Differential reduction of generalized hypergeometric functions in application to Feynman diagrams. HyperDire project.

V. Bytev,
In collaboration with M. Kalmykov, B. Kniehl

CALC 2012
August 1 , Dubna, Russia

Motivation

Oleg Tarasov (1996), and Davydychev-Delbourgo (1997), have suggested two elegant approaches for construction of hypergeometric representation of one-loop Feynman Diagrams. One of main achievement of these approaches are the essential reduction of independent variables. In accordance with Fleischer-Jegerlehner-Tarasov, 2003.

Type of 1-loop diagram	# 1	# 2
N=2 (propagator)	3	1
N=3 (vertex)	6	2
N=4 (box)	10	3
N = 5 (pentagon)	15	4
N = k	k(k+1)/2	k-1

where
#1 the number of kinematic invariants (non-zero masses/momenta)
and
#2 the number of variables in hypergeometric representation.
In this approach, the one-loop N-point function is expressible in terms of hypergeometric functions of $N-1$ variables.
New results for 1-loop Feynman integrals: Kniehl-Tarasov, 2009, 2010.

1-loop diagrams: Finite part

The program of constructing the analytical coefficients of the ε-expansion is a more complicated matter.
The finite parts of one-loop diagrams in $d = 4$ dimension are expressible in terms of the Spence dilogarithm function
t'Hooft, Veltman, 1979;
Denner, Nierste, Scharf, 1991;
Ellis, Zanderighi, 2007;
Denner, Dittmaier, 2010

$$FeynmanDiagramm = \frac{1}{\varepsilon^2}A + \frac{1}{\varepsilon}B + C + D\varepsilon + E\varepsilon^2 \ldots$$

The Task

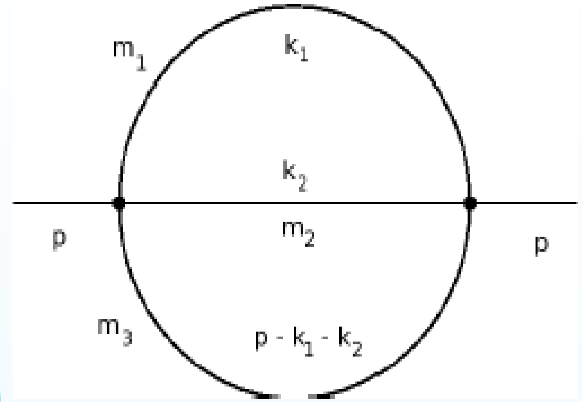
to elaborate the algorithm and implementation for:

- manipulation with multiple hypergeometric (Horn-type) functions (express parameters of arbitrary values in terms of ones that differ from original by integers)
- construction of analytical coefficients of ε-expansion of multiple hypergeometric (Horn-type) functions

Finally:

Package for Numerical Evaluation of finite, $O(\varepsilon)$ and $O(\varepsilon^2)$ parts of one-loop Feynman Diagrams with an arbitrary set of kinematic invariants

Example: Sunset Diagram



$$F_G = \int \frac{d^d k_1 d^d k_2}{[(k_1 - p)^2 - m_1^2][k_2^2 - m_2^2][(k_1 - k_2)^2 - m_3^3]}$$

$$= \int_{-i\infty}^{i\infty} ds_1 ds_2 ds_3 \frac{m_1^{2s_1} m_2^{2s_2} m_3^{2s_3}}{(-p^2)^{s_1+s_2+s_3}} \Gamma(-s_1)\Gamma(-s_2)\Gamma(-s_3)$$

$$\Gamma(3-d+s_1+s_2+s_3) \frac{\Gamma(d/2-1-s_1)\Gamma(d/2-1-s_2)\Gamma(d/2-1-s_3)}{\Gamma(3d/2-3-s_1-s_2-s_3)}$$

$$\begin{aligned}
\sim\, & z_1^{d/2-1} z_2^{d/2-1} F_c^{(3)}(1, d/2, d/2, d/2, d/2; z_1, z_2, z_3) \\
& - z_1^{d/2-1} \Gamma^2(1-d/2) F_c^{(3)}(1, 2-d/2, d/2, 2-d/2, d/2, z_1, z_2, z_3) \\
& - z_2^{d/2-1} \Gamma^2(1-d/2) F_c^{(3)}(1, 2-d/2, d/2, 2-d/2, d/2, z_1, z_2, z_3) \\
& - \Gamma(d/2-1)\Gamma(1-d/2)\Gamma(3-d) F_c^{(3)}(3-d, 2-d/2, 2-d/2, 2-d/2, d/2, z_1, z_2, z_3)\,,
\end{aligned}$$

in terms of the hypergeometric function (in the case $n = 3$)

$$F_c^{(n)}(a, b; c_1, \cdots, c_n; z_1, \cdots z_n) = \sum_{k_1,\ldots k_n} \frac{(a)_{k_1+\cdots+k_n} (b)_{k_1+\cdots+k_n}}{(c_1)_{k_1} \cdots (c_n)_{k_n}} \frac{z_1^{k_1} \cdots z_n^{k_n}}{k_1! \cdots k_n!}$$

with arguments $z_1 = m_1^2/m_3^2, \quad z_2 = m_2^2/m_3^2, \quad z_3 = p^2/m_3^2.$

Horn-type Hypergeometric Functions

In accordance with Horn definition, a formal (Laurent) power series in r variables,

$$\Phi(\vec{z}) = \sum C(\vec{m})\vec{z}^{m} \equiv \sum_{m_1,m_2,\cdots,m_r} C(m_1,m_2,\cdots,m_r)x_1^{m_1}\cdots x_r^{m_r},$$

is called hypergeometric if for each $i = 1,\cdots,r$ the ratio

$$\frac{C(\vec{m}+e_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})} \Rightarrow C(\vec{m}) = \Pi_{i=1}^{r}\lambda_i^{m_i}R(\vec{m})\left(\frac{\Pi_{j=1}^{N}\Gamma(\mu_j(\vec{m})+\gamma_j)}{\Pi_{k=1}^{M}\Gamma(\nu_k(\vec{m})+\delta_k)}\right) .$$

P, Q, R are the rational functions in the index of summation: $\vec{m} = (m_1,\cdots,m_r)$, and $\vec{e}_j$ is unit vector with unity in the j^{th} place.
The Horn hypergeometric function satisfies the following system of equation

$$Q_j\left(\sum_{k=1}^{r} x_k\frac{\partial}{\partial x_k}\right)\frac{1}{x_j}\Phi(\vec{z}) = P_j\left(\sum_{k=1}^{r} x_k\frac{\partial}{\partial x_k}\right)\Phi(\vec{z}) .$$

Current status

The systematic algorithms for construction of analytical coefficients of ε-expansion for a large class of Horn-type hypergeometric functions around integer values of parameters was suggested by Moch-Uwer-Weinzierl, 2001.

Hypergeometric function	Algorithm
${}_pF_{p-1}$	A
F_1	A, B
F_2, F_3	A,C,D
F_4	does not work

Two of these algorithms $A \Rightarrow {}_pF_{p-1}$ and $B \Rightarrow F_1$
was extended for zero-balance case: Weinzierl, 2004.
It was a few attempts to extend these approach
to hypergeometric functions like F_4
Del Duca,Duhr,Glover,Smirnov, 2009
or to another (different from zero-balance case) set of parameters
Rottmann-Reina, 2011
Ablinger-Blümlein-Schneider , 2011

HyperDire project

(HYPERgeometric DIFFerential REduction)

HYPERDIRE is a set of Wolfram Mathematica based programs for differential reduction of Horn type hypergeometrical functions.

HYPERDIRE includes the following packages:

- -- **pfq** is relevant to manipulation with hypergeometrical functions ${}_{p+1}\boldsymbol{F}_{p}$
- -- **AppellF1F4** is relevant to manipulation with Appell`s hypergeometric functions of two variables **F1, F2,F3,F4**.
- Fd multiple variable function

The package is available at:
https://sites.google.com/site/loopcalculations/home

Horn-type Hypergeometric Functions: reduction to the basis

Let us consider the series

$$\Phi(\vec{\gamma};\vec{\sigma};\vec{z}) = \sum_{m_1,m_2,\cdots,m_r=0}^{\infty} \left(\frac{\Pi_{j=1}^{K} \Gamma\left(\sum_{a=1}^{r} \mu_{ja} m_a + \gamma_j\right)}{\Pi_{k=1}^{L} \Gamma\left(\sum_{b=1}^{r} \nu_{kb} m_b + \sigma_k\right)} \right) x_1^{m_1} \cdots x_r^{m_r} \, ,$$

The sequences $\vec{\gamma} = (\gamma_1, \cdots, \gamma_K)$ and $\vec{\sigma} = (\sigma_1, \cdots, \sigma_L)$ are called *upper* and *lower* parameters of the hypergeometric function, respectively. Two functions with sets of parameters shifted by a unit, $\Phi(\vec{\gamma}+\vec{e_c};\vec{\sigma};\vec{z})$ and $\Phi(\vec{\gamma};\vec{\sigma};\vec{z})$, are related by a linear differential operator:

$$\Phi(\vec{\gamma}+\vec{e_c};\vec{\sigma};\vec{z}) = \left(\sum_{a=1}^{r} \mu_{ca} x_a \frac{\partial}{\partial x_a} + \gamma_c \right) \Phi(\vec{\gamma};\vec{\sigma};\vec{z})$$

$$\Phi(\vec{\gamma};\vec{\sigma}-\vec{e_c};\vec{z}) = \left(\sum_{b=1}^{r} \nu_{cb} x_b \frac{\partial}{\partial x_b} + \sigma_c \right) \Phi(\vec{\gamma};\vec{\sigma};\vec{z}) \, .$$

Horn-type Hypergeometric Functions: Inverse Operators

Staring from homogeneous system of PDE and direct differential operators, the inverse differential operators can be constructed:

$$\Phi(\vec{\gamma} - \vec{e_c}; \vec{\sigma}; \vec{z}) = \sum_a S_a(\vec{z}, \vec{\partial_x}) \Phi(\vec{\gamma}; \vec{\sigma}; \vec{z})$$

$$\Phi(\vec{\gamma}; \vec{\sigma} + \vec{e_c}; \vec{z}) = \sum_b L_b(\vec{z}, \vec{\partial_x}) \Phi(\vec{\gamma}; \vec{\sigma}; \vec{z}) \; .$$

In this way, any Horn-type function can be written as follows:

$$P_0(\vec{z}) \Phi(\vec{\gamma} + \vec{k}; \vec{\sigma} + \vec{l}; \vec{z}) = \sum_{m_1, \cdots, m_p = 0} P_{m_1, \cdots, m_r}(\vec{z}) \left(\frac{\partial}{\partial \vec{z}} \right)^{\vec{m}} \Phi(\vec{\gamma}; \vec{\sigma}; \vec{z}) \; ,$$

where $P_0(\vec{z})$ and $P_{m_1, \cdots, m_p}(\vec{z})$ are polynomials with respect to $\vec{\gamma}, \vec{\sigma}$ and $\vec{z}$ and $\vec{k}, \vec{l}$ are lists of integers.

Simplify the procedure of Factorization.

Differential reduction algorithm for $_{p+1}F_p$ hypergeometric funct.

- Differential identities:

$$_pF_q(a_1+1,\vec{a};\vec{b};z) = B^+_{a_1}\,{}_pF_q(a_1,\vec{a};\vec{b};z) = \frac{1}{a_1}(\theta+a_1)\,{}_pF_q(a_1,\vec{a};\vec{b};z)$$

$$_pF_q(\vec{a};b_1-1,\vec{b};z) = H^-_{b_1}\,{}_pF_q(\vec{a};b_1,\vec{b};z) = \frac{1}{b_1-1}(\theta+b_1-1)\,{}_pF_q(\vec{a};b_1,\vec{b};z)$$

$$_{p+1}F_p(\vec{a};b_i+1,\vec{b};z) = H^+_{b_i}\,{}_{p+1}F_p(\vec{a};b_1,\vec{b};z)\ ,$$

$$H^+_{a_i} = \frac{b_i-1}{d_i}\left[\frac{d}{dz}\Pi_{j\neq i}(\theta+b_j-1)-s_i(\theta)\right]\Bigg|_{b_i\to b_i+1}\ ,$$

$$d_i = \Pi_{j=1}^{p+1}(1+a_j-b_i)\ ,$$

$$s_i(x) = \frac{\Pi_{j=1}^{p+1}(x+a_j)-d_i}{x+b_i-1}\ ,$$

Inverse operators:

$$_{p+1}F_p(a_i-1,\vec{a};\vec{b};z) = B^-_{a_i}\,{}_{p+1}F_p(a_i,\vec{a};\vec{b};z)\ ,$$

$$B^-_{a_i} = -\frac{a_i}{c_i}\left[t_i(\theta)-z\Pi_{j\neq i}(\theta+a_j)\right]\Big|_{a_i\to a_i-1}\ ,$$

$$c_i = -a_i\Pi_{j=1}^{p}(b_j-1-a_i)\ ,$$

$$t_i(x) = \frac{x\Pi_{j=1}^{p}(x+b_j-1)-c_i}{x+a_i}\ ,$$

Differential reduction algorithm for $_{p+1}F_p$ hypergeometric funct.

- Example of differential reduction:

$$
{}_3F_2\left(\left.\begin{array}{c} a_1-1, a_2, a_3 \\ b_1, b_2 \end{array}\right| z\right)(b_1-a_1)(b_2-a_1) = \Big\{ (1-z)\theta^2
$$

$$
+ \left[(b_1+b_2-1-a_1)-z(a_2+a_3)\right]\theta + (b_1-a_1)(b_2-a_1) - za_2a_3 \Big\}\, {}_3F_2\left(\left.\begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array}\right| z\right)
$$

- In reduction on more units the structure of equality will be the same

Implementation of algorithm

- The package called HYPERDIRE (HYPERgeometric DIFFerential REduction), based on language of program ‘’Mathematica’’
- Key feature is that the product of non-commutative step-up and step-down operators of differential reduction turn into product of special 2-dimensional matrices and vectors which greatly simplify and reduce the time of calculation
- The functional programming style reduce the calculation time

```
MapAt[ReplacePart[#, Join[#[[1, 1]], Table[0, {i, 1, Length[#[[2, 1]]] - Length[#[[1, 1]]]}]], {1, 1}] &,
 Map[ReplacePart[#, {Table[SymmetricPolynomial[-i, #[[1]]], {i, -Length[#[[1]]], 0}]}, 1] &,
  Nest[If[(numberOfAdBup = 1 + Sum[Length[#[[i, 1]]], {i, 1, Length[#]}];
          changevar = AdBupvector[#[[-1, 2]], listOfAdownAndBupch[[numberOfAdBup]]];
          Mod[Length[#[[-1, 1]]], Length[#[[-1, 2, 2]]]] == 0
         ), Append[#, {{changevar[[1]]}, changevar[[2]], 1 / changevar[[1]], 0}],
     ReplacePart[ReplacePart[Insert[#, changevar[[1]], {-1, 1, 1}], changevar[[2]], {-1, 2} ], #[[-1, 3]] / changevar[[1]], {-1, 3}]] & ,
   initialVector, Length[listOfAdownAndBupch]]]
 , -1]
```

Example of module PFQ

ToGroebnerBasis [{{1+a_1,1+a_2, a_3,a_4},{1+b_1, b_2+1,b_3},x}] ,

IntegerPart={1,1,0,0,1,1,0}   changeVector={-1,-1,0,0,0,0,1}

$$\left\{1, \frac{1}{a_2} + \frac{1}{b_3} + \frac{1}{a_1}, \frac{a_1+a_2+b_3}{a_1 a_2 b_3}, \frac{1}{a_1 a_2 b_3}\right\}, \{\{a_1, a_2, a_3, a_4\}, \{b_1+1, b_2+1, b_3+1\}, x\}, 1\}$$

Hypergeometric function parameters transformation

$$ {}_4F_3\left(\begin{array}{c} 1+a_1, 1+a_2, a_3, a_4 \\ 1+b_1, 1+b_2, b_3 \end{array} \middle| z\right) = \left[1 + \left(\frac{1}{a_2} + \frac{1}{b_3} + \frac{1}{a_1}\right)\theta + \frac{a_1+a_2+b_3}{a_1 a_2 b_3}\theta^2 + \frac{1}{a_1 a_2 b_3}\theta^3\right] {}_4F_3\left(\begin{array}{c} a_1, a_2, a_3, a_4 \\ b_1+1, b_2+1, b_3+1 \end{array} \middle| x\right)$$

Example of module PFQ, reducibility

From differential reduction formulas could be derived reducibility criteria: under which conditions the hypergeometric function could be expressed in terms of hyp. function of lower order (four criteria)

$$ {}_pF_q\left(\begin{matrix} b_1+m_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots b_q \end{matrix}\middle| z\right) = \sum_{j=0}^{m_1} z^j \binom{m_1}{j} \frac{(a_2)_j\cdots(a_p)_j}{(b_1)_j\cdots(b_q)_j} {}_{p-1}F_{q-1}\left(\begin{matrix} a_2+j, \cdots a_p+j \\ b_2+j, \cdots b_q+j \end{matrix}\middle| z\right) $$

$$ {}_pF_q\left(\begin{matrix} a_1, \cdots a_n, a_{n+1}\cdots a_p \\ a_1+1+m_1, \cdots, a_n+1+m_n, b_{n+1}, \cdots, b_q \end{matrix}\middle| z\right) \prod_{r=1}^{n} \frac{1}{(a_r)_{m_r+1}} $$

$$ = \sum_{i=1}^{n}\sum_{j=0}^{m_i} \frac{(-m_i)_j}{j!(a_i+j)m_i!} \left(\prod_{r=1, r\neq i}^{n} \frac{1}{(a_r-a_i-j)_{m_r+1}}\right) $$

$$ \times {}_{p-n+1}F_{q-n+1}\left(\begin{matrix} a_i+j, a_{n+1}, \cdots, a_p \\ a_i+1+j, b_{n+1}, \cdots, b_q \end{matrix}\middle| z\right), $$

Example of module PFQ, reducibility

ToGroebnerBasis [{{3+b$_1$,1+a$_2$,1+a$_3$},{2+b$_1$,2+b$_2$},x}];

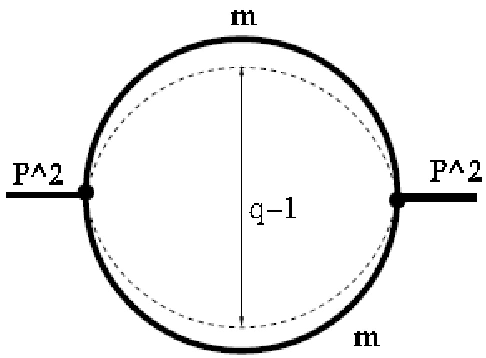
IntegerPart={3,1,1,2,2}   changeVector={-1,-1,-1}

$$\left\{\left\{-\frac{b_2+1}{(x-1)(b_1+2)},-\frac{(a_2x+a_3x-b_1x-x+b_1-b_2+1)(b_2+1)}{(x-1)xa_2a_3(b_1+2)}\right\},\{\{a_2,a_3\},\{b_2+1\},x\},1\right\}$$

Hypergeometric function parameters transformation

$$ {}_3F_2\left(\begin{array}{c} 3+b_1,1+a_2,1+a_3 \\ 2+b_1,2+b_2 \end{array}\middle| x\right) $$

$$ =\left[-\frac{b_2+1}{(x-1)(b_1+2)}-\frac{(a_2x+a_3x-b_1x-x+b_1-b_2+1)(b_2+1)}{(x-1)xa_2a_3(b_1+2)}\theta\right]{}_2F_1\left(\begin{array}{c} a_2,a_3 \\ b_2+1 \end{array}\middle| x\right) $$



Sunset type diagram J_{22}^{q}

$$J_{22}^{q}(m^2,p^2,\alpha_1,\alpha_2,\sigma_1,\cdots,\sigma_{q-1}) = \left[i^{1-n}\pi^{n/2}\right]^{q} \frac{(-m^2)^{\frac{n}{2}q-\alpha_{1,2}-\sigma}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left\{\Pi_{k=1}^{q-1}\frac{\Gamma(\frac{n}{2}-\sigma_k)}{\Gamma(\sigma_k)}\right\}$$

$$\times \frac{\Gamma\left(\alpha_1+\sigma-\frac{n}{2}(q-1)\right)\Gamma\left(\alpha_2+\sigma-\frac{n}{2}(q-1)\right)\Gamma\left(\sigma-\frac{n}{2}(q-2)\right)\Gamma\left(\alpha_{1,2}+\sigma-\frac{n}{2}q\right)}{\Gamma\left(\alpha_{1,2}+2\sigma-n(q-1)\right)\Gamma\left(\frac{n}{2}\right)}$$

$${}_4F_3\left(\begin{array}{c} \alpha_1+\sigma-\frac{n}{2}(q-1), \alpha_2+\sigma-\frac{n}{2}(q-1), \sigma-\frac{n}{2}(q-2), \alpha_{1,2}+\sigma-\frac{n}{2}q \\ \frac{n}{2}, \frac{1}{2}(\alpha_{1,2}-n(q-1))+\sigma, \frac{1}{2}(1+\alpha_{1,2}-n(q-1))+\sigma, \end{array}\middle| \frac{p^2}{4m^2}\right) \cdot ($$

- Criteria of reducibility:
- q=1 $\quad {}_2F_1\left(\begin{array}{c} 1, I_1-\frac{n}{2}, \\ I_2 \end{array}\middle| z\right)$ IBP gives 1 MI
- q=2 $\quad (1,\theta)\times\ {}_3F_2\left(\begin{array}{c} 1, I_1-\frac{n}{2}, I_2-n \\ I_3+\frac{n}{2}, I_4+\frac{1}{2}-\frac{n}{2} \end{array}\middle| z\right)$ IBP gives 2 MI
- q=3,4,5…. $\quad (1,\theta,\theta^2)\times\ {}_3F_2\left(\begin{array}{c} I_1-\frac{n}{2}(q-1), I_2-\frac{n}{2}(q-2), I_3-\frac{n}{2}q \\ \frac{n}{2}, I_4+\frac{1}{2}-\frac{n}{2}(q-1) \end{array}\middle| z\right)$ IBP gives ???

Possible applications

pFq package could work even with ${}_{11}F_{10}$ and reduce it to the function of type ${}_{7}F_{6}$

In[1]:= $b1 = \text{HypergeometricPFQ}\left[\left\{2, 2, 2, 2, 2, \frac{7}{5}+\alpha, \frac{8}{5}+\alpha, \frac{9}{5}+\alpha, \frac{11}{6}+\alpha, \frac{13}{6}+\alpha, \frac{11}{5}+\alpha\right\}, \left\{1, 1, 1, 1, \frac{23}{10}+\alpha, \frac{5}{2}+\alpha, \frac{27}{10}+\alpha, \frac{29}{10}+\alpha, 3+\alpha, \frac{31}{10}+\alpha\right\}, \frac{27}{64}\right];$

In[4]:= `b2 = Simplify[ToGroebnerBasis[{b1[[1]], b1[[2]], x}]]`

```
IntegerPart={2, 2, 2, 2, 2, 1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3}     changeVector={-1, 0, 0, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1}
```

$\text{workingvector}=\left\{\{\{2, 1\}, \{2, 1\}, \{2, 1\}, \{2, 1\}\}, \left\{\left\{2, \frac{7}{5}+\alpha, \frac{8}{5}+\alpha, \frac{9}{5}+\alpha, \frac{11}{6}+\alpha, \frac{13}{6}+\alpha, \frac{11}{5}+\alpha\right\}, \left\{\frac{23}{10}+\alpha, \frac{5}{2}+\alpha, \frac{27}{10}+\alpha, \frac{29}{10}+\alpha, 3+\alpha, \frac{31}{10}+\alpha\right\}, x\right\}\right\}$

Out[4]=

$$\Big\{\Big\{\Big\{\frac{1}{1\,680\,000\,x\,(6+5\alpha)\,(7+6\alpha)}(42+41\alpha+10\alpha^2)\big(-9\,(2\,821\,085\,553+24\,602\,815\,175\alpha+95\,787\,533\,702\alpha^2+219\,256\,207\,280\alpha^3+326\,755\,476\,200\alpha^4+331\,263\,103\,000\alpha^5+231\,344\,230\,000\alpha^6+109\,884\,200\,000\alpha^7+33\,968\,000\,000\alpha^8+6\,170\,000\,000\alpha^9+500\,000\,000\alpha^{10})+8x\,(2\,289\,316\,512+20\,526\,085\,978\alpha+82\,179\,706\,211\alpha^2+193\,548\,285\,390\alpha^3+297\,057\,184\,225\alpha^4+310\,533\,744\,000\alpha^5+223\,966\,777\,500\alpha^6+110\,064\,475\,000\alpha^7+35\,277\,125\,000\alpha^8+6\,660\,000\,000\alpha^9+562\,500\,000\alpha^{10})\big),$$

$$\frac{1}{1\,680\,000\,x\,(6+5\alpha)\,(7+6\alpha)}(42+41\alpha+10\alpha^2)\big(16x\,(4\,134\,953\,189+34\,034\,836\,761\alpha+123\,451\,772\,560\alpha^2+259\,114\,946\,450\alpha^3+346\,970\,643\,125\alpha^4+307\,496\,426\,250\alpha^5+180\,405\,287\,500\alpha^6+67\,580\,062\,500\alpha^7+14\,670\,000\,000\alpha^8+1\,406\,250\,000\alpha^9)-9\,(9\,575\,224\,025+76\,961\,167\,654\alpha+272\,380\,200\,240\alpha^2+557\,274\,636\,800\alpha^3+726\,459\,610\,000\alpha^4+625\,775\,630\,000\alpha^5+356\,190\,400\,000\alpha^6+129\,174\,000\,000\alpha^7+27\,080\,000\,000\alpha^8+2\,500\,000\,000\alpha^9)\big),$$

$$\frac{1}{210\,000\,x\,(6+5\alpha)\,(7+6\alpha)}(42+41\alpha+10\alpha^2)\big(-9\,(1\,613\,184\,619+11\,876\,667\,955\alpha+37\,847\,120\,650\alpha^2+68\,217\,762\,500\alpha^3+76\,091\,650\,000\alpha^4+53\,791\,775\,000\alpha^5+23\,536\,750\,000\alpha^6+5\,827\,500\,000\alpha^7+625\,000\,000\alpha^8)+x\,(11\,882\,339\,311+89\,172\,862\,070\alpha+290\,043\,854\,350\alpha^2+534\,339\,175\,000\alpha^3+610\,160\,225\,000\alpha^4+442\,408\,225\,000\alpha^5+198\,970\,125\,000\alpha^6+50\,760\,000\,000\alpha^7+5\,625\,000\,000\alpha^8)\big),$$

$$\frac{1}{42\,000\,x\,(6+5\alpha)\,(7+6\alpha)}(42+41\alpha+10\alpha^2)\big(-9\,(215\,831\,767+1\,445\,084\,370\alpha+4\,097\,067\,500\alpha^2+6\,381\,165\,000\alpha^3+5\,899\,300\,000\alpha^4+3\,237\,950\,000\alpha^5+977\,000\,000\alpha^6+125\,000\,000\alpha^7)+4x\,(424\,389\,117+2\,880\,426\,845\alpha+8\,293\,156\,875\alpha^2+13\,137\,846\,250\alpha^3+12\,376\,143\,750\alpha^4+6\,936\,043\,750\alpha^5+2\,142\,000\,000\alpha^6+281\,250\,000\alpha^7)\big),$$

$$\frac{1}{2800\,x\,(6+5\alpha)\,(7+6\alpha)}(42+41\alpha+10\alpha^2)\big(-6\,(7\,166\,113+43\,262\,875\alpha+107\,451\,750\alpha^2+140\,705\,000\alpha^3+102\,525\,000\alpha^4+39\,425\,000\alpha^5+6\,250\,000\alpha^6)+x\,(40\,189\,883+244\,308\,000\alpha+612\,175\,500\alpha^2+809\,880\,000\alpha^3+597\,150\,000\alpha^4+232\,800\,000\alpha^5+37\,500\,000\alpha^6)\big),$$

$$\frac{3\,(42+41\alpha+10\alpha^2)\big(-188\,949-1\,017\,950\alpha-2\,169\,000\alpha^2-2\,290\,000\alpha^3-1\,200\,000\alpha^4-250\,000\alpha^5+x\,(189\,229+1\,017\,950\alpha+2\,169\,000\alpha^2+2\,290\,000\alpha^3+1\,200\,000\alpha^4+250\,000\alpha^5)\big)}{280\,x\,(6+5\alpha)\,(7+6\alpha)}\Big\},$$

$$\Big\{\Big\{1, \frac{7}{5}+\alpha, \frac{8}{5}+\alpha, \frac{9}{5}+\alpha, \frac{11}{6}+\alpha, \frac{7}{6}+\alpha, \frac{6}{5}+\alpha\Big\}, \Big\{\frac{23}{10}+\alpha, \frac{5}{2}+\alpha, \frac{27}{10}+\alpha, \frac{29}{10}+\alpha, 2+\alpha, \frac{21}{10}+\alpha\Big\}, x\Big\},$$

$$1\Big\},$$

$$\Big\{\frac{3\,(1+\alpha)\,(3+2\alpha)\,(11+10\alpha)\,(13+10\alpha)\,(17+10\alpha)\,(19+10\alpha)\,(855\,078+4\,478\,639\alpha+9\,531\,550\alpha^2+10\,533\,000\alpha^3+6\,360\,000\alpha^4+1\,985\,000\alpha^5+250\,000\alpha^6)}{560\,000\,x\,(6+5\alpha)\,(7+6\alpha)},$$

$$\{\},$$

$$1\Big\}\Big\}$$

Appell Function F1,F2,F3,F4 the case of two variables

Let us consider the system of linear differential equations of the second order for the functions $\omega(\vec{z})$:

$$\theta_{11}\omega(\vec{z}) = \left\{P_0(\vec{z})\theta_{12} + P_1(\vec{z})\theta_1 + P_2(\vec{z})\theta_2 + P_3(\vec{z})\right\}\omega(\vec{z}) ,$$

$$\theta_{22}\omega(\vec{z}) = \left\{R_0(\vec{z})\theta_{12} + R_1(\vec{z})\theta_1 + R_2(\vec{z})\theta_2 + R_3(\vec{z})\right\}\omega(\vec{z}) ,$$

$$\theta_j = z_j \partial_{z_j}$$

The differential reduction algorithm in application to the Appell function could be done in similar way as for the case of one variable hypergeometrical function

$$R(x,y)F_1(\vec{A}+\vec{m};x,y) = \left[P_0(x,y) + P_1(x,y)\theta_x + P_2(x,y)\theta_y\right] F_1(\vec{A};x,y) , \quad (76)$$

$$S(x,y)F_j(\vec{A}+\vec{m};x,y) = \left[Q_0(x,y)+Q_1(x,y)\theta_x+Q_2(x,y)\theta_y+Q_3(x,y)\theta_x\theta_y\right] F_j(\vec{A};x,y) , \quad (77)$$

Differential reduction for F_1

the direct differential expressions reads:

$$\begin{aligned}
aF_1(a+1,b_1,b_2,c;x,y) &= (\theta_x+\theta_y+a)F_1(a,b_1,b_2,c;x,y)\ , \\
b_1F_1(a,b_1+1,b_2,c;x,y) &= (\theta_x+b_1)F_1(a,b_1,b_2,c;x,y)\ , \\
(c-1)F_1(a,b_1,b_2,c-1;x,y) &= (\theta_x+\theta_y+c-1)F_1(a,b_1,b_2,c;x,y)\ .
\end{aligned}$$

Inverse differential relations:

$$\begin{aligned}
&(c-a)F_1(a-1,b_1,b_2,c;x,y) = \\
&[c-a-b_1x-b_2y+(1-x)\theta_x+(1-y)\theta_y]\,F_1(a,b_1,b_2,c;x,y)\ , \\
&(c-b_1-b_2)F_1(a,b_1-1,b_2,c;x,y) = \\
&\left[c-b_1-b_2-ax+(1-x)\theta_x-x\left(1-\frac{1}{y}\right)\theta_y\right]F_1(a,b_1,b_2,c;x,y)\ , \\
&(c-a)(c-b_1-b_2)F_1(a,b_1,b_2,c+1;x,y) = \\
&c\left[(c-a-b_1-b_2)-\left(1-\frac{1}{x}\right)\theta_x-\left(1-\frac{1}{y}\right)\theta_y\right]F_1(a,b_1,b_2,c;x,y)\ .
\end{aligned}$$

Example of module AppellF1F4

$\mathbf{F1IndexChange}[\{1,-1,0,0\},\{a,b_1,b_2,c,z_1,z_2\}]$

$$\left\{\left\{\frac{a(-\mathrm{z1})+a+\mathrm{b1}\mathrm{z}_1+\mathrm{b}_2\mathrm{z}_2-c-\mathrm{z}_1+1}{a-c+1},-\frac{(\mathrm{z}_1-1)(a-\mathrm{b}_1+1)}{(\mathrm{b}_1-1)(a-c+1)},\frac{\mathrm{z}_2-1}{a-c+1}\right\},\{a+1,\mathrm{b}_1-1,\mathrm{b}_2,c,\mathrm{z}_1,\mathrm{z}_2\},\mathrm{AppellF1}\right\}$$

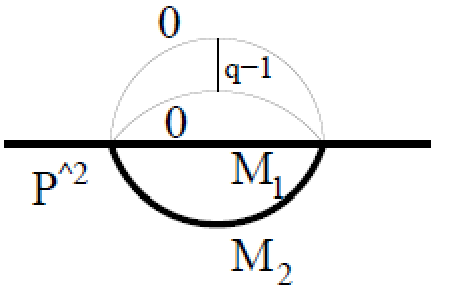
In explicit form:

$$F_1(a,b_1,b_2,c;z_1,z_2)=$$
$$\left[\frac{-az_1+a+b_1z_1+b_2z_2-c-z_1+1}{a-c+1}-\frac{(z_1-1)(a-b_1+1)}{(b_1-1)(a-c+1)}\theta_1+\frac{z_2-1}{a-c+1}\theta_2\right]$$
$$\times F_1(a+1,b_1-1,b_2,c;z_1,z_2).$$

The similar procedures are implemented for Appell function F_2, F_3, F_4

F2IndexChange[], **F3IndexChange**[], and **F4IndexChange**[].

Application AppellF1F4



massive q-loop propagator
could be expressed trough the F4 hypergeometrical function.

J^{q}_{023}

$$J^{q}_{023}(M_1^2, M_2^2, \alpha_1, \alpha_2, \sigma_1, \cdots, \sigma_{q-1}) = \frac{[i^{1-n}\pi^{q/2}]^q(-M_1^2)^{\frac{n}{2}q-a_{\alpha_1,\alpha_2,\sigma}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma\left(\frac{n}{2}\right)}\left\{\Pi_{i=1}^{q-1}\frac{\Gamma\left(\frac{n}{2}-\sigma_i\right)}{\Gamma(\sigma_i)}\right\}$$

$$\left[\Gamma\left(\frac{n}{2}-\alpha_2\right)\Gamma\left(a_{\alpha_1,\alpha_2,\sigma}-\frac{n}{2}q\right)\Gamma\left(a_{\alpha_2,\sigma}-\frac{n}{2}(q-1)\right)\right.$$

$$\times F_4\left(a_{\alpha_1,\alpha_2,\sigma}-\frac{n}{2}q, a_{\alpha_2,\sigma}-\frac{n}{2}(q-1), \frac{n}{2}, 1+\alpha_2-\frac{n}{2}\middle|\,\frac{p^2}{M_1^2}, \frac{M_2^2}{M_1^2}\right)$$

$$+\left(\frac{M_2^2}{M_1^2}\right)^{\frac{n}{2}-\alpha_2}\Gamma\left(\alpha_2-\frac{n}{2}\right)\Gamma\left(a_{\alpha_1,\sigma}-\frac{n}{2}(q-1)\right)\Gamma\left(\sigma-\frac{n}{2}(q-2)\right)$$

$$\left.\times F_4\left(a_{\alpha_1,\sigma}-\frac{n}{2}(q-1), \sigma-\frac{n}{2}(q-2), \frac{n}{2}, 1-\alpha_2+\frac{n}{2}\middle|\,\frac{p^2}{M_1^2}, \frac{M_2^2}{M_1^2}\right)\right]. \tag{132}$$

The case of multiple variables

- Functions F_A, F_B, F_C, F_D are the extensions of two variable functions F_1, F_2, F_3, F_4 to the mutivariable case.
- In HyperDire project now is implemented only F_D differential reduction for any number of argument:

```
In[4]:=
answer = FdIndexChange[{-1,{1,-1,0,0,0},0}, {a,{b1,b2,b3,b4,b5},c,{z1,z2,z3,z4,z5}}];
Simplify[answer]
```

$$\text{Out[5]=}\ \left\{\left\{\frac{-1+z1}{-1+z2}, \frac{-1+z1}{(-1+a)\,(-1+z2)}, \frac{a\,(z1-z2)+(-1+b2)\,(-1+z1)\,z2+c\,(-z1+z2)}{(-1+a)\,(-1+b2)\,(-1+z2)\,z2}, \frac{-1+z1}{(-1+a)\,(-1+z2)}, \frac{-1+z1}{(-1+a)\,(-1+z2)}, \frac{-1+z1}{(-1+a)\,(-1+z2)}\right\}, \{-1+a, \{1+b1, -1+b2, b3, b4, b5\}, c, \{z1, z2, z3, z4, z5\}\}\right\}$$

$F_D(a; b1, b2, b3, b4, b5; c; z1, z2, z3, z4, z5)$ is expressed in the terms of the function

$F_D(a-1; b1+1, b2-1, b3, b4, b5; c; z1, z2, z3, z4, z5)$ and its five derivatives thank You for an attention!