \quad Z_\Gamma = Z_\psi^{-1}$$

# On-shell renormalization

$$A_0 = [Z_A^{\text{os}}(e_0)]^{1/2} A_{\text{os}} \quad \psi_0 = [Z_\psi^{\text{os}}(e_0, a_0)]^{1/2} \psi_{\text{os}}$$

$$e_0 = [Z_\alpha^{\text{os}}(e_0)]^{1/2} e_{\text{os}} \quad a_0 = Z_A^{\text{os}}(e_0) a_{\text{os}}$$

$$p^2 \rightarrow 0 : \quad D_\perp^{\text{os}}(p^2) \rightarrow D_\perp^0(p^2) \quad S_{\text{os}}(p) \rightarrow S_0(p)$$

$$Z_A^{\text{os}}(e_0) = \frac{1}{1 - \Pi(0)} \quad Z_\psi^{\text{os}}(e_0, a_0) = \frac{1}{1 - \Sigma_V(0)}$$

# On-shell renormalization

$$\begin{aligned}A_0 &= [Z_A^{\text{os}}(e_0)]^{1/2} A_{\text{os}} & \psi_0 &= [Z_\psi^{\text{os}}(e_0, a_0)]^{1/2} \psi_{\text{os}} \\e_0 &= [Z_\alpha^{\text{os}}(e_0)]^{1/2} e_{\text{os}} & a_0 &= Z_A^{\text{os}}(e_0) a_{\text{os}} \\p^2 \rightarrow 0 : & D_\perp^{\text{os}}(p^2) \rightarrow D_\perp^0(p^2) & S_{\text{os}}(p) &\rightarrow S_0(p) \\Z_A^{\text{os}}(e_0) &= \frac{1}{1 - \Pi(0)} & Z_\psi^{\text{os}}(e_0, a_0) &= \frac{1}{1 - \Sigma_V(0)}\end{aligned}$$

At  $q \rightarrow 0$ ,  $p$  on mass shell, physical polarizations

$$e_{\text{os}} \gamma^\mu = e_0 \Gamma^\mu Z_\psi^{\text{os}} [Z_A^{\text{os}}(e_0)]^{1/2}$$

$$\Gamma^\mu = Z_\Gamma^{\text{os}} \gamma^\mu$$

$$Z_\alpha^{\text{os}} = [Z_\Gamma^{\text{os}} Z_\psi^{\text{os}}]^{-2} [Z_A^{\text{os}}]^{-1}$$

Ward identity

$$\Gamma^\mu = \frac{\partial S^{-1}(p)}{\partial p_\mu} = \frac{\partial}{\partial p_\mu} \left[ \frac{Z_\psi^{\text{os}}}{\not{p}} \right]^{-1} = [Z_\psi^{\text{os}}]^{-1} \gamma^\mu \quad Z_\Gamma^{\text{os}} = [Z_\psi^{\text{os}}]^{-1}$$

# $\overline{\text{MS}}$ renormalized fields and parameters

$$\begin{aligned} A(\mu) &= \zeta_A^{-1/2}(\mu) A'(\mu) & \psi(\mu) &= \zeta_\psi^{-1/2}(\mu) \psi'(\mu) \\ e(\mu) &= \zeta_\alpha^{-1/2}(\mu) e'(\mu) & a(\mu) &= \zeta_A^{-1}(\mu) a'(\mu) \end{aligned}$$

where

$$\zeta_A(\mu) = \frac{Z_A(\alpha(\mu))}{Z'_A(\alpha'(\mu))} \zeta_A^0$$

$$\zeta_\psi(\mu) = \frac{Z_\psi(\alpha(\mu), a(\mu))}{Z'_\psi(\alpha'(\mu), a'(\mu))} \zeta_\psi^0$$

$$\zeta_\alpha(\mu) = \frac{Z_\alpha(\alpha(\mu))}{Z'_\alpha(\alpha'(\mu))} \zeta_\alpha^0$$

# RG equations

RG equations

$$\frac{d \log \zeta_A(\mu)}{d \log \mu} = \gamma_A(\alpha(\mu)) - \gamma'_A(\alpha'(\mu))$$

$$\frac{d \log \zeta_\psi(\mu)}{d \log \mu} = \gamma_\psi(\alpha(\mu), a(\mu)) - \gamma'_\psi(\alpha'(\mu), a'(\mu))$$

$$\frac{d \log \zeta_\alpha(\mu)}{d \log \mu} = 2 [\beta(\alpha(\mu)) - \beta'(\alpha'(\mu))]$$

where

$$\frac{d \log Z_A(\alpha(\mu))}{d \log \mu} = \gamma_A(\alpha(\mu)) \qquad \frac{d \log a(\mu)}{d \log \mu} = -\gamma_A(\alpha(\mu))$$

$$\frac{d \log Z_\psi(\alpha(\mu), a(\mu))}{d \log \mu} = \gamma_\psi(\alpha(\mu), a(\mu))$$

$$\frac{d \log Z_\alpha(\alpha(\mu))}{d \log \mu} = 2\beta(\alpha(\mu)) \qquad \frac{d \log \alpha(\mu)}{d \log \mu} = -2\varepsilon - 2\beta(\alpha(\mu))$$



# Photon field

$$D_{\perp}^{\text{os}}(p^2) = D'_{\perp}{}^{\text{os}}(p^2) [1 + \mathcal{O}(p^2)]$$

$$A_{\text{os}} = A'_{\text{os}} + \mathcal{O}\left(\frac{1}{M^2}\right)$$

$$\zeta_A^0 = \frac{Z'_A{}^{\text{os}}(e'_0)}{Z_A{}^{\text{os}}(e_0)}$$

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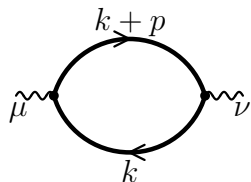
$$\zeta_A^0 = \frac{Z'_A{}^{\text{os}}(e'_0)}{Z_A{}^{\text{os}}(e_0)}$$

$$Z_A{}^{\text{os}}(e_0) = \frac{1}{1 - \Pi(0)}$$

$$Z'_A{}^{\text{os}}(e'_0) = \frac{1}{1 - \Pi'(0)} = 1 \quad \Pi'(0) = 1$$

$$\zeta_A^0 = 1 - \Pi(0)$$

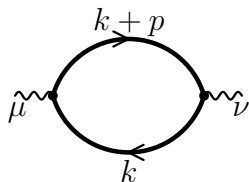
1 loop



The diagram shows a one-loop bubble diagram. Two external wavy lines are attached to the vertices of the loop. The left wavy line is labeled with the Greek letter  $\mu$  and the right one with  $\nu$ . The loop consists of two fermion lines forming a circle. The top fermion line has an arrow pointing to the right and is labeled with the momentum  $k + p$ . The bottom fermion line has an arrow pointing to the left and is labeled with the momentum  $k$ .

$$= i (p^2 g_{\mu\nu} - p_\mu p_\nu) \Pi(p^2)$$

# 1 loop

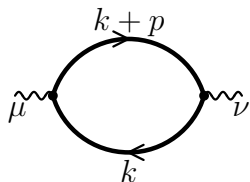

$$= i (p^2 g_{\mu\nu} - p_\mu p_\nu) \Pi(p^2)$$

$$\Pi(p^2) = \frac{4ie_0^2}{(d-1)p^2} \int \frac{d^d k}{(2\pi)^d} \frac{N}{D_1 D_2}$$

$$D_1 = M_0^2 - (k+p)^2 \quad D_2 = M_0^2 - k^2$$

$$N = \frac{1}{4} \text{Tr} \gamma_\mu (\not{k} + \not{p} + M_0) \gamma^\mu (\not{k} + M_0)$$

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$$\Pi(0) = -\frac{4}{3} \frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon)$$

# Photon field decoupling

$$\zeta_A^0 = 1 + \frac{4}{3} \frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \dots$$

# Photon field decoupling

$$\begin{aligned}\zeta_A^0 &= 1 + \frac{4 e_0^2 M_0^{-2\varepsilon}}{3 (4\pi)^{d/2}} \Gamma(\varepsilon) + \dots \\ &= 1 + \frac{4 \alpha(\mu)}{3 4\pi\varepsilon} Z_\alpha(\alpha(\mu)) Z_m^{-2\varepsilon}(\alpha(\mu)) \left( \frac{\mu}{M(\mu)} \right)^{2\varepsilon} \Gamma(1 + \varepsilon) e^{\gamma\varepsilon} + \dots\end{aligned}$$

where

$$M_0 = Z_m(\alpha(\mu)) M(\mu)$$

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where

$$M_0 = Z_m(\alpha(\mu)) M(\mu)$$

$$Z_A^{(\prime)}(\alpha) = 1 - \frac{4}{3} n_f \frac{\alpha}{4\pi\varepsilon}$$

$$\zeta_A(\mu) = 1 + \frac{4}{3} L \frac{\alpha(\mu)}{4\pi} + \dots \quad L = 2 \log \frac{\mu}{M(\mu)}$$

It is convenient to do choose  $\mu = \bar{M}$

$$M(\bar{M}) = \bar{M}$$



# Electron charge

Scattering: on-shell electron (physical polarization),  $q \rightarrow 0$

$$e_0 \Gamma^\mu Z_\psi^{\text{os}} [Z_A^{\text{os}}]^{1/2} = e'_0 \Gamma'^\mu Z'_\psi^{\text{os}} [Z'_A{}^{\text{os}}]^{1/2}$$

$$\Gamma^\mu = Z_\Gamma^{\text{os}} \gamma^\mu \quad \Gamma'^\mu = Z'_\Gamma{}^{\text{os}} \gamma^\mu$$

$$\zeta_\alpha^0 = \frac{[Z_\Gamma^{\text{os}} Z_\psi^{\text{os}}]^2 Z_A^{\text{os}}}{[Z'_\Gamma{}^{\text{os}} Z'_\psi{}^{\text{os}}]^2 Z'_A{}^{\text{os}}} = \frac{Z'_\alpha{}^{\text{os}}}{Z_\alpha^{\text{os}}}$$

$$Z'_\Gamma{}^{\text{os}} = Z'_\psi{}^{\text{os}} = Z'_A{}^{\text{os}} = 1 \quad Z_\Gamma^{\text{os}} Z_\psi^{\text{os}} = 1$$

$$\zeta_\alpha^0 = [\zeta_A^0]^{-1}$$

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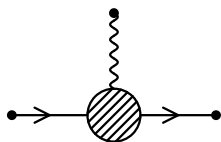
$$\zeta_\alpha^0 = [\zeta_A^0]^{-1}$$

The on-shell charge (measured at large distances) is the same in both theories

$$\alpha_{\text{os}} = \alpha'_{\text{os}}$$

# Electron charge

Green function  $\bar{\psi}_0, \psi_0, A_0$



$$e_0 \Gamma S S D = [\zeta_\psi^0]^{-1} [\zeta_A^0]^{-1/2} e'_0 \Gamma' S' S' D'$$

$$e_0 \Gamma^\mu = \zeta_\psi^0 [\zeta_A^0]^{1/2} e'_0 \Gamma'^\mu$$

$$\Gamma^\mu = [\zeta_\Gamma^0]^{-1} \Gamma'^\mu \quad \zeta_\Gamma^0 = \frac{Z'_\Gamma{}^{\text{os}}}{Z_\Gamma^{\text{os}}}$$

$$\zeta_\alpha^0 = [\zeta_\Gamma^0 \zeta_\psi^0]^{-2} [\zeta_A^0]^{-1}$$

# Electron charge

$\overline{\text{MS}}$  decoupling

$$\alpha(\mu) = \zeta_\alpha^{-1}(\mu) \alpha'(\mu) \quad \zeta_\alpha(\mu) = \frac{Z_\alpha(\alpha(\mu))}{Z'_\alpha(\alpha'(\mu))} \zeta_\alpha^0$$

$$Z_\alpha^{(\prime)} = \left[ Z_A^{(\prime)} \right]^{-1} \quad \zeta_\alpha(\mu) = \zeta_A^{-1}(\mu)$$

# Electron charge

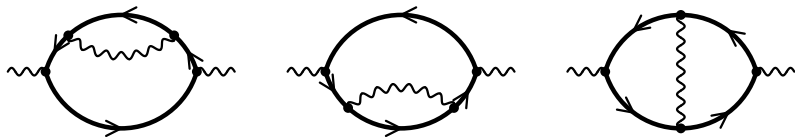
$\overline{\text{MS}}$  decoupling

$$\alpha(\mu) = \zeta_\alpha^{-1}(\mu) \alpha'(\mu) \quad \zeta_\alpha(\mu) = \frac{Z_\alpha(\alpha(\mu))}{Z'_\alpha(\alpha'(\mu))} \zeta_\alpha^0$$

$$Z_\alpha^{(l)} = \left[ Z_A^{(l)} \right]^{-1} \quad \zeta_\alpha(\mu) = \zeta_A^{-1}(\mu)$$

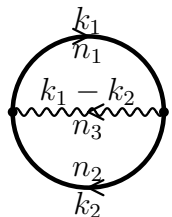
$$\zeta_A(\mu) = 1 - \frac{4}{3} L \frac{\alpha(\mu)}{4\pi} + \dots$$

## 2 loops



$$\begin{aligned}\Pi(0) = & -\frac{4 e_0^2 M_0^{-2\varepsilon}}{3 (4\pi)^{d/2}} \Gamma(\varepsilon) \\ & -\frac{2 (d-4)(5d^2 - 33d + 34)}{3 d(d-5)} \left( \frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \dots\end{aligned}$$

# A. Vladimirov (1980)



The diagram shows a circular loop with two external legs. The top leg is a straight line with momentum \$k\_1\$ and index \$n\_1\$. The bottom leg is a straight line with momentum \$k\_2\$ and index \$n\_2\$. A wavy internal line connects the two vertices, with momentum \$k\_1 - k\_2\$ and index \$n\_3\$.

$$\int \frac{d^d k_1 d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} = -\pi^d M^{2(d-n_1-n_2-n_3)} V(n_1, n_2, n_3)$$

$$D_1 = M^2 - k_1^2 \quad D_2 = M^2 - k_2^2 \quad D_3 = -(k_1 - k_2)^2$$

$$V(n_1, n_2, n_3) =$$

$$\frac{\Gamma\left(\frac{d}{2} - n_3\right) \Gamma\left(n_1 + n_3 - \frac{d}{2}\right) \Gamma\left(n_2 + n_3 - \frac{d}{2}\right) \Gamma(n_1 + n_2 + n_3 - d)}{\Gamma\left(\frac{d}{2}\right) \Gamma(n_1) \Gamma(n_2) \Gamma(n_1 + n_2 + 2n_3 - d)}$$

## Photon field decoupling: 2 loops

$$\begin{aligned}\zeta_A^0 &= 1 + \frac{4}{3} \frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \\ &\quad + \frac{2}{3} \frac{(d-4)(5d^2 - 33d + 34)}{d(d-5)} \left( \frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \dots \\ &= 1 + \frac{4}{3} e^{L\varepsilon} \frac{\alpha(\mu)}{4\pi\varepsilon} Z_\alpha(\alpha(\mu)) Z_m^{-2\varepsilon}(\alpha(\mu)) \\ &\quad - \varepsilon \left( 6 - \frac{13}{3}\varepsilon + \dots \right) e^{2L\varepsilon} \left( \frac{\alpha(\mu)}{4\pi\varepsilon} \right)^2 + \dots\end{aligned}$$



## Photon field decoupling: 2 loops

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$$Z_A^{(\prime)}(\alpha) = 1 - \frac{4}{3} n_f \frac{\alpha}{4\pi\varepsilon} - 2\varepsilon n_f \left( \frac{\alpha}{4\pi\varepsilon} \right)^2 + \dots$$

$$Z_\alpha = Z_A^{-1} = 1 + 2 \cdot \frac{4}{3} \frac{\alpha(\mu)}{4\pi\varepsilon} + \dots \quad Z_m = 1 - 3 \frac{\alpha(\mu)}{4\pi\varepsilon} + \dots$$

## 2-loop renormalized decoupling

$$\zeta_{\alpha}^{-1}(\mu) = \zeta_A(\mu) = 1 + \frac{4}{3}L \frac{\alpha(\mu)}{4\pi} + \left(-4L + \frac{13}{3}\right) \left(\frac{\alpha(\mu)}{4\pi}\right)^2 + \dots$$

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$$\mu = M_{\text{os}}$$

$$\frac{M(\mu)}{M_{\text{os}}} = 1 - 6 \left(\log \frac{\mu}{M_{\text{os}}} + \frac{2}{3}\right) \frac{\alpha}{4\pi} + \dots \quad L = 8 \frac{\alpha}{4\pi}$$

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For any  $\mu = \bar{M}(1 + \mathcal{O}(\alpha))$ ,  $\zeta_\alpha = 1 + \mathcal{O}(\alpha^2)$

## Electron field

The electron propagators in the two theories are related by

$$\not{p}S(p) = [\zeta_\psi^0]^{-1} \not{p}S'(p) + \mathcal{O}\left(\frac{p^2}{M^2}\right)$$

Matching at  $p \rightarrow 0$  — power-suppressed terms play no role.

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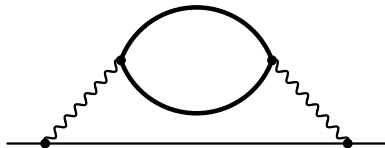
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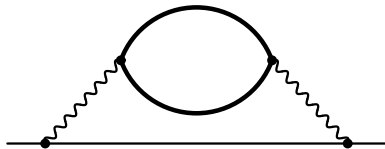
## 2 loops

The first diagram contributing to  $\Sigma_V(0)$  appears at 2 loops



## 2 loops

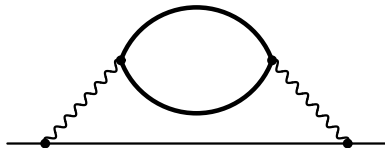
The first diagram contributing to  $\Sigma_V(0)$  appears at 2 loops



$$-i\not{p}\Sigma_V(p^2) = \int \frac{d^d k}{(2\pi)^d} i e_0 \gamma^\mu i \frac{\not{k} + \not{p}}{(k+p)^2} i e_0 \gamma^\nu \left( \frac{-i}{k^2} \right)^2 i(k^2 g_{\mu\nu} - k_\mu k_\nu) \Pi(k^2)$$

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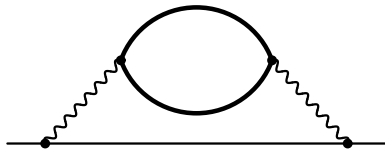


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$$\zeta_\psi^0 = 1 - \frac{2(d-1)(d-4)(d-6)}{d(d-2)(d-5)(d-7)} \left( \frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \dots$$

# Renormalized decoupling coefficient

$$\zeta_\psi(\mu) = \frac{Z_\psi(\alpha(\mu), a(\mu))}{Z'_\psi(\alpha'(\mu), a'(\mu))} \zeta_\psi^0$$

$$\zeta_\psi^0 = 1 - \varepsilon \left( 1 - \frac{5}{6}\varepsilon + \dots \right) \left( \frac{\alpha}{4\pi\varepsilon} \right)^2 + \dots$$

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$$\alpha'(M) = \alpha(M) [1 + \mathcal{O}(\alpha^2)], \quad a'(M) = a(M) [1 + \mathcal{O}(\alpha^2)]$$

$$\gamma_\psi^{(\prime)}(\alpha, a) = 2a \frac{\alpha}{4\pi} - (4n_f + 3) \left( \frac{\alpha}{4\pi} \right)^2 + \dots$$

$$\frac{Z_\psi(\alpha(\bar{M}), a(\bar{M}))}{Z'_\psi(\alpha'(\bar{M}), a'(\bar{M}))} = 1 + \varepsilon \left( \frac{\alpha}{4\pi\varepsilon} \right)^2$$

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$$\zeta_\psi(\bar{M}) = 1 + \frac{5}{6} \left( \frac{\alpha(\bar{M})}{4\pi} \right)^2 + \dots$$



# Electron mass

$$\Sigma(p) = \not{p}\Sigma_V(p^2) + m_0\Sigma_S(p^2)$$

$$S(p) = \frac{1}{\not{p} - m_0 - \Sigma(p)} = \frac{1}{1 - \Sigma_V(p^2)} \frac{1}{\not{p} - \frac{1 + \Sigma_S(p^2)}{1 - \Sigma_V(p^2)}m_0}$$

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$$\begin{aligned} & \frac{1}{1 - \Sigma_V(p^2)} \frac{1}{\not{p} - \frac{1 + \Sigma_S(p^2)}{1 - \Sigma_V(p^2)}m_0} \\ &= [\zeta_\psi^0]^{-1} \frac{1}{1 - \Sigma'_V(p^2)} \frac{1}{\not{p} - \frac{1 + \Sigma'_S(p^2)}{1 - \Sigma'_V(p^2)}m'_0} \end{aligned}$$

# Electron mass

Linear approximation in  $m$

$$\frac{1 + \Sigma_S(0)}{1 - \Sigma_V(0)} m_0 = \frac{1 + \Sigma'_S(0)}{1 - \Sigma'_V(0)} m'_0$$

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Linear approximation in  $m$

$$\frac{1 + \Sigma_S(0)}{1 - \Sigma_V(0)} m_0 = \frac{1 + \Sigma'_S(0)}{1 - \Sigma'_V(0)} m'_0$$

$$m_0 = [\zeta_m^0]^{-1} m'_0$$

$$\zeta_m^0 = [\zeta_q^0]^{-1} \frac{1 + \Sigma_S(0)}{1 + \Sigma'_S(0)} = \frac{1 + \Sigma_S(0)}{1 - \Sigma_V(0)}$$

$$\Sigma'_V(0) = \Sigma'_S(0) = 0$$

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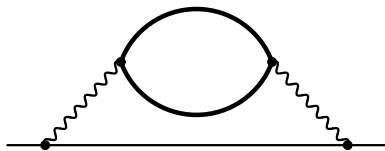
$$\Sigma'_V(0) = \Sigma'_S(0) = 0$$

On-shell masses coincide

$$m_{\text{os}} = m'_{\text{os}}$$

(less convenient for calculation)

## 2 loops



$$\Sigma_S(0) = -\frac{2(d-1)(d-6)}{(d-2)(d-5)(d-7)} \left( \frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \dots$$

$$\zeta_m^0 = 1 - \frac{8(d-1)(d-6)}{d(d-2)(d-5)(d-7)} \left( \frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \dots$$

$$= 1 + \left( 2 - \frac{5}{3}\varepsilon + \frac{89}{18}\varepsilon^2 + \dots \right) \left( \frac{\alpha}{4\pi\varepsilon} \right)^2 + \dots$$

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RG equation

$$\frac{d \log \zeta_m(\mu)}{d \log \mu} + \gamma_m(\alpha(\mu)) - \gamma'_m(\alpha'(\mu)) = 0$$

## QCD with a heavy flavor

$$L = \sum_{i=1}^{n_l} \bar{q}_{0i} \not{D}_0 q_{0i} + \bar{Q}_0 (\not{D}_0 - M_0) Q_0 \\ - \frac{1}{4} G_{0\mu\nu}^a G^{0a\mu\nu} - \frac{1}{2a_0} (\partial_\mu A_0^{a\mu})^2 + (\partial_\mu \bar{c}_0^a) (D_0^\mu c_0^a)$$

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Low-energy effective theory

$$L' = \sum_{i=1}^{n_l} \bar{q}'_{0i} \not{D}'_0 q'_{0i} - \frac{1}{4} G_{0\mu\nu}^{\prime a} G^{\prime 0a\mu\nu} - \frac{1}{2a'_0} (\partial_\mu A_0^{\prime a\mu})^2 + (\partial_\mu \bar{c}'_0{}^a) (D_0^{\prime\mu} c_0^{\prime a})$$

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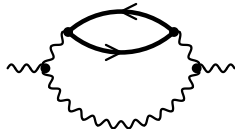
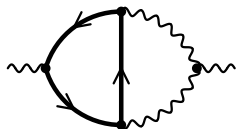
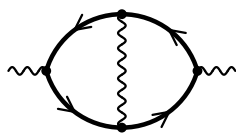
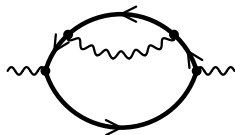
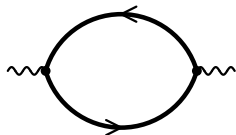
Low-energy effective theory

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Decoupling

$$A_0 = [\zeta_A^0]^{-1/2} A'_0 \quad q_0 = [\zeta_q^0]^{-1/2} q'_0 \quad c_0 = [\zeta_c^0]^{-1/2} c'_0 \\ g_0 = [\zeta_\alpha^0]^{-1/2} g'_0 \quad a_0 = [\zeta_A^0]^{-1} a'_0$$

# Gluon self energy

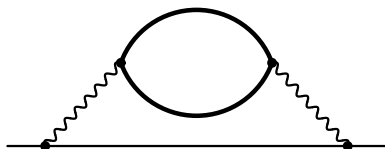


## Gluon field

$$\begin{aligned}\zeta_A^0 &= 1 + \frac{4}{3} T_F \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \\ &+ \frac{1}{d(d-5)} \left[ \frac{2}{3} (d-4)(5d^2 - 33d + 34) C_F \right. \\ &\quad \left. - \frac{d^5 - 20d^4 + 145d^3 - 458d^2 + 588d - 232}{(d-2)(d-7)} C_A \right] \\ &\times T_F \left( \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \dots\end{aligned}$$

$C_F$  term trivially follows from QED

# Quark self energy



$$\Sigma_V(0) = \frac{2(d-1)(d-4)(d-6)}{d(d-2)(d-5)(d-7)} C_F T_F \left( \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \dots$$

$$\Sigma_S(0) = -\frac{2(d-1)(d-6)}{(d-2)(d-5)(d-7)} C_F T_F \left( \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \dots$$

## Quark field and mass

$$\zeta_q^0 = 1 - \frac{2(d-1)(d-4)(d-6)}{d(d-2)(d-5)(d-7)} C_F T_F \left( \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \dots$$

$$\zeta_m^0 = 1 - \frac{8(d-1)(d-6)}{d(d-2)(d-5)(d-7)} C_F T_F \left( \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \dots$$

Trivially follows from QED results



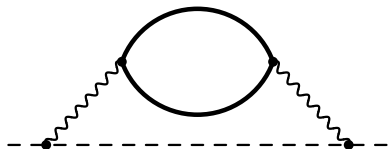
# Ghost field

$$G(p) = \frac{1}{p^2 - \Sigma_c(p^2)} \quad Z_c^{\text{os}} = \frac{1}{1 - \frac{d\Sigma_c}{dp^2}(0)}$$

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$$Z_c^{\text{os}} = \frac{1}{1 - \frac{d\Sigma_c}{dp^2}(0)}$$



$$\frac{d\Sigma_c}{dp^2}(0) = -\frac{2(d-1)(d-6)}{d(d-2)(d-5)(d-7)} C_A T_F \left( \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \dots$$

## Decoupling: $\alpha_s$

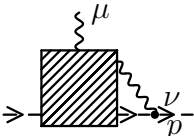
Expand vertex functions in their external momenta up to the first non-vanishing term

- ▶ Quark–gluon:  $\gamma^\mu$
- ▶ 3–gluon:  $f^{a_1 a_2 a_3} (g^{\mu_1 \mu_2} (k_1 - k_2)^{\mu_3} + \text{cycle}),$   
 $d^{a_1 a_2 a_3} (g^{\mu_1 \mu_2} k_3^{\mu_3} + \text{cycle}).$  Slavnov–Taylor identity  
 $\langle T \{ \partial^\mu A_\mu(x), \partial^\nu A_\nu(y), \partial^\lambda A_\lambda(z) \} \rangle = 0 \Rightarrow$   
 $\Gamma_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3} k_1^{\mu_1} k_2^{\mu_2} k_3^{\mu_3} = 0$
- ▶ Ghost–gluon:  $p^\mu$  (outgoing ghost momentum)

Low–energy effective Lagrangian has the QCD form (up to power corrections)

$$\begin{aligned} \zeta_\alpha^0(g_0) &= \Gamma_{A\bar{c}c}^2 [Z_c^{\text{os}}]^2 Z_A^{\text{os}} = \Gamma_{A\bar{q}q}^2 [Z_q^{\text{os}}]^2 Z_A^{\text{os}} \\ &= \Gamma_{AAA}^2 [Z_A^{\text{os}}]^3 \end{aligned}$$

# Ghost-gluon vertex



The diagram shows a square vertex with diagonal hatching. A wavy line labeled  $\mu$  enters from the top. A dashed line labeled  $\nu$  enters from the right, with a dot at the vertex and momentum  $p$  below it. A dashed line exits from the left, and another dashed line exits from the bottom. The diagram is equated to  $A^{\mu\nu} p_\nu$ .

$$= A^{\mu\nu} p_\nu \quad A^{\mu\nu}(0) = Ag^{\mu\nu}$$

# Ghost-gluon vertex

$$= A^{\mu\nu} p_\nu \quad A^{\mu\nu}(0) = Ag^{\mu\nu}$$

$$\sim k^\lambda$$

All corrections vanish in Landau gauge

$$= 0$$

# Ghost-gluon vertex

The image displays two equations for the ghost-gluon vertex. Each equation shows a triangle diagram on the left, equated to a coefficient  $a_0$  multiplied by a difference of two diagrams in large square brackets on the right.

The top equation shows a triangle diagram with a ghost loop (dashed line) and a gluon loop (wavy line). The right side shows the difference between two diagrams: one with a ghost loop and a gluon loop, and another with a gluon loop and a ghost loop.

The bottom equation shows a triangle diagram with a ghost loop (dashed line) and a gluon loop (wavy line). The right side shows the difference between two diagrams: one with a gluon loop and a ghost loop, and another with a ghost loop and a gluon loop.

# Ghost-gluon vertex

$$\begin{aligned} & \left[ \text{Triangle diagram with ghost loop} \right] = a_0 \left[ \text{Triangle diagram with gluon loop} - \text{Triangle diagram with ghost loop} \right] \\ & \left[ \text{Triangle diagram with gluon loop} \right] = a_0 \left[ \text{Triangle diagram with gluon loop} - \text{Triangle diagram with ghost loop} \right] \end{aligned}$$

The diagrams show the ghost-gluon vertex  $a_0$  as a sum of two terms. Each term is a triangle diagram with a ghost loop (dashed line) and a gluon loop (wavy line). The first term has a ghost loop on the left and a gluon loop on the right. The second term has a gluon loop on the left and a ghost loop on the right. The diagrams are enclosed in large square brackets.

$$\left[ \text{Triangle diagram with gluon loop} \right] = a_0 \left[ \text{Triangle diagram with gluon loop} \right]$$

The diagram shows a triangle diagram with a gluon loop (wavy line) and a ghost loop (dashed line). The diagram is enclosed in large square brackets.

## Decoupling: $\alpha_s$

$$\begin{aligned} [\zeta_\alpha^0]^{-1} &= 1 + \frac{4}{3} T_F \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \\ &+ \frac{d-4}{d(d-5)} \left[ \frac{2}{3} (5d^2 - 33d + 34) C_F - \frac{d^3 - 14d^2 + 53d - 32}{d-7} C_A \right] \\ &\times T_F \left( \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \dots \end{aligned}$$



## Decoupling: $\alpha_s$

$$\begin{aligned} [\zeta_\alpha^0]^{-1} &= 1 + \frac{4}{3} T_F \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \\ &+ \frac{d-4}{d(d-5)} \left[ \frac{2}{3} (5d^2 - 33d + 34) C_F - \frac{d^3 - 14d^2 + 53d - 32}{d-7} C_A \right] \\ &\times T_F \left( \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \dots \end{aligned}$$

$$g_0'^2 = \zeta_\alpha^0(g_0) g_0^2$$

# Step 1

Via renormalized quantities

$$\frac{g_0^2}{(4\pi)^{d/2}} \Gamma(\varepsilon) = \mu^{2\varepsilon} \frac{\alpha_s(\mu)}{4\pi\varepsilon} Z_\alpha(\alpha_s(\mu)) \Gamma(1 + \varepsilon) e^{\gamma\varepsilon}$$

$$Z_\alpha(\alpha) = 1 - \beta_0 \frac{\alpha}{4\pi\varepsilon} + \left( \beta_0^2 - \frac{1}{2} \beta_1 \varepsilon \right) \left( \frac{\alpha}{4\pi\varepsilon} \right)^2 + \dots$$

$$M_0 = Z_m(\alpha_s(\mu)) M(\mu)$$

$g_0^2$  via  $\alpha_s(\mu)$

## Step 2

Inverting the series

$$\frac{g_0'^2}{(4\pi)^{d/2}} \Gamma(\varepsilon) = \mu'^{2\varepsilon} \frac{\alpha'_s(\mu')}{4\pi\varepsilon} Z'_\alpha(\alpha'_s(\mu')) \Gamma(1 + \varepsilon) e^{\gamma\varepsilon}$$

we obtain

$$\begin{aligned} \frac{\alpha'_s(\mu')}{4\pi\varepsilon} &= \frac{g_0'^2 \mu'^{-2\varepsilon}}{(4\pi)^{d/2} \varepsilon} e^{-\gamma\varepsilon} \left[ 1 + \beta_0' \frac{g_0'^2 \mu'^{-2\varepsilon}}{(4\pi)^{d/2} \varepsilon} e^{-\gamma\varepsilon} \right. \\ &\quad \left. + \left( \beta_0'^2 + \frac{1}{2} \beta_1' \varepsilon \right) \left( \frac{g_0'^2 \mu'^{-2\varepsilon}}{(4\pi)^{d/2} \varepsilon} e^{-\gamma\varepsilon} \right)^2 + \dots \right] \end{aligned}$$

$\alpha'_s(\mu')$  via  $\alpha_s(\mu)$

# Renormalized decoupling coefficient

$$\alpha'_s(\mu) = \zeta_\alpha(\mu) \alpha_s(\mu)$$

$$\begin{aligned} \zeta_\alpha(\mu) = & 1 - \frac{4}{3} L T_F \frac{\alpha_s(\mu)}{4\pi} \\ & + \left[ \frac{16}{9} T_F L^2 + 4 \left( C_F - \frac{5}{3} C_A \right) L \right. \\ & \quad \left. - \left( \frac{13}{3} C_F - \frac{32}{9} C_A \right) \right] T_F \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 + \dots \end{aligned}$$

$$L = 2 \log \frac{\mu}{M(\mu)}$$

# Renormalized decoupling coefficient

Convenient to use  $\mu = \bar{M}$  ( $M(\bar{M}) = \bar{M}$ )

$$\alpha'_s(\bar{M}) = \zeta_\alpha(\bar{M}) \alpha_s(\bar{M})$$

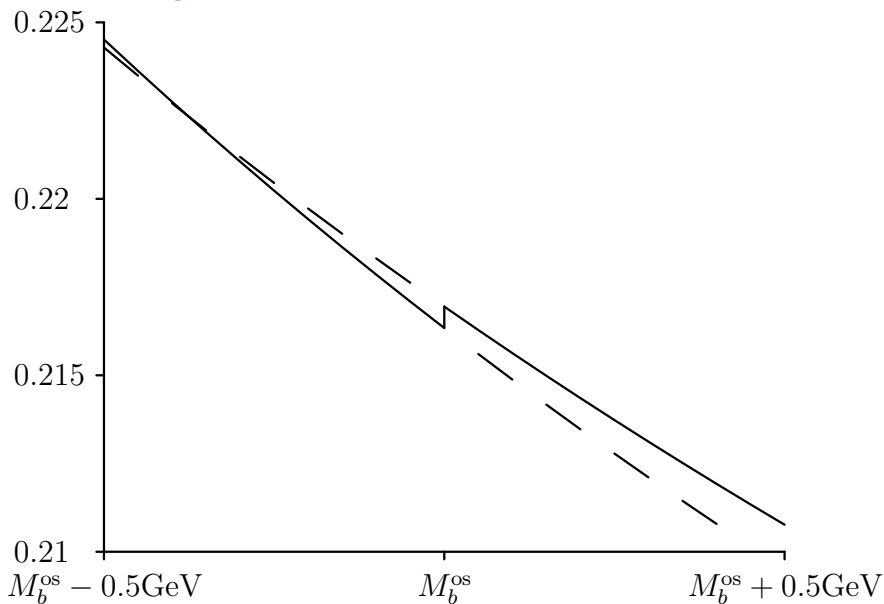
$$\zeta_\alpha(\bar{M}) = 1 - \left( \frac{13}{3} C_F - \frac{32}{9} C_A \right) T_F \left( \frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \dots$$

$C_F$  term from QED

For other values of  $\mu$  — RG

$$\frac{d \log \zeta_\alpha(\mu)}{d \log \mu} = 2 [\beta(\alpha(\mu)) - \beta'(\alpha'(\mu))]$$

# Decoupling: $\alpha_s$



# Light-quark masses

Trivially obtained from QED results

$$m'(\bar{M}) = \zeta_m(\bar{M}) m(\bar{M})$$

$$\zeta_m(\bar{M}) = 1 - \frac{89}{18} C_F T_F \left( \frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \dots$$

For other values of  $\mu, \mu'$  — RG

## Gluon field and $a$

$$a'_0 = \zeta_A^0 a_0$$

$$\zeta_A^0 = 1 + \frac{4}{3} T_F \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \dots$$



## Gluon field and $a$

$$a'_0 = \zeta_A^0 a_0$$

$$\zeta_A^0 = 1 + \frac{4}{3} T_F \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \dots$$

Re-expressing via renormalized quantities

$$a_0 = Z_A(\alpha_s(\mu), a(\mu)) a(\mu)$$

$a'_0$  via  $a(\mu)$ ,  $\alpha_s(\mu)$

## Gluon field and $a$

$$a'_0 = \zeta_A^0 a_0$$
$$\zeta_A^0 = 1 + \frac{4}{3} T_F \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \dots$$

Re-expressing via renormalized quantities

$$a_0 = Z_A(\alpha_s(\mu), a(\mu)) a(\mu)$$

$a'_0$  via  $a(\mu)$ ,  $\alpha_s(\mu)$

Solving by iterations

$$a'_0 = Z'_A(\alpha'_s(\mu'), a'(\mu')) a'(\mu')$$

$a'(\mu')$  via  $a(\mu)$ ,  $\alpha_s(\mu)$

## Gluon field and $a$

Convenient to use  $\mu = \bar{M}$

$$a'(\bar{M}) = \zeta_A(\bar{M}) a(\bar{M})$$

$$\zeta_A(\bar{M}) = 1 + \frac{13}{12}(4C_F - C_A)T_F \left( \frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \dots$$

Gauge-dependent at  $\alpha_s^3$

For other values of  $\mu, \mu'$  — RG

# Quark and ghost fields

From QED

$$\zeta_q(\bar{M}) = 1 + \frac{5}{6}C_F T_F \left( \frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \dots$$

Gauge dependent from 3 loops

# Quark and ghost fields

From QED

$$\zeta_q(\bar{M}) = 1 + \frac{5}{6} C_F T_F \left( \frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \dots$$

Gauge dependent from 3 loops

$$\zeta_c(\bar{M}) = 1 - \frac{89}{72} C_A T_F \left( \frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \dots$$

# Conclusion

- ▶ Relate  $\alpha_s(m_\tau)$  to  $\alpha_s(m_Z)$
- ▶ Retate  $m_s(m_\tau)$  to higher  $\mu$
- ▶ Parton distribution functions (and their moments)
- ▶ Other quantities needed in a wide range of  $\mu$  — QCD decoupling effects are rather large (unlike QED)

A nice and simple example of low-energy effective theories (Higgs–gluon interaction is similar).