Evolution of cusped light-like Wilson loops and dynamical equations in the loop space

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OUTLINE:

- **Rapidity divergences in transverse-momentum dependent PDFs:** classification and origin
- **Rapidity divergences in light-like Wilson loops:** explicit results for quadrilateral contours
- **Renormalization properties of the light-like Wilson loops:** how to work with light-cone singular objects?
- **Null-polygons and cusps:** dynamics enters via obstructions of the Wilson loops
- **Makeenko-Migdal approach:** geometrical properties of the loop space, dynamics in terms of cusped Wilson loops and energy/rapidity evolution
- Modified **Schwinger approach:** RG and rapidity evolution of cusped light-like objects
- **Universality further conjectures:** evolution equations and geometrical properties of the loops space

GENERIC TMD

"Trial" TMD with the light-like and transverse gauge links:

$$\Phi(x, \boldsymbol{k}_\perp) = \frac{1}{2} \int \frac{dz^- d^2 z_\perp}{2\pi(2\pi)^2} \, \mathrm{e}^{-ikz} \cdot \langle p, s | \bar{\psi}(z) \, [z^-, z_\perp; 0^-, 0_\perp]_{n \cup l_\perp} \psi(0) | p, s \rangle |_{z^+=0}$$

Generic straight Wilson line $[\infty, z]_w$:

$$[\infty; z]_w \equiv \mathcal{P} \exp \left[-ig \int_0^\infty d\tau \, w_\mu \, A_a^\mu t^a (z + w\tau) \right]$$

Tree-level, formally:

$$\Phi^{(0)}(x, \boldsymbol{k}_\perp) = \delta(1-x)\delta^{(2)}(\boldsymbol{k}_\perp) \; , \; \int d^2 k_\perp \Phi(x, \boldsymbol{k}_\perp) = f(x) = \text{integrated PDF}$$

$$f_a(x, \mu^2) = \frac{1}{2} \int \frac{dz^-}{2\pi} \, \mathrm{e}^{-ik^+ z^-} \left\langle p \, | \bar{\psi}_a(z^-, \mathbf{0}_\perp)[z^-, 0^-]_n \gamma^+ \psi_a(0^-, \mathbf{0}_\perp) | \, p \right\rangle$$

One-loop corrections: $\longrightarrow$ emergent (light-cone/rapidity) singularities!

CLASSIFICATION of SINGULARITIES

1. Ultraviolet poles $\sim \frac{1}{\varepsilon}$: removed by the standard renormalization procedure;

2. Overlapping divergences: contain the UV and rapidity poles simultaneously $\sim \frac{1}{\varepsilon} \ln \theta$: generalized renormalization procedure

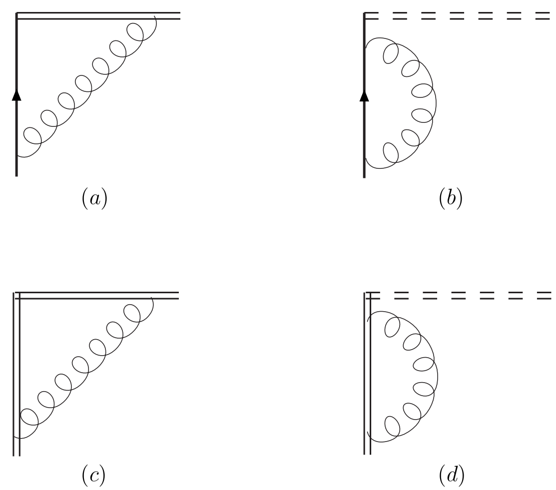
3. Pure rapidity divergences: $\sim \ln^{1,2} \theta$: can be safely summed up by means of the Collins-Soper equation.

4. Specific self-energy divergences: stem from the gauge links, do not affect rapidity evolution; treated by modifications of the soft factors

Collins (2003, 2008, 2011 etc.); Chiu, Jain, Neill, Rothstein (2011, 2012); Avsar (2012) Idilbi, Scimemi (2011, 2012); Cherednikov, Stefanis (2008, 2009, 2010)

RAPIDITY DIVERGENCES in TMDs

One-loop corrections: Source of the rapidity divergences in the covariant and light-cone gauges; Collins (2003); Cherednikov, Stefanis (2011)



Covariant gauge

$$[\text{left panel}] = -\frac{\alpha_s}{\pi} C_{\rm F} \Gamma(\epsilon) \left[4\pi \frac{\mu^2}{-p^2} \right]^{\epsilon} \delta(1-x) \delta^{(2)}(\boldsymbol{k}_\perp) \int_0^1 dx \, \frac{x^{1-\epsilon}}{(1-x)^{1+\epsilon}}$$

Light-cone gauge

$$[\text{right panel}] = -\frac{\alpha_s}{\pi} C_{\rm F} \, \Gamma(\epsilon) \left[4\pi \frac{\mu^2}{-p^2} \right]^{\epsilon} \delta(1-x) \delta^{(2)}(\boldsymbol{k}_\perp) \int_0^1 dx \frac{(1-x)^{1-\epsilon}}{x^{\epsilon}[x]}$$

RAPIDITY DIVERGENCES in TMDs
One-loop corrections

Regularization of the “spurious” singularity in the light-cone gluon propagator (the same is valid for the light-like Wilson line in covariant gauges)

$$D^{\mu\nu}(q) = \frac{i}{q^2 + i0}\left(-g^{\mu\nu} + \frac{(n^-)^\mu q^\nu + (n^-)^\nu q^\mu}{[q^+]}\right)$$

$$\frac{1}{[q^+]_\eta^{\mathrm{PV}}} = \lim_{\eta\to 0}\frac{1}{2}\left(\frac{1}{q^+ + i\eta} + \frac{1}{q^+ - i\eta}\right) \ , \ \bar{\eta} = \eta/p^+ \ , \ \theta = \ln \eta/p^+$$

$$\Sigma_{\mathrm{UV}}^{(a,b)}(\epsilon, \bar{\eta}) = \frac{\alpha_s}{\pi} C_{\mathrm{F}} \frac{1}{\epsilon} \ln \bar{\eta} \ , \ \Sigma_{\mathrm{UV}}^{(c,d)}(\epsilon, \bar{\eta}, \bar{\eta}_+) = -\frac{\alpha_s}{\pi} C_{\mathrm{F}} \frac{1}{\epsilon} \ln \bar{\eta}\bar{\eta}_+$$

$$\frac{d\Sigma_{\mathrm{UV}}^{(a,b)}}{d\theta} = -\frac{d\Sigma_{\mathrm{UV}}^{(c,d)}}{d\theta} = \frac{\alpha_s}{\pi} C_{\mathrm{F}} \frac{1}{\epsilon}$$

$$\Gamma_{\mathrm{cusp}} = \frac{\alpha_s}{\pi} C_{\mathrm{F}}$$

RAPIDITY DIVERGENCES in TMDs
Subtractions and factorization

ICh-Stefanis factorization conjecture:

$$W^{\mu\nu} = |H(Q,\mu)^2|^{\mu\nu} \cdot \frac{\mathcal{F}^{[\mathrm{An}]}_{\mathrm{unsub.}}(x, \boldsymbol{k}_\perp; \mu, \theta)}{S_F(n^+, n^-; \theta) L_F^{-1}(n^+)} \otimes \frac{\mathcal{D}^{[\mathrm{An}]}_{\mathrm{unsub.}}(z, z\boldsymbol{k}'_\perp; \mu, \theta)}{S_D(n^+, n^-; \theta) L_D^{-1}(n^-)} + \ldots$$

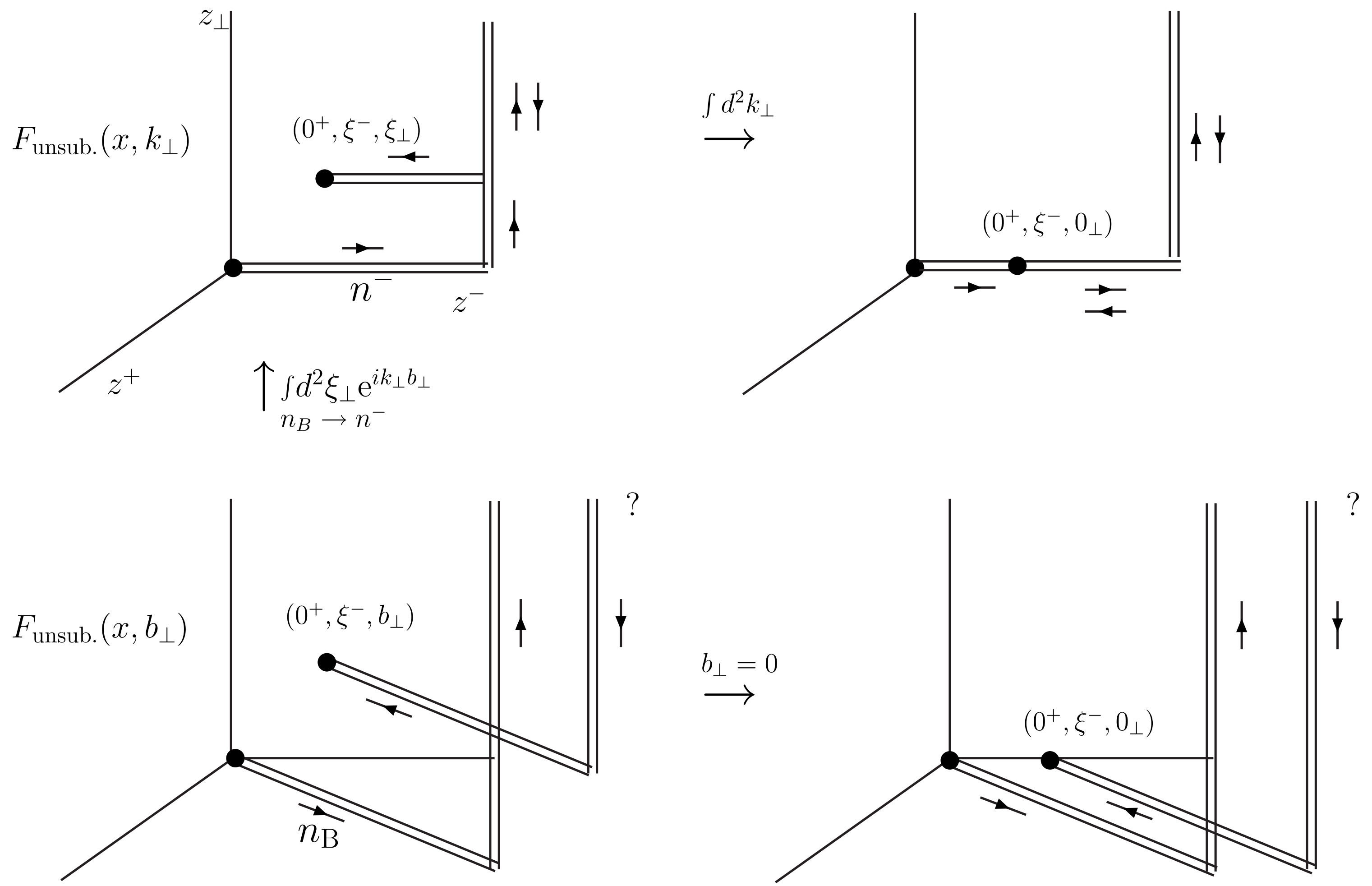
Collins definition:

$$W^{\mu\nu} = |H(Q,\mu)^2|^{\mu\nu} \cdot \mathcal{F}^{[\mathrm{Col.}]}(x, \boldsymbol{k}_\perp; \mu, \zeta_F) \otimes \mathcal{D}^{[\mathrm{Col.}]}(z, z\boldsymbol{k}'_\perp; \mu, \zeta_D) + \ldots$$

$$\mathcal{F}^{[\mathrm{Col.}]}(x, \boldsymbol{b}_\perp; \mu, \zeta_F) = \mathcal{F}^{[\mathrm{An}]}_{\mathrm{unsub.}}(x, \boldsymbol{b}_\perp; \mu) \cdot \sqrt{\frac{S(n^+, n_B)}{S(n^+, n^-) S(n_A, n^-)}}$$

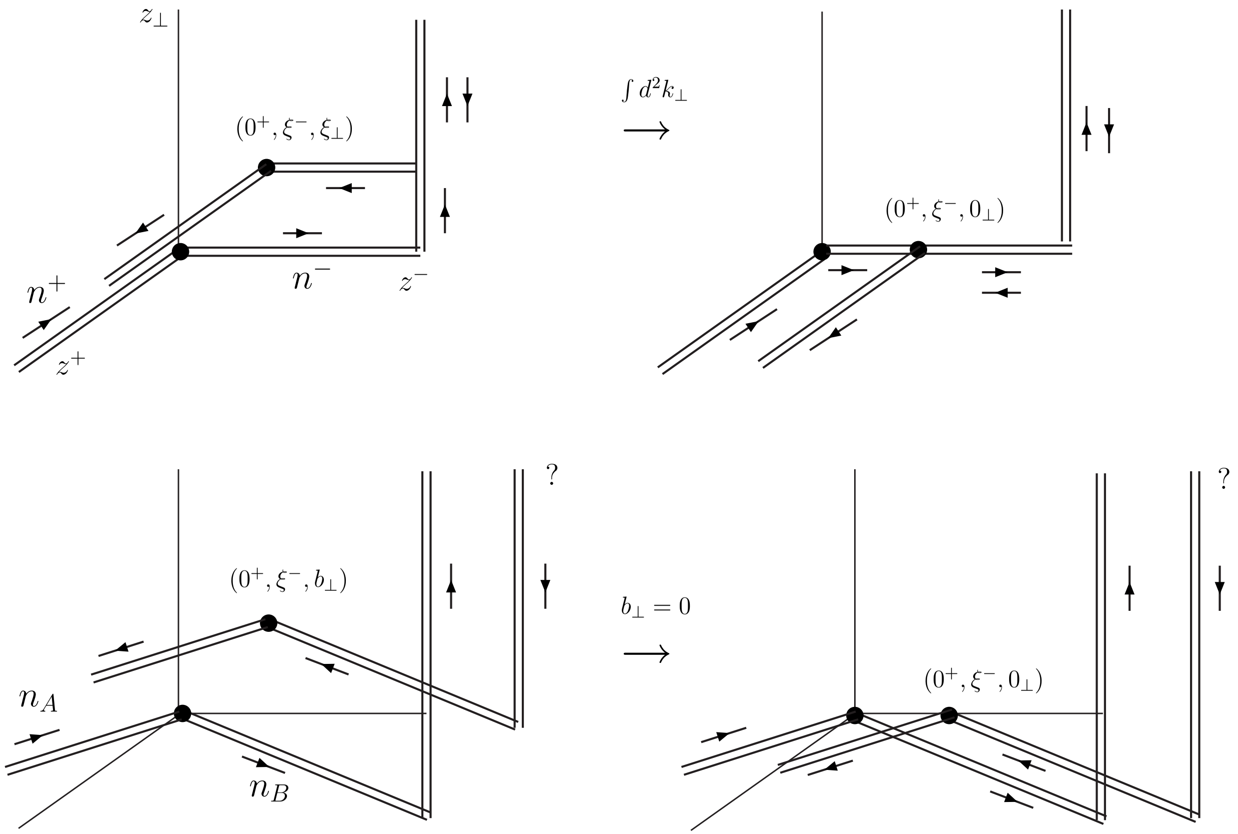
RAPIDITY DIVERGENCES in TMDs

Geometrical structure of integration contours in the unsubtracted TMDs with the light-like and off-the-light-cone longitudinal gauge links and their symbolic reduction to the collinear PDFs



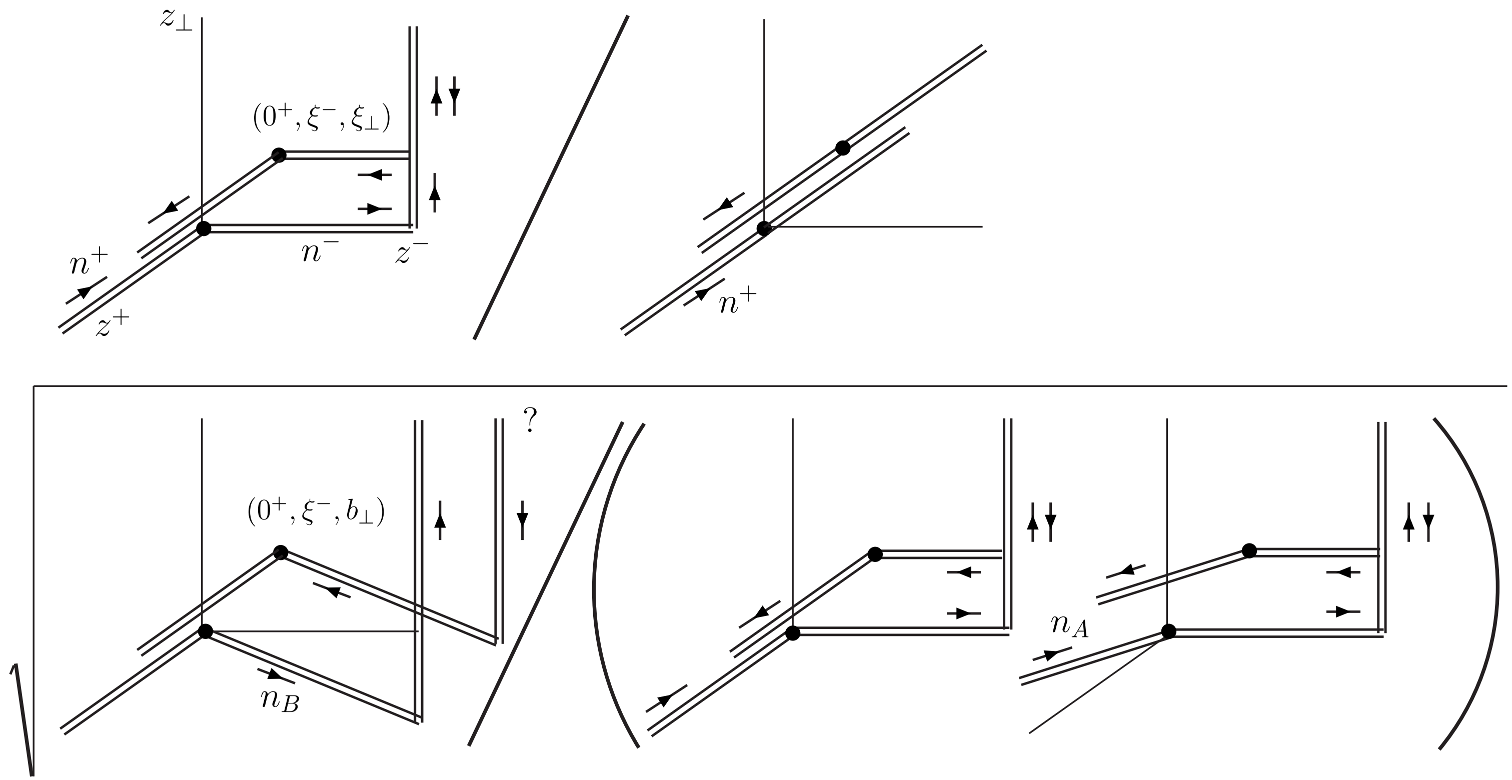
RAPIDITY DIVERGENCES in TMDs

Geometrical structure of integration contours in the unsubtracted soft factors with the light-like and off-the-light-cone longitudinal gauge links and their symbolic reduction to the collinear PDFs



RAPIDITY DIVERGENCES in TMDs

Geometrical structure of integration contours in the full soft factors with the light-like and off-the-light-cone longitudinal gauge links and their symbolic reduction to the collinear PDFs



RAPIDITY DIVERGENCES in TMDs
Evolution equations

+ Rapidity evolution (Collins-Soper eq.) in the impact parameter space:

$$\mathcal{F}\left(x, \mathbf{b}_\perp; \mu, \eta\right) = \int d^2\mathbf{k}_\perp \, \mathrm{e}^{i\boldsymbol{b}_\perp \cdot \boldsymbol{k}_\perp} \, \mathcal{F}\left(x, \mathbf{k}_\perp; \mu, \eta\right)$$

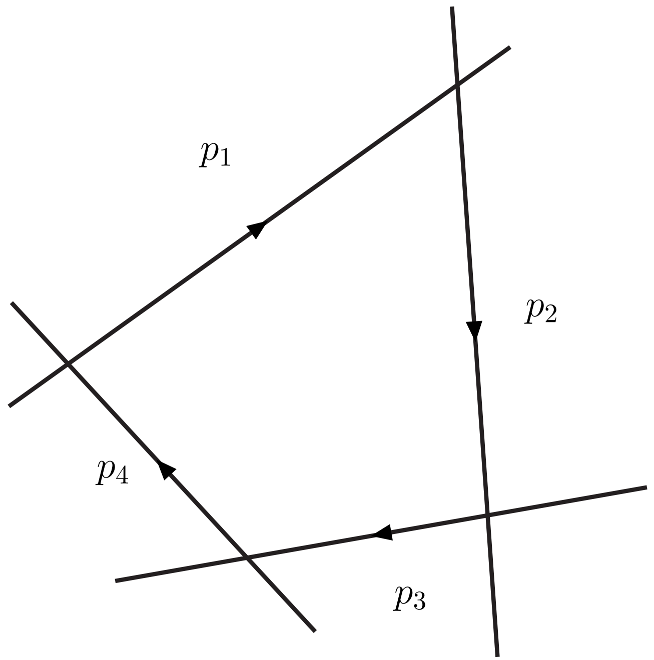
$$\zeta \frac{\partial}{\partial \zeta} \mathcal{F}\left(x, \mathbf{k}_\perp; \mu, \zeta\right) = \left[K_{\mathrm{v}}(\mu, b_\perp) + G_{\mathrm{v}}(\mu, \zeta)\right] \, \mathcal{F}\left(x, \mathbf{k}_\perp; \mu, \zeta\right)$$

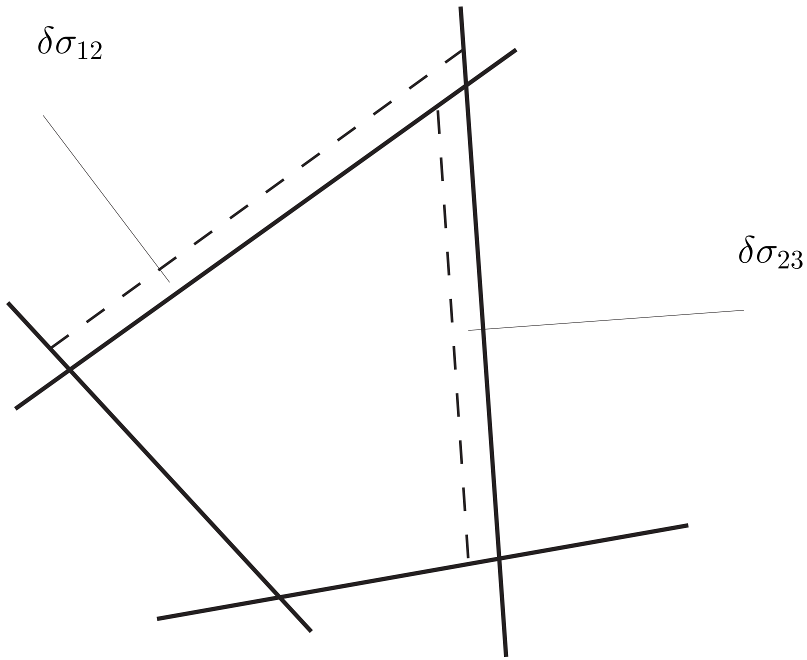
Works, but: too complicated and "arbitrary" soft factors; problems with reduction to the integrated PDF; too much "evolutions"...

→ Looking for more elegant ideas: properties of the cusped light-like Wilson lines/loops.

RAPIDITY DIVERGENCES in LIGHT-LIKE WILSON LOOPS

Generic light-like quadrilateral contour





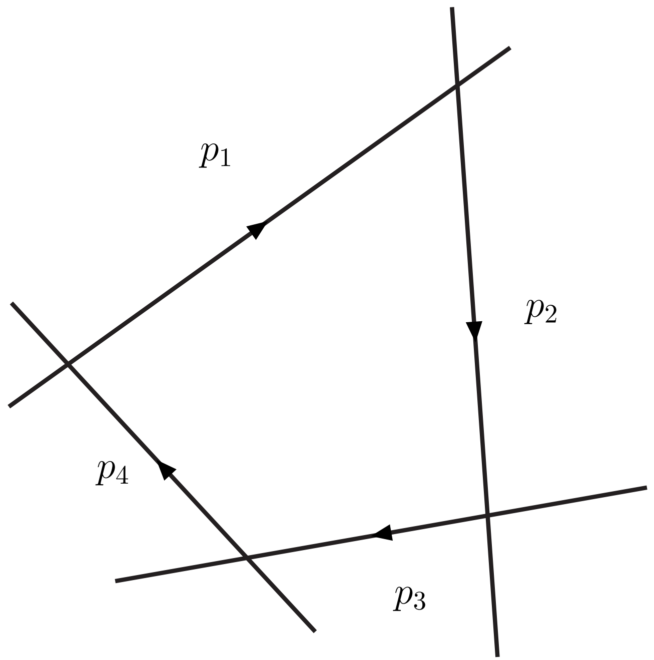
Motivation: **duality** between 4-gluon planar scattering amplitude in $\mathcal{N} = 4$ SYM and the Wilson loop made up from four light-like segments: $x_i - x_{i+1} = p_i$ are equal to the external momenta of this 4-gluon amplitude. The IR evolution of the former is dual to the UV evolution of the latter: governed by the cusp anomalous dimension.

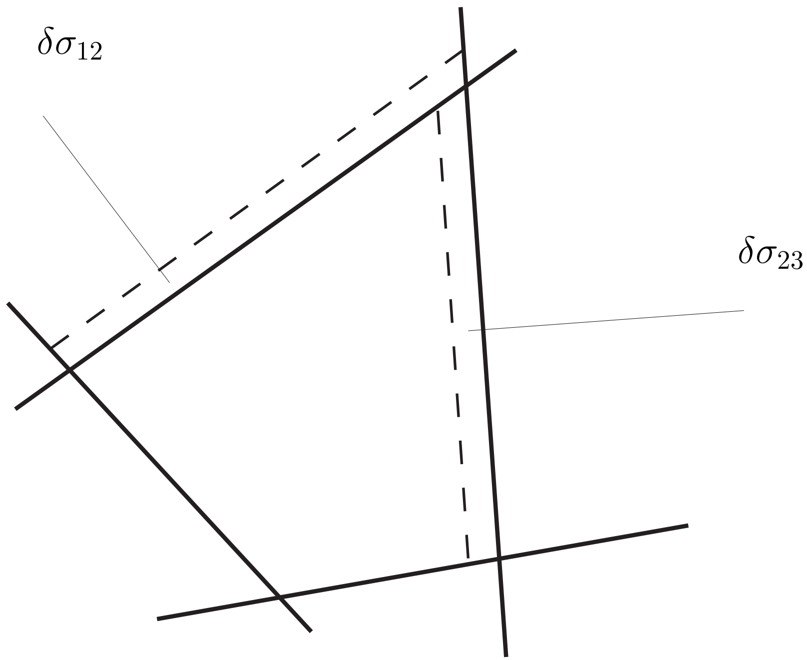
Alday, Maldacena (2007); Makeenko (2003); Korchemsky, Drummond, Sokatchev (2008); Alday, Eden, Korchemsky, Maldacena, Sokatchev (2011); Beisert et al. (2012); Belitsky (2012) etc.

RAPIDITY DIVERGENCES in LIGHT-LIKE WILSON LOOPS

Generic light-like quadrilateral contour:

Angle-conserving infinitesimal contour deformations, area differentials





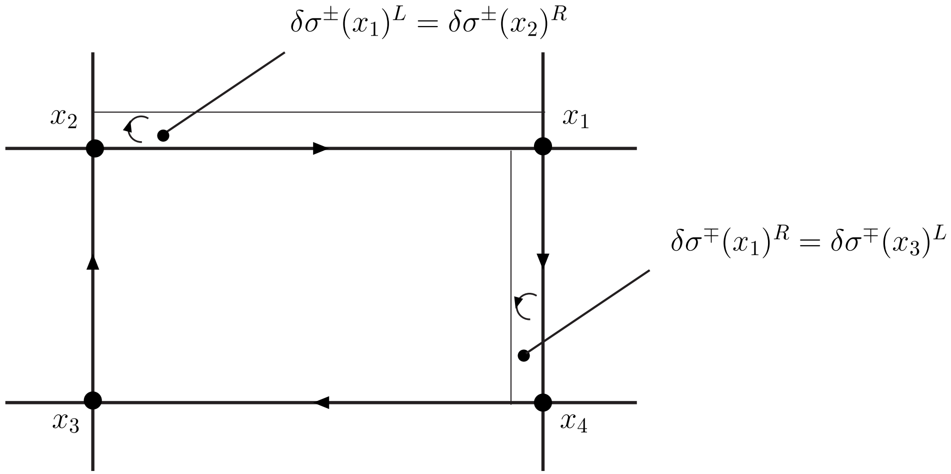
$$W_{\rm SYM}(\Gamma) = \langle 0 | \mathcal{P}\ \exp\left[\oint_\Gamma d\tau \left(i\dot{z}^\mu \hat{A}_\mu(z) + \Phi_i \theta_i |\dot{z}| \right) \right] |0\rangle$$

"Area" differentials: $|\delta\sigma_{12}| = \frac{1}{2}\left(p_1 \delta p_2 + p_1 \delta p_4\right) \to$ however, the structure of the world-sheet is complicated.

RAPIDITY DIVERGENCES in LIGHT-LIKE WILSON LOOPS

Null-plane light-like rectangular contour:

Angle-conserving infinitesimal contour deformations



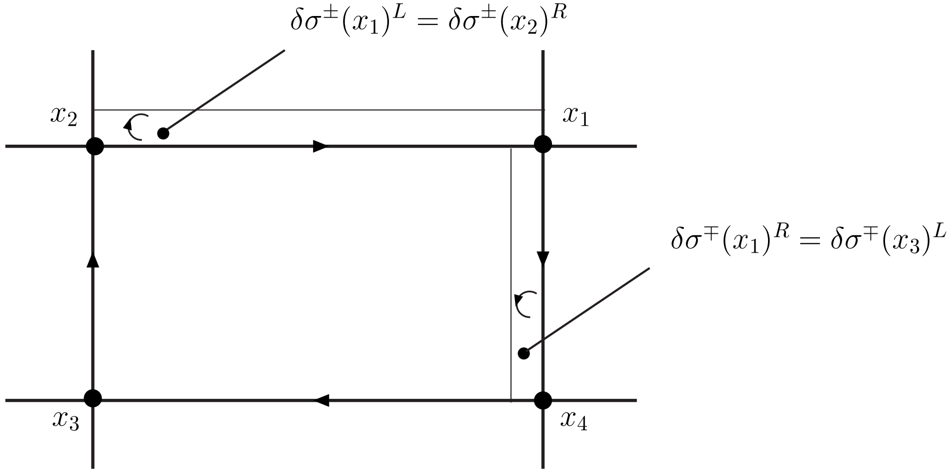
Null-plane is determined by $\boldsymbol{z}_\perp = 0$; therefore, the minimal (oriented) area is well-defined:

$$\delta\sigma^{+-} = \oint_{\delta\Gamma^{\pm}} z^+ dz^- = N^+ \delta N^-$$

$$\delta\sigma^{-+} = \oint_{\delta\Gamma^{\mp}} z^- dz^+ = N^- \delta N^+$$

RENORMALIZATION PROPERTIES of LIGHT-LIKE WILSON LOOPS

Mandelstam rapidity/energy variables



Large-N_C limit:

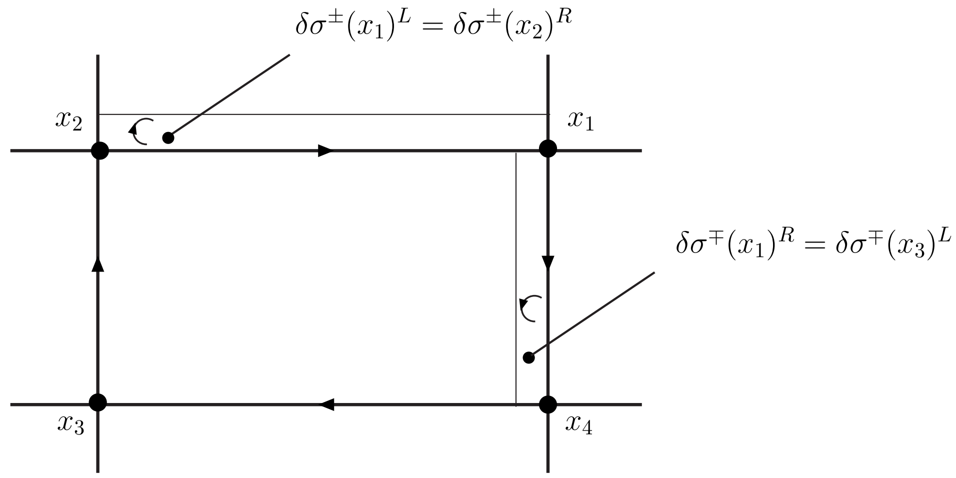
$$W(\Gamma) = 1 + \frac{\alpha_s N_c}{2\pi} \left\{ -\frac{1}{\epsilon^2} \left(\left[\frac{-s + i0}{\mu^2} \right]^{\epsilon} + \left[\frac{-t + i0}{\mu^2} \right]^{\epsilon} \right) + \frac{1}{2} \ln^2 \frac{s}{t} + 2\zeta_2 + O(\epsilon) \right\} + O(\alpha_s N_c)$$

Energy/rapidity variables: $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$ vs **area** variables (rectangular contour in the null-plane): $\Sigma = (p_1 \cdot p_2) = s/2 = -t/2$

Korchemskaya, Korchemsky (1992); Bassetto, Korchemskaya, Korchemsky, Nardelli (1993)

RENORMALIZATION PROPERTIES of LIGHT-LIKE WILSON LOOPS

Null-plane light-like rectangular contour:
Mandelstam rapidity/energy variables

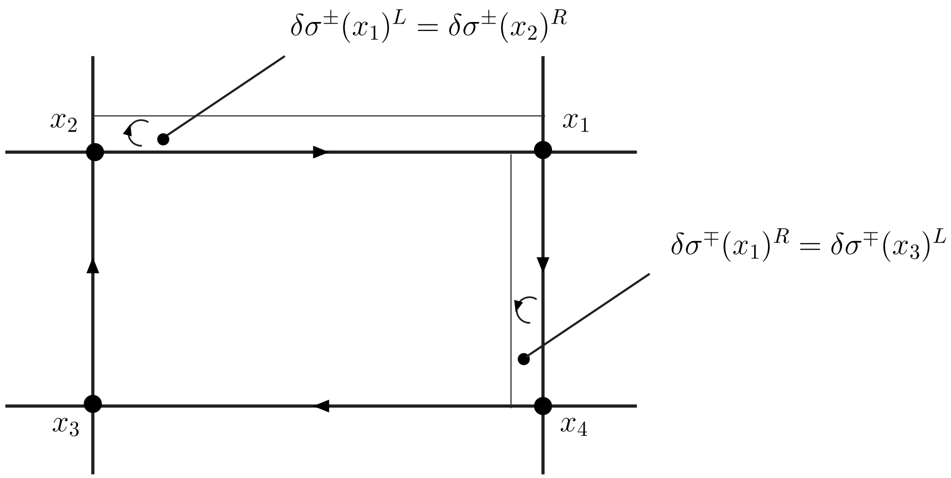


$W(\Gamma)$ is not multiplicatively renormalizable due to light-cone extra divergences—dual to the TMD case.

However, energy/area logarithmic derivative does the job ($\bar{s} = s/\mu^2$):

$$\frac{d \ln W(\Gamma)}{d \ln \bar{s}} = \frac{1}{2} \frac{d \ln W(\Gamma)}{d \ln \bar{\Sigma}} = -\frac{\alpha_s N_c}{2\pi} \frac{1}{\epsilon} \left([\bar{s} + i0]^{\epsilon} - [-\bar{s} + i0]^{\epsilon} \right)$$

RENORMALIZATION PROPERTIES of LIGHT-LIKE WILSON LOOPS



Anomalous dimension results from (large-N_c):

$$\mu \frac{d}{d\mu} \frac{d \ln W(\Gamma)}{d \ln \bar{s}} \sim -4\, \Gamma_{\rm cusp} \ , \ \Gamma_{\rm cusp} = \frac{\alpha_s N_c}{2\pi}$$

We get finite result by means of the area derivative: **dynamical properties** of the light-like Wilson loop are encoded in the cusp anomalous dimension: Korchemsky, Radyushkin (1987). Local quantity: behavior in vicinity of an obstruction. Path-dependence shows up in finite terms. We related the **geometry** of the loop space (area differentials) and the **dynamics** of the fundamental d.o.f., that is the light-like Wilson loops.

MAKEENKO-MIGDAL APPROACH

Wilson loops as the (fundamental) gauge-invariant degrees of freedom:

$$W_n(\Gamma_1, \ldots \Gamma_n) = \langle 0 | \mathcal{T} \frac{1}{N_c} \Phi(\Gamma_1) \cdots \frac{1}{N_c} \Phi(\Gamma_n) | 0 \rangle$$

$$\Phi(\Gamma_i) = \mathcal{P} \exp \left[ig \int_{\Gamma_i} dz^\mu \hat{A}_\mu(z) \right]$$

The Wilson functionals obey the Makeenko-Migdal loop equations:

$$\partial^\nu \frac{\delta}{\delta \sigma_{\mu\nu}(x)} W_1(\Gamma) = N_c g^2 \oint_\Gamma dz^\mu \, \delta^{(4)}(x - z) W_2(\Gamma_{xz} \Gamma_{zx})$$

The equation is exact and non-perturbative, but not closed and difficult to solve in general.

Polyakov (1979); Makeenko, Migdal (1979, 1981); Kazakov, Kostov (1980); Brandt, Neri, Sato (1981); Brandt, Gocksch, Sato, Neri (1982) etc.

MAKEENKO-MIGDAL APPROACH

Area derivative:

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\Phi(\Gamma) = \lim_{|\delta\sigma_{\mu\nu}(x)|} \frac{\Phi(\Gamma\delta\Gamma_x) - \Phi(\Gamma)}{|\delta\sigma_{\mu\nu}(x)|}$$

Path derivative:

$$\partial_\mu\Phi(\Gamma) = \lim_{|\delta x_\mu|} \frac{\Phi(\delta x_\mu^{-1}\Gamma\delta x_\mu) - \Phi(\Gamma)}{|\delta x_\mu|}$$

Mandelstam formula:

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\mathrm{Tr}\Phi(\Gamma) = ig\mathrm{Tr}\left[F_{\mu\nu}\Phi(\Gamma)\right]$$

No information about cusps, divergences, etc.

Problems:

- Most interesting loops are divergent and have obstructions: we are particularly interested in **cusped** loops. In that case, renormalized version of the MM equation is not available.
- The area functional derivative is not well-defined operation for arbitrary contour. In particular, the area differentiation for cusped loops is not (at least) straightforward.
- Problems with continuous deformation of the loops in the Minkowski space: consistent definition of the derivatives obscure.
- Connection of the loop functionals to observables.
- Solution of the MM equations in the four-dimensional Minkowskian space-time is not known.

MAKEENKO-MIGDAL APPROACH

Simplifications:

- Large-N_c limit: factorization property $W_2(C_1, C_2) \approx W_1(C_1) \cdot W_2(C_2)$
- Null-plane light-cone rectangular contours are effectively two-dimensional
- Light-like polygons with conserved angles: no angle-dependent contributions which may break MM-equation
- Area differentiation: the power of divergency decreases

Therefore, the MM approach relates cusp dynamics, renormalization properties and geometry of the loop space. The problem now is how to extract reliable information.

MAKEENKO-MIGDAL APPROACH / preliminary!

Non-Abelian exponentiation of the regularized (but not renormalized) Wilson loops with cusps: Gatheral (1983); Frenkel, Taylor (1984); Korchemsky, Radyushkin (1987)

$$W(\Gamma; \epsilon; g; s, t) = \exp\left[\sum_{k=1} \alpha_s^k \, C_k(W) F_k(W)\right]$$

"Maximally non-Abelian" coefficients $C_k \sim C_F \, N_c^{k-1} \to \frac{N_c^2}{2}$

Perturbative expansion of the MM equation:

$$\mu \frac{d}{d\mu} \frac{dW}{d\ln s} = Z \, \alpha_s \, W \quad , \quad W(\epsilon; g; s, t) = 1 + \alpha_s C_1 F_1 + \alpha_s^2 \left(C_2 F_2 + \frac{1}{2!} C_1^2 F_1^2 \right) + O(\alpha_s)$$

—a closed chain of perturbative equations:

$$C_1 \frac{dF_1}{d\ln s} = Z(\epsilon; s, t)\alpha_s \; , \; C_2 \frac{dF_2}{d\ln s} = Z \; C_1 F_1 - \frac{1}{2!} C_1^2 \frac{dF_1^2}{d\ln s} \ldots$$

$Z(\epsilon; s, t)$ is universal factor related to the cusp anomalous dimension: Cherednikov, Mertens, Van der Veken (2012) (in preparation)

SCHWINGER APPROACH

Schwinger (1951)

Fundamental quantum dynamical principle

$$\delta\langle\, '\, |\, ''\, \rangle = \frac{i}{\hbar}\langle\, '\, |\delta S|\, ''\, \rangle$$

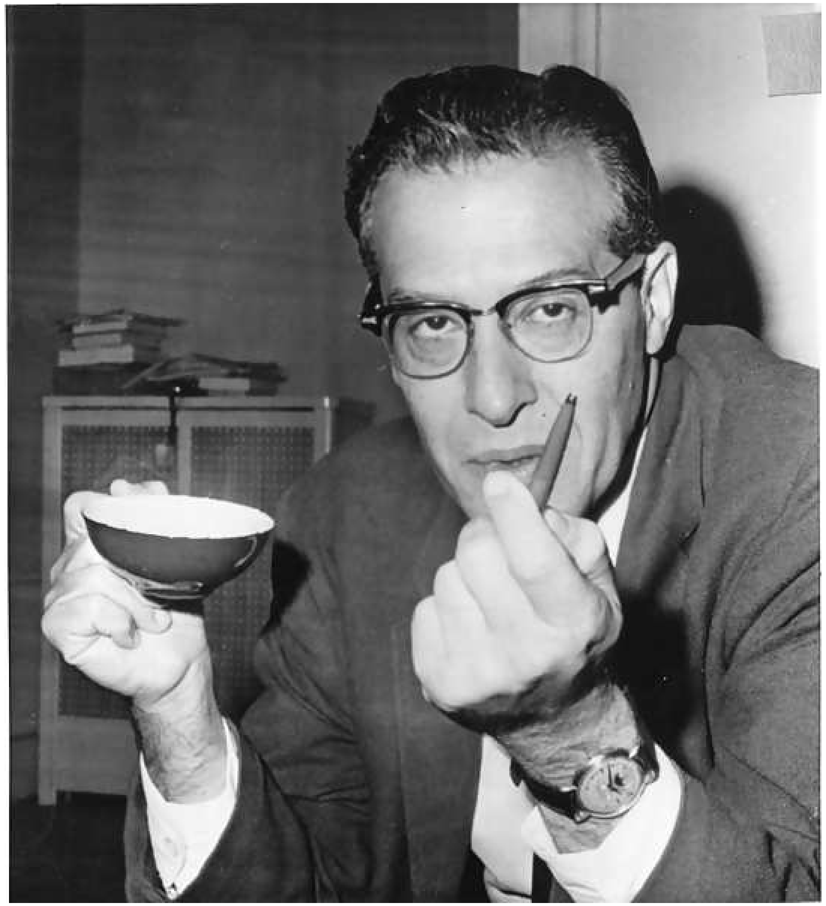
Application to the Wilson functionals $\Phi(\Gamma)$ + Mandelstam formula + Stokes theorem yields MM Eq.

$$\langle 0|\nabla_\mu F^{\mu\nu}\, \mathrm{Tr}\Phi(\Gamma)|0\rangle = i\hbar\langle 0|\frac{\delta}{\delta A_\nu}\, \mathrm{Tr}\Phi(\Gamma)|0\rangle$$

$$\partial^\nu\, \frac{\delta}{\delta\sigma_{\mu\nu}(x)}\, W_1(\Gamma) = N_c g^2 \oint_\Gamma dz^\mu\, \delta^{(4)}(x-z) W_2(\Gamma_{xz}\Gamma_{zx})$$

SCHWINGER APPROACH

Schwinger caution:



Did you take care of singularities?

MODEFIED SCHWINGER APPROACH
for light-like rectangular contours

Return to the definition of the area derivative and consider special area differentials (do not make use of the Stokes theorem or Mandelstam formula)

Take into account renormalization group invariance

$$\mu \frac{d}{d\mu} \left[\sigma_{\mu\nu} \frac{\delta}{\delta \sigma_{\mu\nu}} \ \ln \ W(\Gamma) \right] = - \sum \Gamma_{\text{cusp}}$$

Works for the rectangular light-like Wilson loop in the null-plane;

Works for the TMD "on the light-cone":

$$\mu \frac{d}{d\mu} \left[\frac{d}{d \ln \theta} \ \ln \ \Phi(x, \boldsymbol{k}_\perp) \right] = 2\Gamma_{\text{cusp}}$$

→ complete evolution of the TMDs + further development...

UNIVERSALITY of the MAKEENKO-MIGDAL APPROACH:

DISCUSSION, CONCLUSIONS and OUTLOOK

- Makeenko-Migdal approach provides a full and consistent description of the geometrical properties of the loop space. Fundamental degrees of freedom are closed Wilson loops and the MM Eqs. resemble the Schwinger-Dyson Eqs. in the loop space. In general, the system of the MM Eqs. is not closed and cannot be straightforwardly applied to calculate any useful quantity.

- However, in the large-N_c limit, in the null-plane $\boldsymbol{z}_\perp = 0$, for the rectangular light-like Wilson loops, the area functional derivative is reduced to the normal derivative for the dimensionally regularized (not renormalized!) loops and the MM Eqs. appear to be equivalent to the energy/rapidity evolution equations.

- We can then relate the geometrical properties of the loop space and the dynamics encoded in cusps. The latter are introduced by externally-driven obstructions of (initially) smooth Wilson loops. Renormalized MM Eq. for cusped loops are closed dynamical equations for the loop functionals, they can be (in principle) formulated consistently and even solved on the light-cone.

- Conjecture: since the MM approach is universal, it can be applied to construction of the energy/rapidity evolution equations in many interesting situations. Specific properties of the Wilson loops are determined by the contours with cusps (and/or self-intersections).