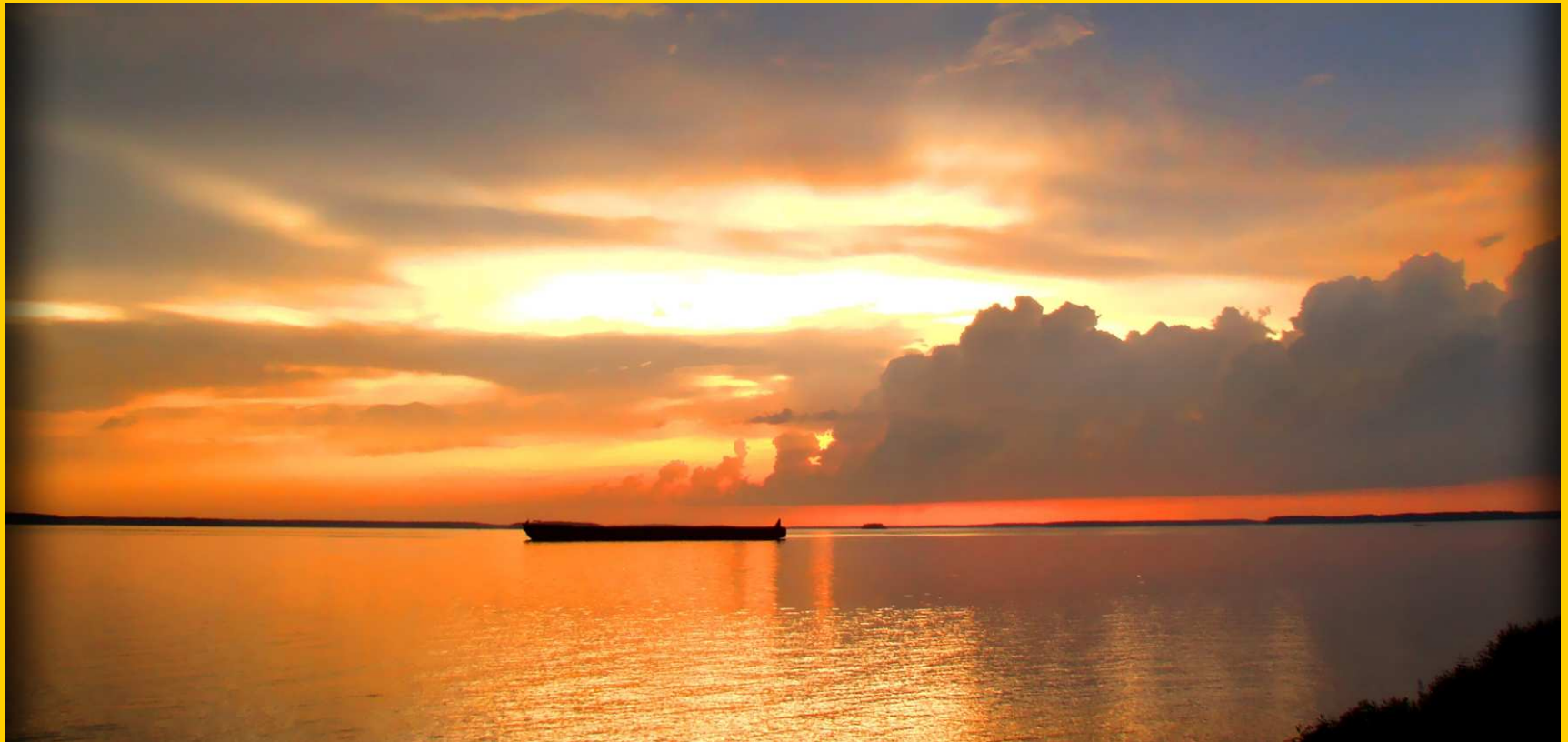

FAPT for Hadronic Observables in QCD: Pion FF, Adler Function, Low- Q^2 Evolution

Alexander P. BAKULEV

Bogoliubov Lab. Theor. Phys., JINR (Dubna, Russia)



OUTLINE

- **Intro**: Asymptotic Series in Perturbative QFT
- **APT and FAPT**
- **Application**: Pion FF $F_\pi(Q^2)$ in FAPT
- **Resummation** in APT and FAPT
 - **One-loop** case
 - **Two- and Three-loop** cases
- **Application**: Adler Function
- **Application**: Higgs Decay $H^0 \rightarrow b\bar{b}$
- **Application**: PDF Evolution at **Low** Q^2
- **Conclusions**

Collaborators & Publications

Collaborators:



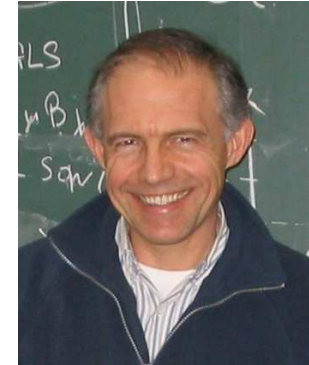
S. V. Mikhailov
JINR (Dubna)



D. V. Shirkov
JINR (Dubna)



I. V. Potapova
JINR (Dubna)



N. G. Stefanis
RUB (Bochum)

Publications:

- **A. B. & Mikhailov — Solovtsov Memorial Seminar, Dubna, Jan. 17–18, 2008, Dubna: JINR (2008) pp. 119–133**
- **A. B. — Phys. Part. Nucl. 40 (2009) 715**
- **A. B., Mikhailov, Stefanis — JHEP 1006 (2010) 085**
- **A. B. & Shirkov — ArXiv:1102.2380[hep-ph]**
- **A. B. & Potapova — NPB (PS) 219–220 (2011) 193–200**

Asymptotic Series in Perturbative QFT

Strength and Weakness of Pert. QFT

A lot of successive pert. calculations in **QM** and **QFT**.
Practically, it is synonym of Quantum Theory.
Feynman diagrams became a symbol of **QFT**.

Nevertheless, power expansion of the quantum amplitude $C(\alpha)$ is not convergent.

Feynman Series $\sum c_k \alpha^k$ is not Convergent !

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Nevertheless, power expansion of the quantum amplitude $C(\alpha)$ is not convergent.

Feynman Series $\sum c_k \alpha^k$ is not Convergent !

Due to

- Essential singularity at $\alpha = 0$
- Factorial growth of coefficients $c_k \sim k!$

Singularity at $g = 0$, factorial growth $c_k \sim k!$

For illustration, take the 0-dim analog $I(g) = \int_{-\infty}^{\infty} e^{-x^2 - gx^4} dx$

Expanding it in power-in- g series:

$$I(g) \sim \sum_{k=0} (-g)^k I_k \quad \text{with} \quad I_k = \frac{\Gamma(2k + 1/2)}{\Gamma(k + 1)} \rightarrow 2^k k!$$

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Meanwhile, $I(g)$ can be expressed via MacDonald function

$$I(g) = \frac{1}{\sqrt{2g}} e^{1/8g} K_{1/4} \left(\frac{1}{8g} \right)$$

with known analytic properties in complex g plane.

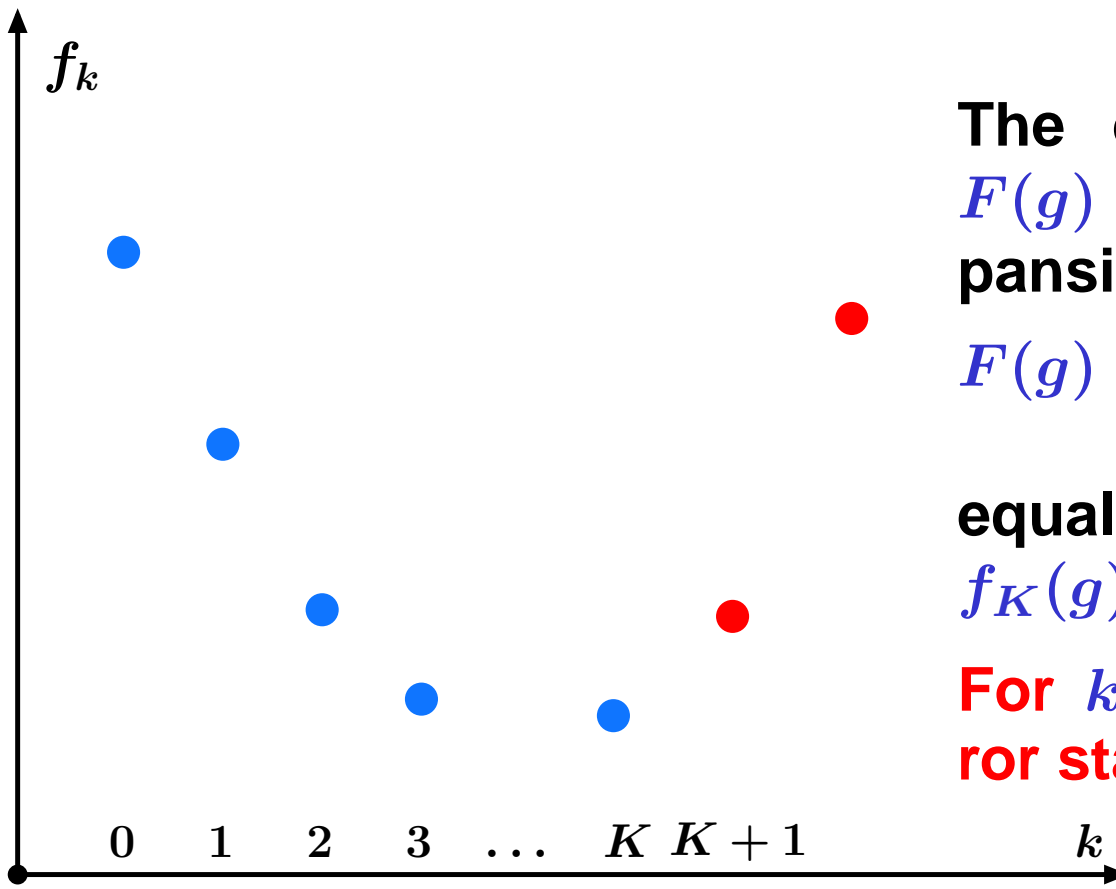
It has an essential singularity $e^{-1/8g}$ near the origin:

$$I(g) = \sqrt{\pi} - \frac{g}{\sqrt{2\pi}} \int_0^{\infty} \frac{d\gamma \exp(-1/4\gamma)}{\gamma(g + \gamma)}$$

Asymptotic Series and 'Practic. Convergence'

The **Henry Poincaré** (end of XIX) analysis of Asymptotic Series (AS) can be summed as follows:

AS can be used for obtaining **quantitative information** on expanded function.



The error of approximating $F(g)$ by first K terms of expansion, $F_K(g)$,

$$F(g) \rightarrow F_K(g) = \sum_{k \leq K} f_k(g) \text{ is}$$

equal to the last retained term $f_K(g)$.

For $k \geq K + 1$ truncation error starts to grow!

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For the power AS, $f_k(g) = f_k g^k$ with factorial growth $f_k \sim k!$

absolute values of $f_k(g)$ cease to diminish at $k \sim 1/g$.

This yields to the natural **best possible accuracy** of a given AS
(in contrast to convergent series!)

Asymptotic Series and 'Practic. Convergence'

$$I(g) = \int_{-\infty}^{\infty} e^{-x^2 - gx^4} dx \quad ? = ? \quad \sum_{k \geq 0} I_k (-g)^k$$

g	K	$(-g)^K I_K$	$(-g)^{K+1} I_{K+1}$	$\Delta_K I(g)$
0.07	5	-0.016(0.5%)	+0.022(0.8%)	0.5%
0.07	7	-0.38(1.4%)	+0.074(2.9%)	1.4%

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Thus, $K_*(g = 0.07) = 5$ and $K_*(g = 0.15) = 2$.

Not possible to get the 1% accuracy at $g = 0.15$ for $I(g)$.

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What to do in QCD?

APT approach delivers a solution!

Analytic Perturbation Theory in QCD

Analytic Perturb Theory (APT): Preamble

1st step: Improving **PT** by **RG** Method (**Bogoliubov–Shirkov [1955-56]**). In QFT, **RG** result obeys unphysical singularity.

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(**Bogoliubov–Logunov–Shirkov [1959]**, **Radyushkin and Krasnikov&Pivovarov [1982]**).

Its minimal (without extra parameters) version was devised by **Jones&Solovtsov&Shirkov [1996–2006]** and is known as **Analytic Perturbation Theory**.

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3rd step: Generalizing **APT** by including **fractional** powers of coupling and its products with **logarithms** due to principle of **analytization “as a whole”** (**Karanikas–Stefanis, [2001]**) in (**A. B.&Mikhailov&Stefanis [2005–2009]**) \Rightarrow **Fractional APT**.

Quite **new** improvement of **FAPT** in **Cvetic&Kotikov [arXiv:1106.4275]**.

Basics of pQCD

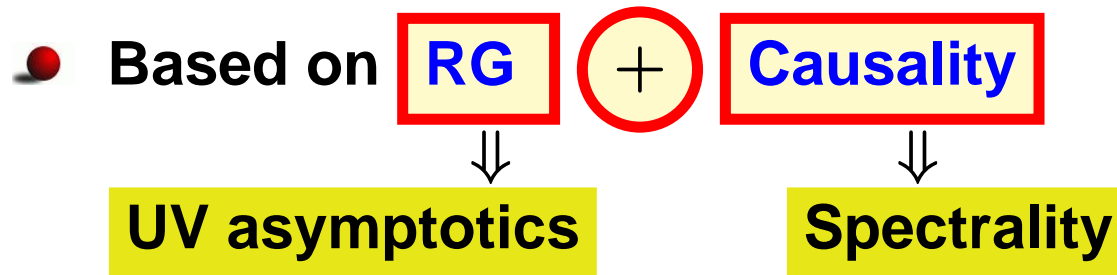
- coupling $\alpha_s(\mu^2) = (4\pi/b_0) a_s[L]$ with $L = \ln(\mu^2/\Lambda^2)$
- RG equation $\frac{d a_s[L]}{d L} = -a_s^2 - c_1 a_s^3 - \dots$
- 1-loop solution generates Landau pole singularity:
 $a_s[L] = 1/L$
- 2-loop solution generates square-root singularity:
 $a_s[L] \sim 1/\sqrt{L + c_1 \ln c_1}$
- PT series: $D[L] = 1 + d_1 a_s[L] + d_2 a_s^2[L] + \dots$

Basics of APT

- Different effective couplings in **Minkowskian (R&K&P[1982])** and **Euclidean (S&S[1996])** regions.

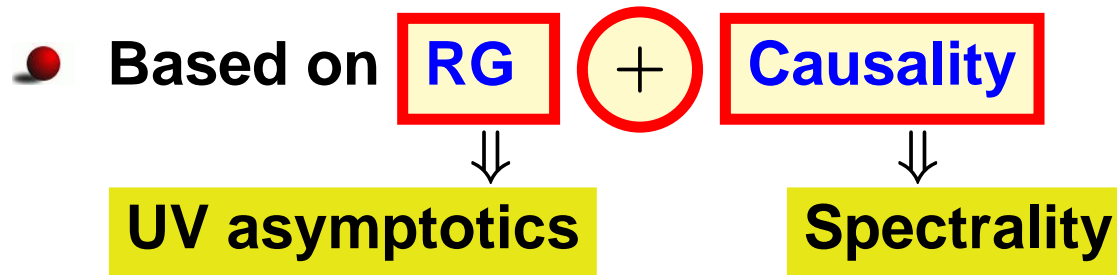
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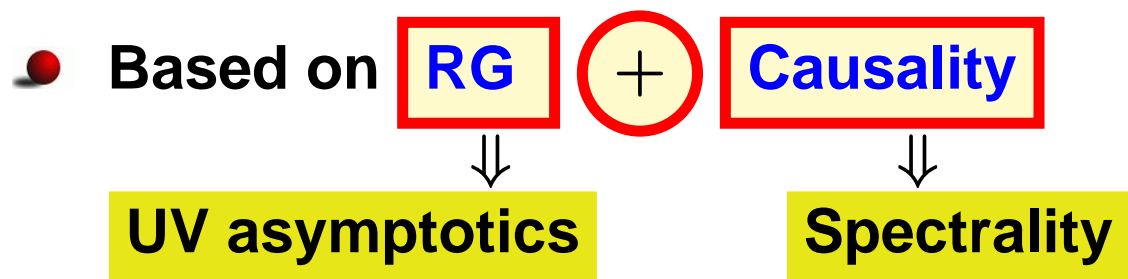
- Different effective couplings in **Minkowskian (R&K&P[1982])** and **Euclidean (S&S[1996])** regions.



- **Euclidean:** $-q^2 = Q^2$, $L = \ln Q^2 / \Lambda^2$, $\{\mathcal{A}_n(L)\}_{n \in \mathbb{N}}$
- **Minkowskian:** $q^2 = s$, $L_s = \ln s / \Lambda^2$, $\{\mathfrak{A}_n(L_s)\}_{n \in \mathbb{N}}$

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- **PT** $\sum_m d_m a_s^m(Q^2)$ \Rightarrow $\sum_m d_m \mathcal{A}_m(Q^2)$ **APT**
 m is power \Rightarrow m is index

Spectral representation

By **analytization** we mean “Källén–Lehmann” representation

$$[f(Q^2)]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon} d\sigma$$

Then (note here **pole remover**):

$$\begin{aligned}\rho(\sigma) &= \frac{1}{L_\sigma^2 + \pi^2} \\ \mathcal{A}_1[L] &= \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2} d\sigma = \frac{1}{L} - \frac{1}{e^L - 1} \\ \mathfrak{A}_1[L_s] &= \int_s^\infty \frac{\rho(\sigma)}{\sigma} d\sigma = \frac{1}{\pi} \arccos \frac{L_s}{\sqrt{\pi^2 + L_s^2}}\end{aligned}$$

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with spectral density $\rho_f(\sigma) = \text{Im} [f(-\sigma)] / \pi$. Then:

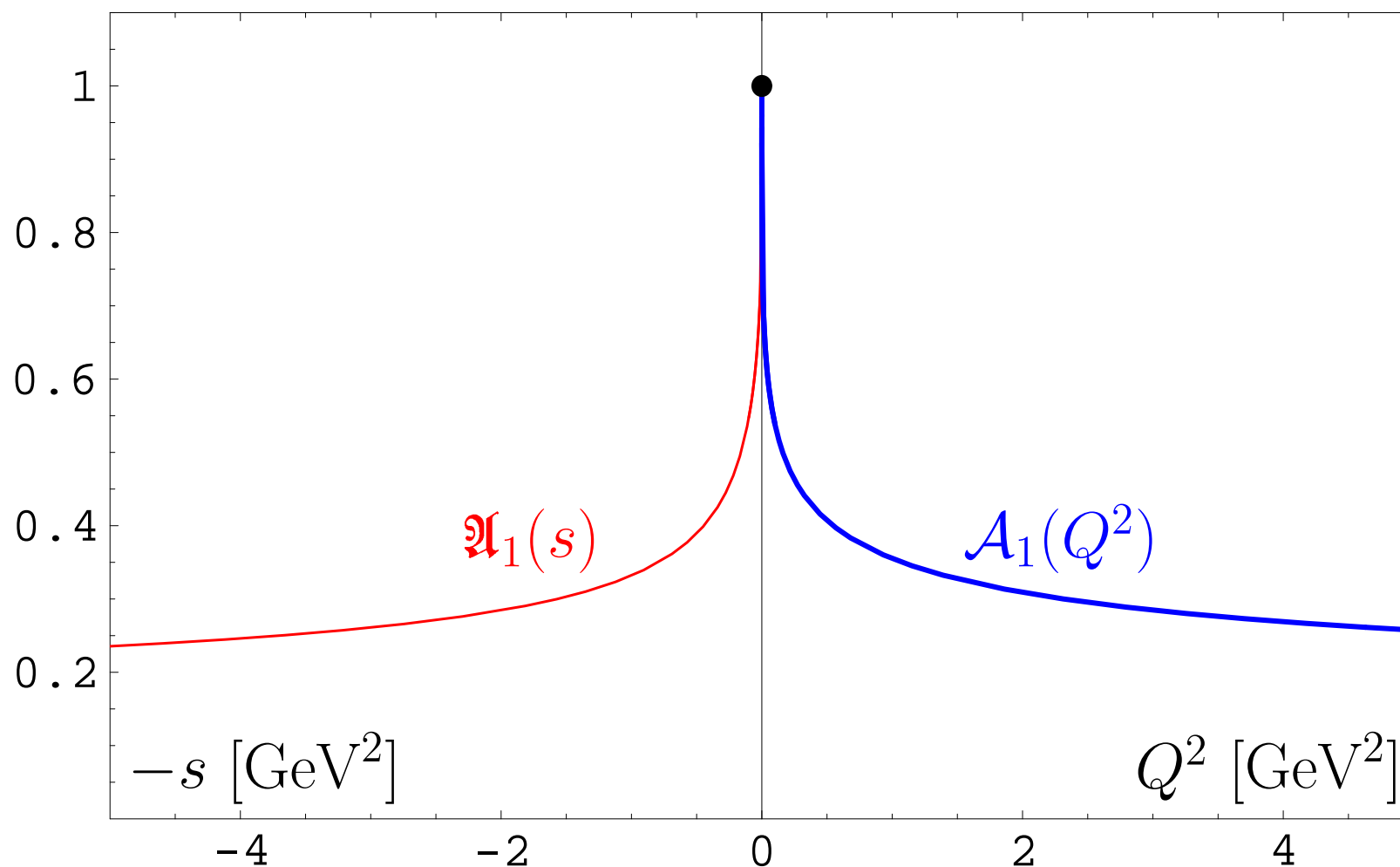
$$\mathcal{A}_n[L] = \int_0^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2} d\sigma = \frac{1}{(n-1)!} \left(-\frac{d}{dL} \right)^{n-1} \mathcal{A}_1[L]$$

$$\mathfrak{A}_n[L_s] = \int_s^\infty \frac{\rho_n(\sigma)}{\sigma} d\sigma = \frac{1}{(n-1)!} \left(-\frac{d}{dL_s} \right)^{n-1} \mathfrak{A}_1[L_s]$$

$$a_s^n[L] = \frac{1}{(n-1)!} \left(-\frac{d}{dL} \right)^{n-1} a_s[L]$$

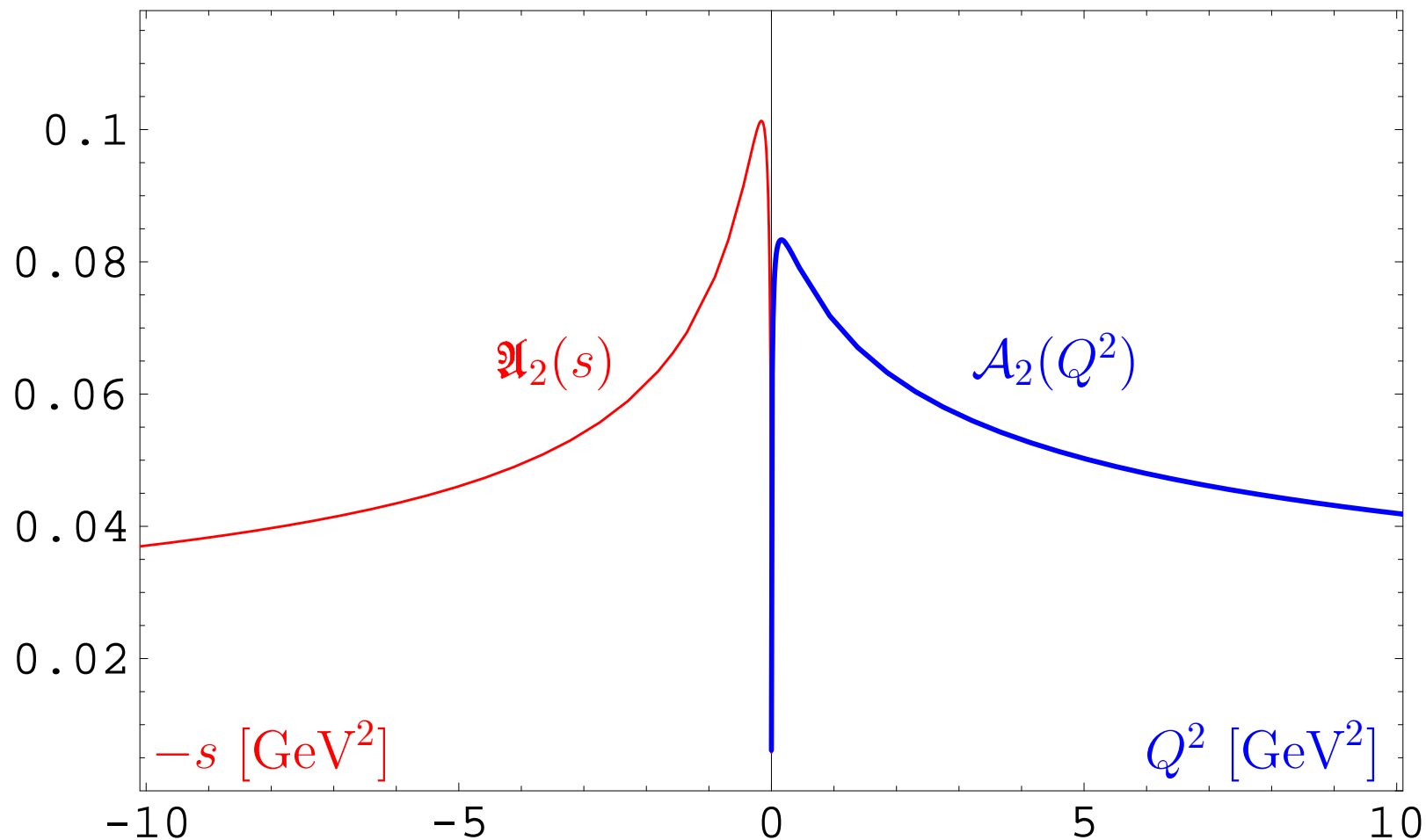
APT graphics: Distorting mirror

First, couplings: $\mathfrak{A}_1(s)$ and $\mathcal{A}_1(Q^2)$



APT graphics: Distorting mirror

Second, square-images: $\mathfrak{A}_2(s)$ and $\mathcal{A}_2(Q^2)$



**Need
to use
Fractional APT**

Problems of APT

In standard QCD PT we have not only power series

$$F[L] = \sum_m f_m a_s^m[L], \text{ but also:}$$

● **Factorization** $\rightarrow (a_s[L])^n L^m$

\Rightarrow **analytization “as a whole”** Karanikas&Stefanis [2001]

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- **RG-improvement to account for higher-orders** \rightarrow

$$Z[L] = \exp \left\{ \int^{a_s[L]} \frac{\gamma(a)}{\beta(a)} da \right\} \xrightarrow{1\text{-loop}} [a_s[L]]^{\gamma_0/(2\beta_0)}$$

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- **Two-loop case** $\rightarrow (a_s)^\nu \ln(a_s)$

New functions: $(a_s)^\nu$, $(a_s)^\nu \ln(a_s)$, $(a_s)^\nu L^m$, ...

Constructing one-loop **FAPT**

In one-loop **APT** we have a very nice recurrence relation

$$\mathcal{A}_n[L] = \frac{1}{(n-1)!} \left(-\frac{d}{dL} \right)^{n-1} \mathcal{A}_1[L]$$

and the same in Minkowski domain

$$\mathfrak{A}_n[L] = \frac{1}{(n-1)!} \left(-\frac{d}{dL} \right)^{n-1} \mathfrak{A}_1[L].$$

We can use it to construct **FAPT**.

FAPT(E): Properties of $\mathcal{A}_\nu[L]$

First, Euclidean coupling ($L = L(Q^2)$):

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1 - \nu)}{\Gamma(\nu)}$$

Here $F(z, \nu)$ is reduced **Lerch** transcendent. function. It is analytic function in ν .

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- $\mathcal{A}_0[L] = 1$;
- $\mathcal{A}_{-m}[L] = L^m$ for $m \in \mathbb{N}$;
- $\mathcal{A}_m[L] = (-1)^m \mathcal{A}_m[-L]$ for $m \geq 2$, $m \in \mathbb{N}$;
- $\mathcal{A}_m[\pm\infty] = 0$ for $m \geq 2$, $m \in \mathbb{N}$;

FAPT(M): Properties of $\mathfrak{A}_\nu[L]$

Now, Minkowskian coupling ($L = L(s)$):

$$\mathfrak{A}_\nu[L] = \frac{\sin \left[(\nu - 1) \arccos \left(L / \sqrt{\pi^2 + L^2} \right) \right]}{\pi (\nu - 1) (\pi^2 + L^2)^{(\nu-1)/2}}$$

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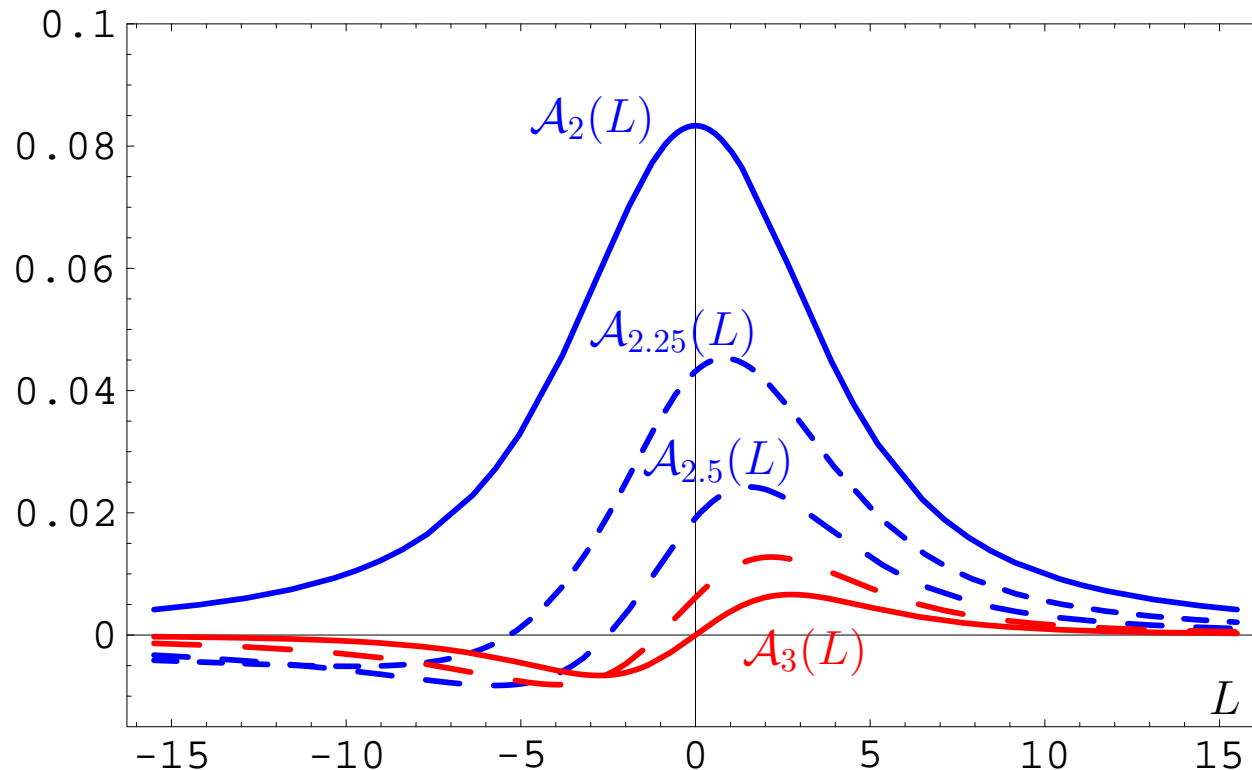
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- $\mathfrak{A}_m[L] = (-1)^m \mathfrak{A}_m[-L]$ for $m \geq 2$, $m \in \mathbb{N}$;
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FAPT(E): Graphics of $\mathcal{A}_\nu[L]$ vs. L

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1 - \nu)}{\Gamma(\nu)}$$

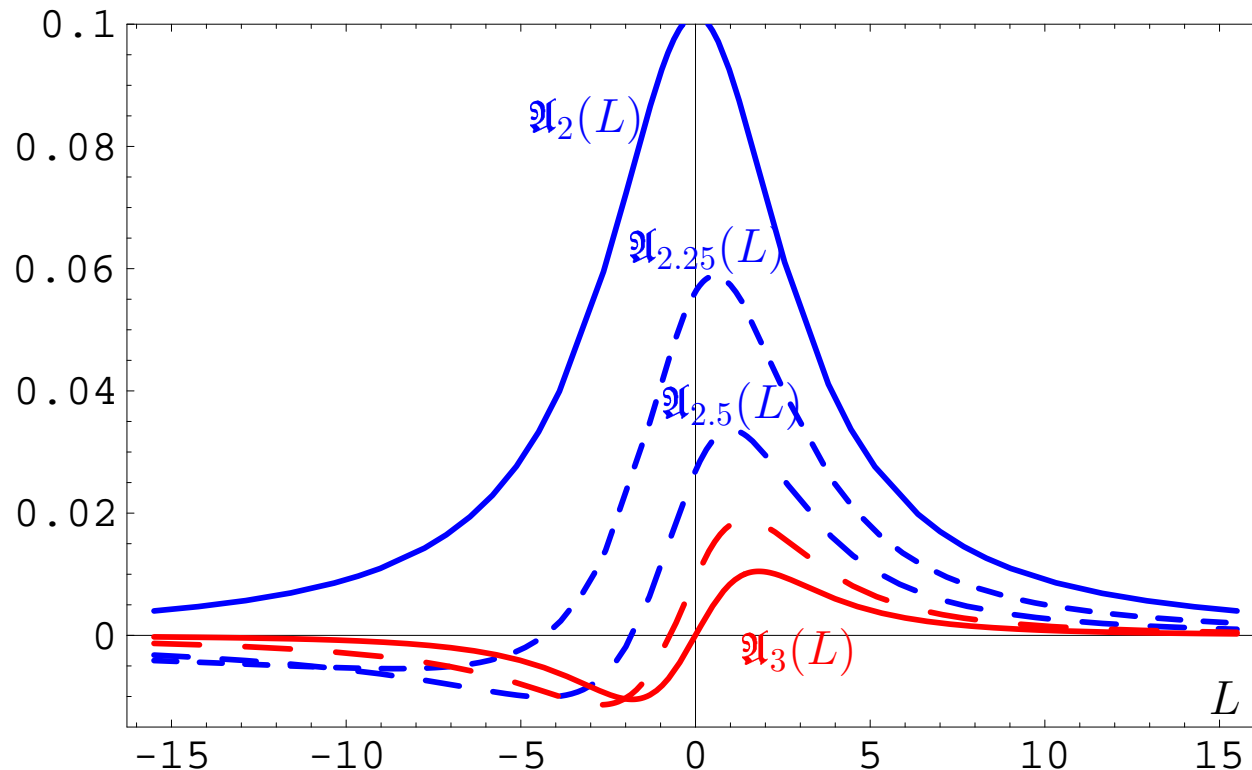
Graphics for fractional $\nu \in [2, 3]$:



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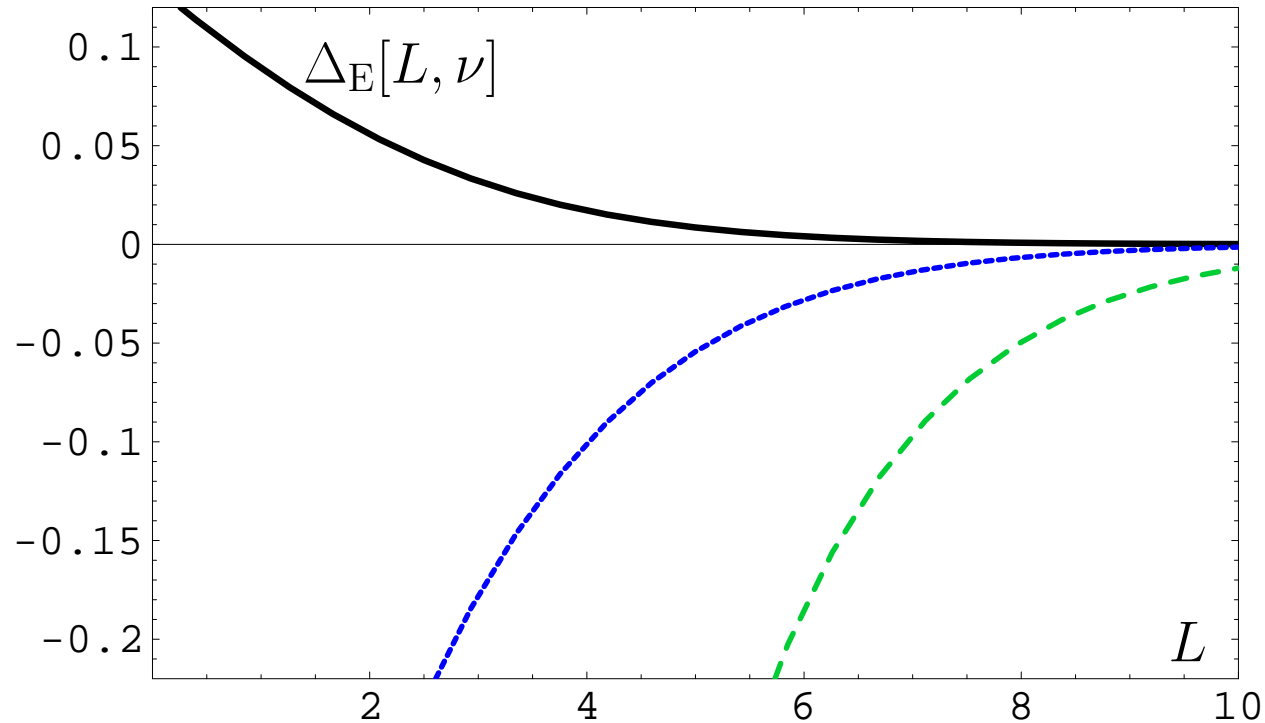
Compare with graphics in Minkowskian region :



FAPT(E): Comparing \mathcal{A}_ν with $(\mathcal{A}_1)^\nu$

$$\Delta_E(L, \nu) = \frac{\mathcal{A}_\nu[L] - (\mathcal{A}_1[L])^\nu}{\mathcal{A}_\nu[L]}$$

Graphics for fractional $\nu = 0.62, 1.62$ and 2.62 :

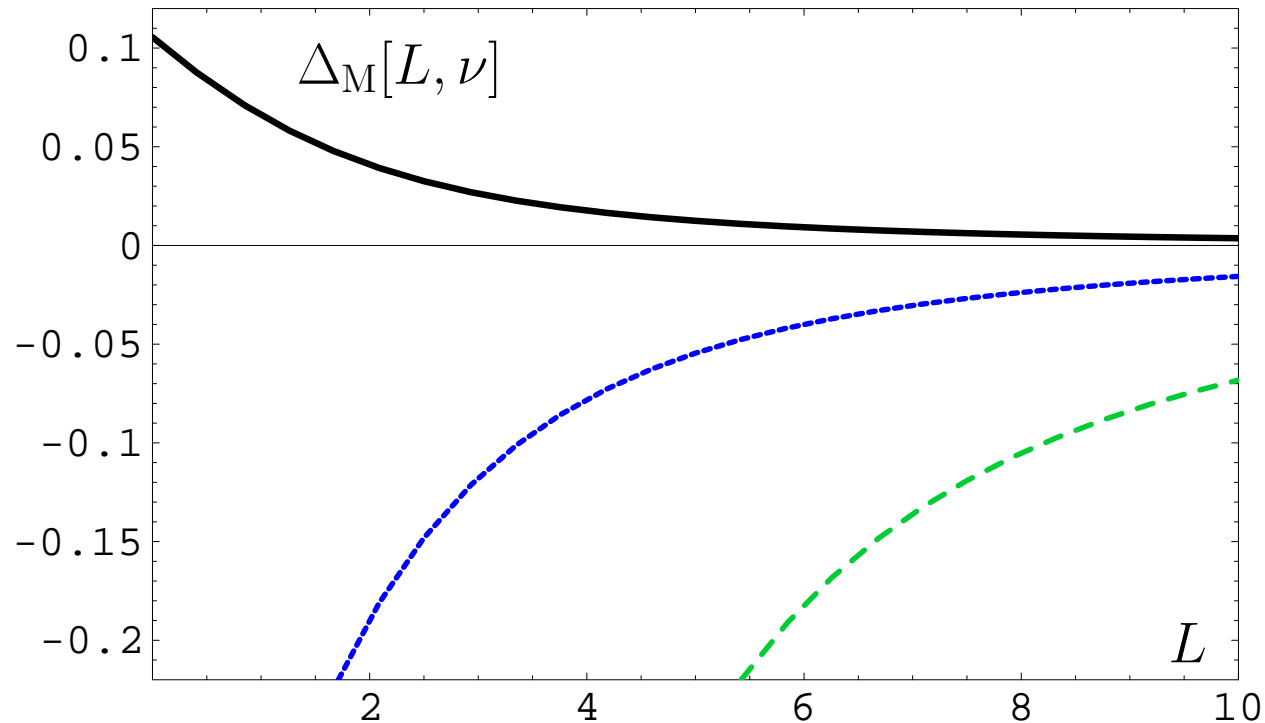


The larger ν is — the more important FAPT becomes!

FAPT(M): Comparing α_ν with $(\alpha_1)^\nu$

$$\Delta_M(L, \nu) = \frac{\alpha_\nu[L] - (\alpha_1[L])^\nu}{\alpha_\nu[L]}$$

Minkowskian graphics for $\nu = 0.62$, **1.62** and **2.62**:



The larger ν is — the more important FAPT becomes!

Application:

Pion FF in FAPT

Factorizable part of pion FF at NLO

Scaled hard-scattering amplitude truncated at NLO and evaluated at renormalization scale $\mu_R^2 = \lambda_R Q^2$ reads

$$\begin{aligned} Q^2 T_H^{\text{NLO}}(x, y, Q^2; \mu_F^2, \lambda_R Q^2) &= \alpha_s(\lambda_R Q^2) t_H^{(0)}(x, y) \\ &+ \frac{\alpha_s^2(\lambda_R Q^2)}{4\pi} C_F t_{H,2}^{(1,F)}\left(x, y; \frac{\mu_F^2}{Q^2}\right) \\ &+ \frac{\alpha_s^2(\lambda_R Q^2)}{4\pi} \left\{ b_0 t_H^{(1,\beta)}(x, y; \lambda_R) + t_H^{(\text{FG})}(x, y) \right\} \end{aligned}$$

with shorthand notation

$$t_{H,2}^{(1,F)}\left(x, y; \frac{\mu_F^2}{Q^2}\right) = t_H^{(0)}(x, y) \left[2 \left(3 + \ln(\bar{x}\bar{y}) \right) \ln \frac{Q^2}{\mu_F^2} \right]$$

Pion Distribution Amplitude

Leading twist 2 pion DA at normalization scale $\mu_0^2 \approx 1 \text{ GeV}^2$ given by

$$\varphi_\pi(x, \mu_0^2) = 6x(1-x) \left[1 + a_2(\mu_0^2) C_2^{3/2}(2x-1) + a_4(\mu_0^2) C_4^{3/2}(2x-1) + \dots \right]$$

All **nonperturbative** information encapsulated in Gegenbauer coefficients $a_n(\mu_0^2)$.

To obtain factorized part of pion FF \Rightarrow convolute pion DA with hard-scattering amplitude:

$$F_\pi^{\text{Fact}}(Q^2) = \varphi_\pi(x, \mu_0^2) \otimes_x T_H^{\text{NLO}}(x, y, Q^2; \mu_F^2, \lambda_R Q^2) \otimes_y \varphi_\pi(y, \mu_0^2)$$

Analyticity of Pion FF at NLO

Naive “analytization” [Stefanis, Schroers, Kim – PLB 449 (1999) 299; EPJC 18 (2000) 137]

$$\begin{aligned} & [Q^2 T_H(x, y, Q^2; \mu_F^2, \lambda_R Q^2)]_{\text{Nai-An}} = \\ & \mathcal{A}_1^{(2)}(\lambda_R Q^2) t_H^{(0)}(x, y) + \frac{\left(\mathcal{A}_1^{(2)}(\lambda_R Q^2)\right)^2}{4\pi} t_H^{(1)}\left(x, y; \lambda_R, \frac{\mu_F^2}{Q^2}\right) \end{aligned}$$

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Naive “analytization” [Stefanis, Schroers, Kim – PLB 449 (1999) 299; EPJC 18 (2000) 137]

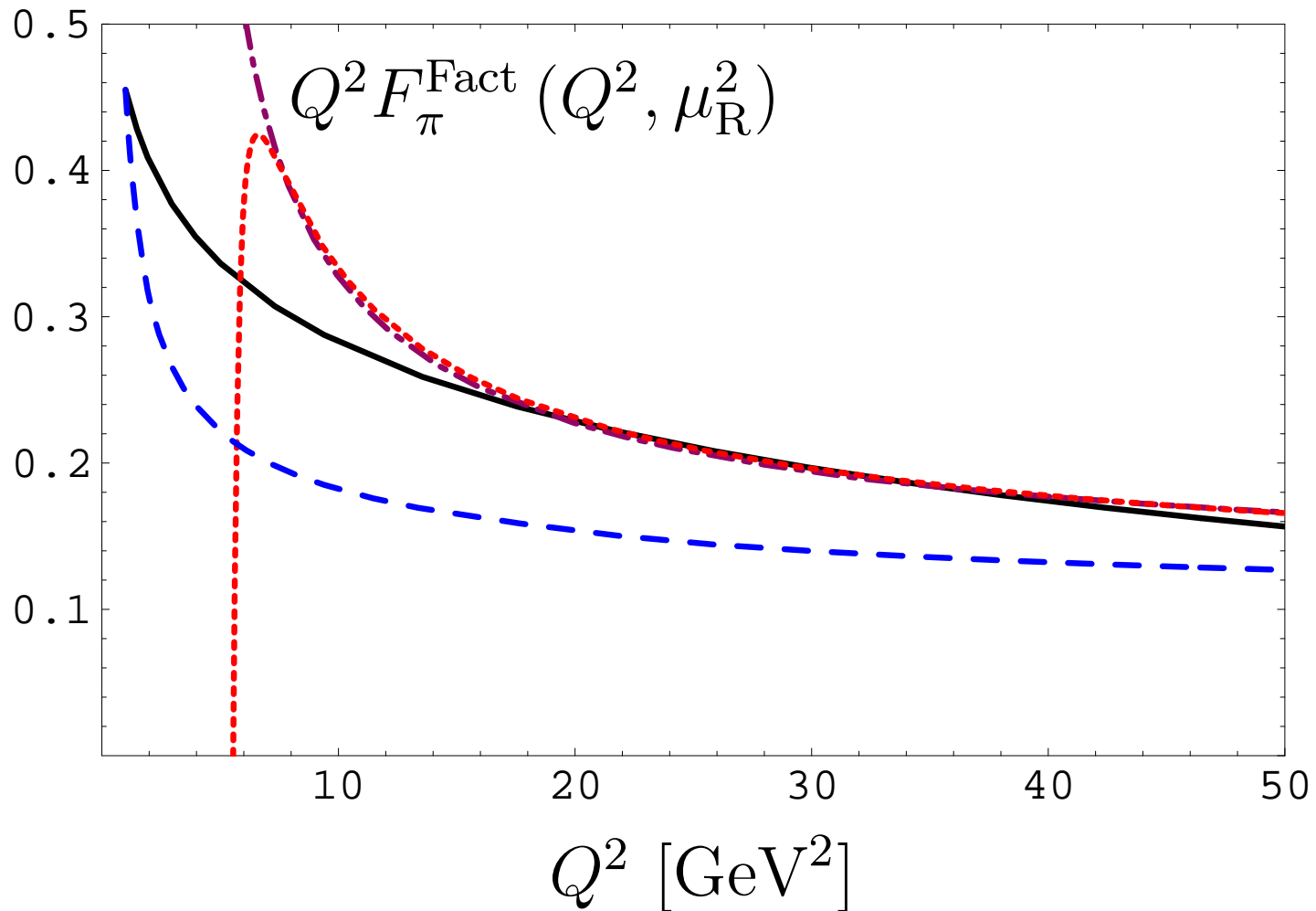
$$\begin{aligned} & [Q^2 T_H(x, y, Q^2; \mu_F^2, \lambda_R Q^2)]_{\text{Nai-An}} = \\ & \mathcal{A}_1^{(2)}(\lambda_R Q^2) t_H^{(0)}(x, y) + \frac{\left(\mathcal{A}_1^{(2)}(\lambda_R Q^2)\right)^2}{4\pi} t_H^{(1)}\left(x, y; \lambda_R, \frac{\mu_F^2}{Q^2}\right) \end{aligned}$$

Maximal “analytization” [A. B., Passek, Schroers, Stefanis – PRD 70 (2004) 033014]

$$\begin{aligned} & [Q^2 T_H(x, y, Q^2; \mu_F^2, \lambda_R Q^2)]_{\text{Max-An}} = \\ & \mathcal{A}_1^{(2)}(\lambda_R Q^2) t_H^{(0)}(x, y) + \frac{\mathcal{A}_2^{(2)}(\lambda_R Q^2)}{4\pi} t_H^{(1)}\left(x, y; \lambda_R, \frac{\mu_F^2}{Q^2}\right) \end{aligned}$$

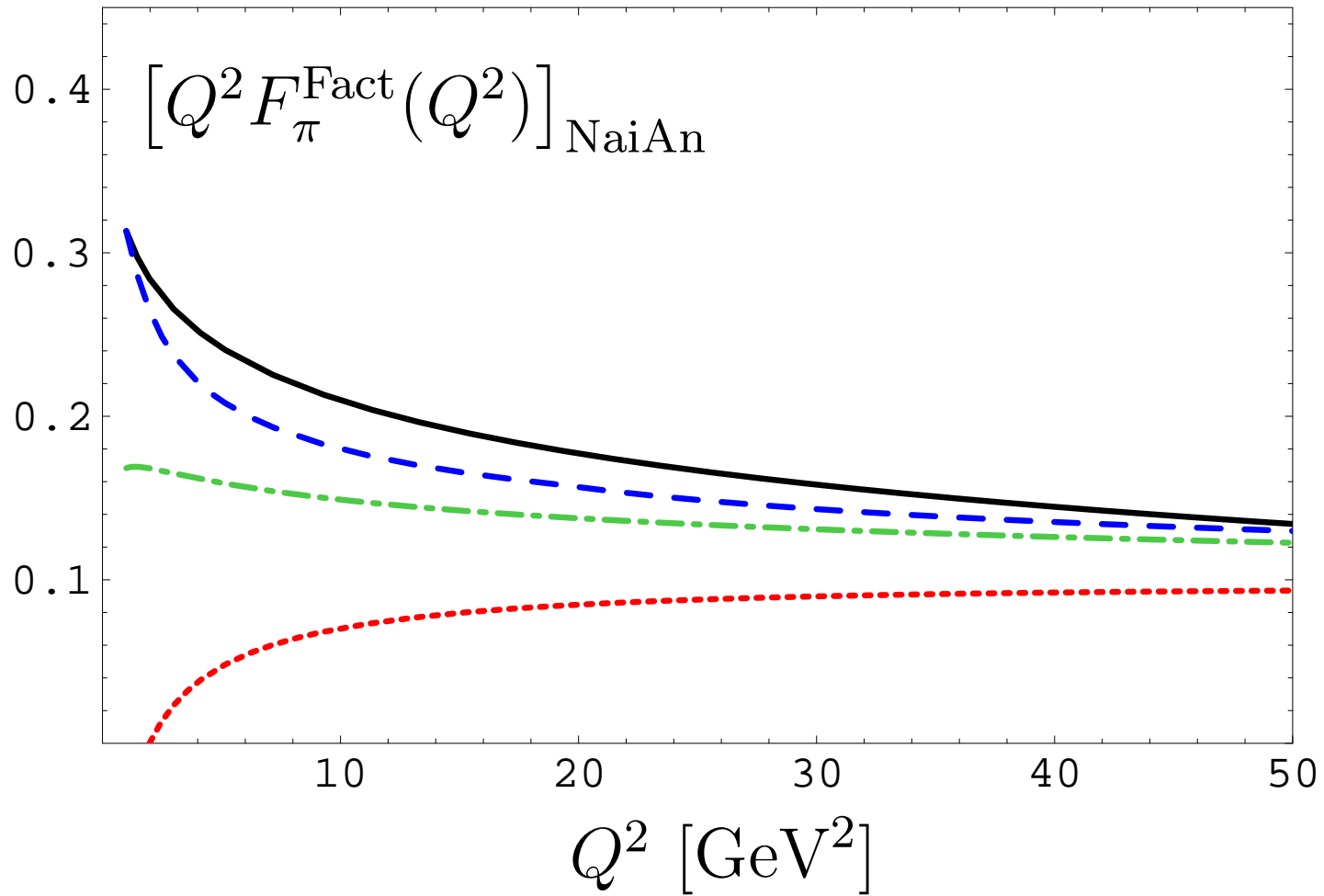
Factorized Pion FF in Standard \overline{MS} scheme

— \overline{BLM} , - - - default, PMS, - · - · - FAC



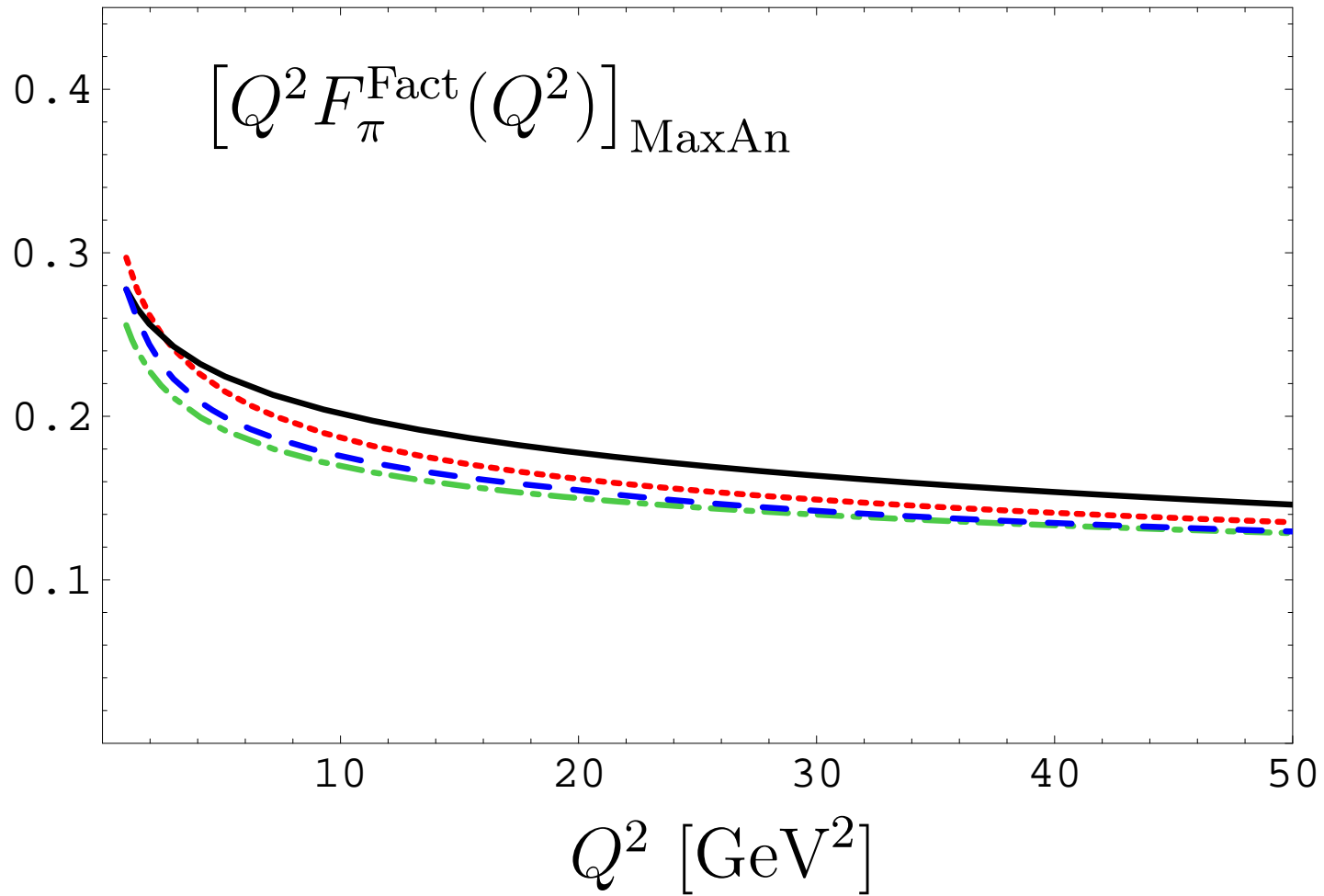
Factorized Pion FF in Naive Analyticization

— $\overline{\text{BLM}}$, - - - default, BLM, - · - · α_V

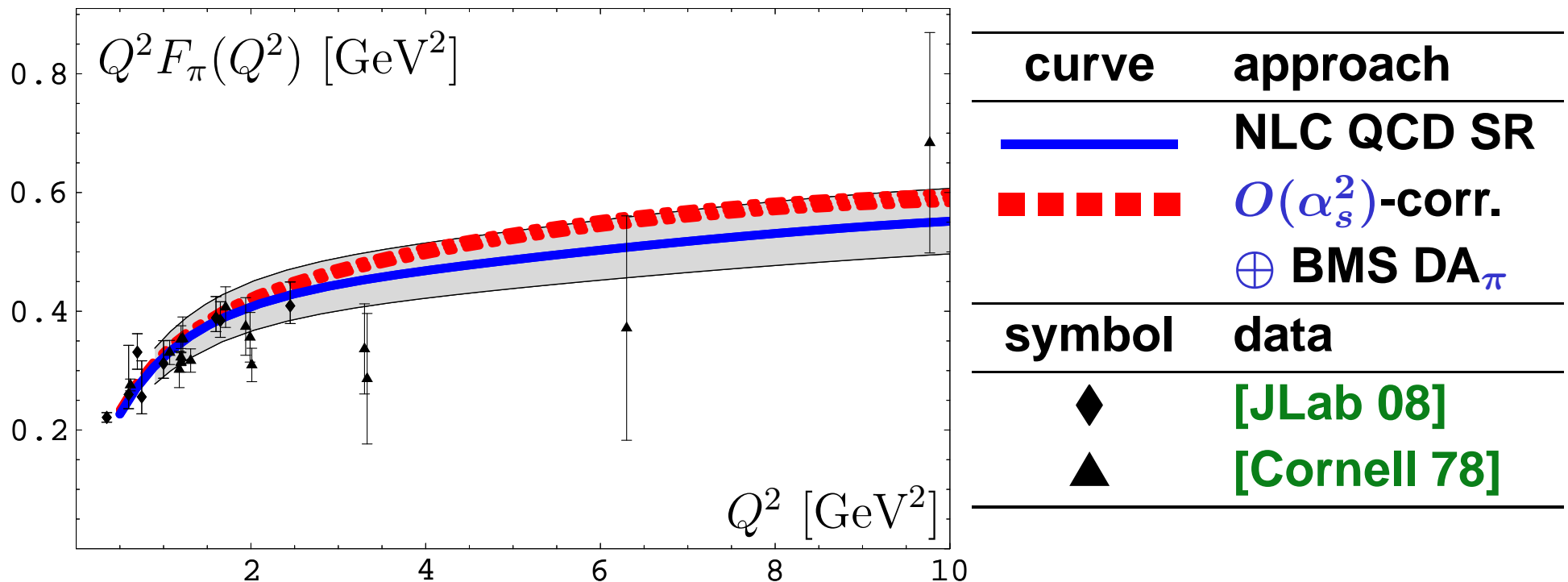


Factorized Pion FF in Max. Analyticization

— $\overline{\text{BLM}}$, - - - default, BLM, - - - α_V

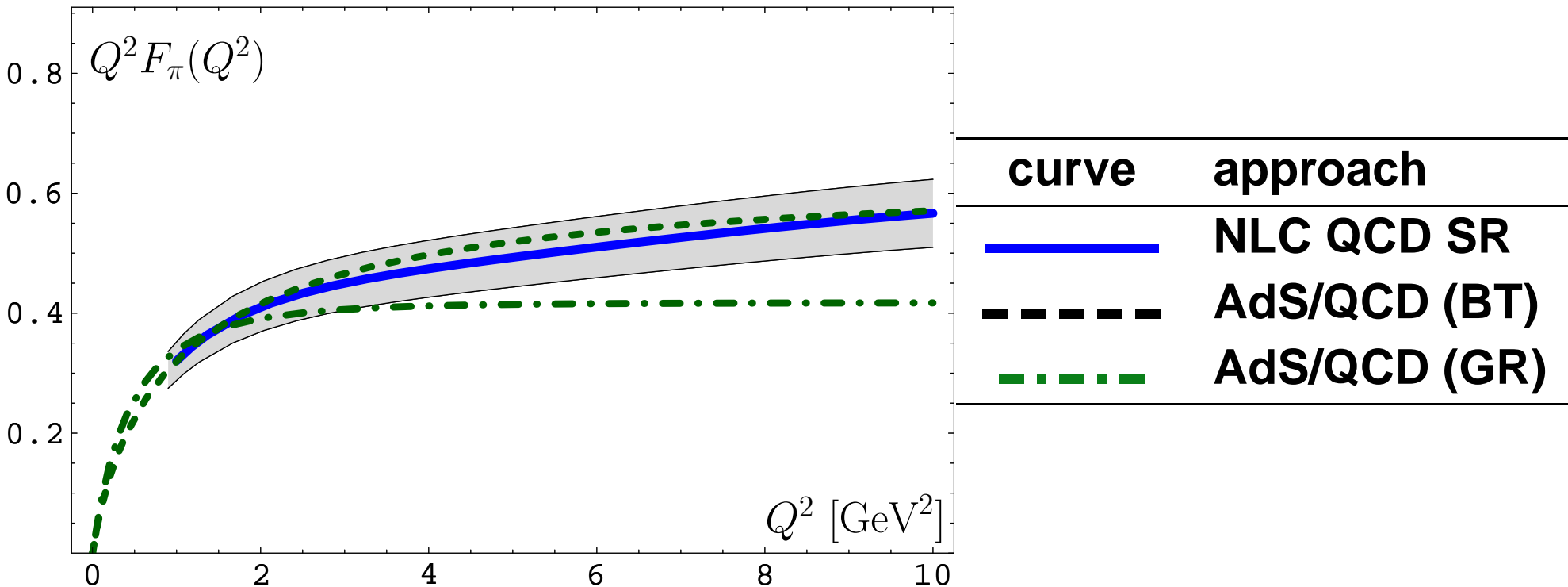


NNLO Correction Estimation using FAPT



Pion FF from SRs with NLC (blue solid line)
 and with NNLO correction using FAPT:
 NNLO correction is of the order of 3 – 10%.

Comparing with AdS/QCD



Pion FF from QCD SRs with NLC (blue solid line)
in comparison with AdS/QCD:
green dashed line = [Brodsky&de Teramond 07] and
green dot-dashed line = [Grigoriyan&Radyushkin 07].

Resummation in one-loop APT and FAPT

Generating function for coefficients

Consider series $\mathcal{D}[L] = d_0 + \sum_{n=1}^{\infty} d_n \mathcal{A}_n[L]$

Generating function for coefficients

Consider series $\mathcal{D}[L] = d_0 + \sum_{n=1}^{\infty} d_n \mathcal{A}_n[L]$

Let exists the generating function $P(t)$ for coefficients:

$$d_n = d_1 \int_0^{\infty} P(t) t^{n-1} dt \quad \text{with} \quad \int_0^{\infty} P(t) dt = 1.$$

We define a shorthand notation

$$\langle\langle f(t) \rangle\rangle_{P(t)} \equiv \int_0^{\infty} f(t) P(t) dt.$$

Then coefficients $d_n = d_1 \langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

Generating function for coefficients

Consider series $\mathcal{D}[L] = d_0 + d_1 \sum_{n=1}^{\infty} \langle \langle t^{n-1} \rangle \rangle_{P(t)} \mathcal{A}_n[L]$

We have one-loop recurrence relation:

$$\mathcal{A}_{n+1}[L] = \frac{1}{\Gamma(n+1)} \left(-\frac{d}{dL} \right)^n \mathcal{A}_1[L].$$

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Result:

$$\mathcal{D}[L] = d_0 + d_1 \langle \langle \mathcal{A}_1[L - t] \rangle \rangle_{P(t)}$$

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Result:

$$\mathcal{D}[L] = d_0 + d_1 \langle \langle \mathcal{A}_1[L - t] \rangle \rangle_{P(t)}$$

and for Minkowski region:

$$\mathcal{R}[L] = d_0 + d_1 \langle \langle \mathcal{A}_1[L - t] \rangle \rangle_{P(t)}$$

Models for perturbative coefficients

Coefficients d_n of the PT series:

Model $P(t)$	d_n
$c\delta(t - c)$	c^n

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Model $P(t)$	d_n
$c \delta(t - c)$	c^n
$\theta(t < 1)$	$\frac{1}{n}$
$(t/c)^{\gamma+1} e^{-t/c}$	$n^\gamma c^n \Gamma(n + 1)$

Resummation in one-loop FAPT

Consider series $\mathcal{R}_\nu[L] = d_0 \mathfrak{A}_\nu[L] + \sum_{n=1}^{\infty} d_n \mathfrak{A}_{n+\nu}[L]$

or $\mathcal{D}_\nu[L] = d_0 \mathcal{A}_\nu[L] + \sum_{n=1}^{\infty} d_n \mathcal{A}_{n+\nu}[L]$

with coefficients $d_n = d_1 \langle \langle t^{n-1} \rangle \rangle_{P(t)}$.

Result:

$$\mathcal{R}_\nu[L] = d_0 \mathfrak{A}_\nu[L] + d_1 \langle \langle \mathfrak{A}_{1+\nu}[L - t] \rangle \rangle_{P_\nu(t)} ;$$

$$\mathcal{D}_\nu[L] = d_0 \mathcal{A}_\nu[L] + d_1 \langle \langle \mathcal{A}_{1+\nu}[L - t] \rangle \rangle_{P_\nu(t)} .$$

where $P_\nu(t) = \int_0^1 P \left(\frac{t}{1-z} \right) \nu z^{\nu-1} \frac{dz}{1-z} .$

Resummation in two- and three-loop FAPT

Resummation in two-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_\nu[L] = \mathcal{A}_\nu^{(2)}[L]$ or $\mathfrak{A}_\nu^{(2)}[L]$ (or $\rho_\nu^{(2)}[L]$ — for global).

Resummation in two-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_\nu[L] = \mathcal{A}_\nu^{(2)}[L]$ or $\mathcal{R}_\nu^{(2)}[L]$ (or $\rho_\nu^{(2)}[L]$ — for global).

We have two-loop recurrence relation ($c_1 = b_1/b_0^2$):

$$-\frac{1}{n+\nu} \frac{d}{dL} \mathcal{F}_{n+\nu}[L] = \mathcal{F}_{n+1+\nu}[L] + c_1 \mathcal{F}_{n+2+\nu}[L].$$

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In order to resum our series we need to define the two-loop

time $\tau_2(t) = t - c_1 \ln \left[1 + \frac{t}{c_1} \right]$ with $\frac{d\tau_2(t)}{dt} = \frac{1}{1 + c_1/t}$

to be compared with standard two-loop evolution time $\tau_{(2)}(t)$

with $\frac{dt}{d\tau_{(2)}(t)} = \frac{1}{1 + c_1/\tau_{(2)}(t)}$

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Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{F}_{n+\nu}[L]$.

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$$-\frac{1}{n+\nu} \frac{d}{dL} \mathcal{F}_{n+\nu}[L] = \mathcal{F}_{n+1+\nu}[L] + c_1 \mathcal{F}_{n+2+\nu}[L].$$

Result (with $\tau_2(t) = t - c_1 \ln(1 + t/c_1)$):

$$\begin{aligned} \mathcal{S}[L] = & \left\langle\left\langle \mathcal{F}_{1+\nu}[L] - \frac{t^2}{c_1 + t} \int_0^1 z^\nu dz \dot{\mathcal{F}}_{1+\nu}[L + \tau_2(tz) - \tau_2(t)] \right. \right. \\ & \left. \left. + \frac{c_1 t}{c_1 + t} \left\{ \mathcal{F}_{2+\nu}[L] - \int_0^1 dz \frac{t^2 z^{\nu+1}}{c_1 + tz} \dot{\mathcal{F}}_{2+\nu}[L + \tau_2(tz) - \tau_2(t)] \right\} \right\rangle\right\rangle_{P(t)} \end{aligned}$$

Resummation in three-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{F}_{n+\nu}[L]$.

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Resummation in three-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_\nu[L] = \mathcal{A}_\nu^{(2)}[L]$ or $\mathcal{R}_\nu^{(2)}[L]$ (or $\rho_\nu^{(2)}[L]$ — for global).

We have three-loop recurrence relation ($c_2 = b_2/b_0^3$):

$$\frac{-d\mathcal{F}_{n+\nu}[L]}{(n+\nu)dL} = \mathcal{F}_{n+1+\nu}[L] + c_1 \mathcal{F}_{n+2+\nu}[L] + c_2 \mathcal{F}_{n+3+\nu}[L].$$

Resummation in three-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{F}_{n+\nu}[L]$.

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Now, to resum our series, we need to define the three-loop

time $\tau_3(t)$ with $\frac{d\tau_3(t)}{dt} = \frac{1}{1 + (c_1/t) + c_2/t^2}$

to be compared with standard three-loop evolution time $\tau_{(3)}(t)$

with $\frac{dt}{d\tau_{(3)}(t)} = \frac{1}{1 + (c_1/\tau_{(3)}(t)) + c_2/\tau_{(3)}(t)^2}$

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Result ($L_{z,t} \equiv L + \tau_3(tz) - \tau_3(t)$):

$$\begin{aligned} \mathcal{S}[L] = & \left\langle\left\langle \mathcal{F}_{1+\nu}[L] + t \mathcal{F}_{2+\nu}[L] - \frac{t^2}{t^2 + c_1 t + c_2} \int_0^1 z^\nu dz \left\{ t \dot{\mathcal{F}}_{1+\nu}[L_{z,t}] \right. \right. \right. \\ & \left. \left. \left. + z t^2 \dot{\mathcal{F}}_{2+\nu}[Lz, t] + (\nu + 1) t \mathcal{F}_{2+\nu}[Lz, t] - \frac{c_2 \nu}{z} \mathcal{F}_{3+\nu}[Lz, t] \right\} \right\rangle\right\rangle_{P(t)} \end{aligned}$$

Resummation for Adler function $D(Q^2)$

Adler function $D(Q^2)$ in vector channel

Adler function $D(Q^2)$ can be expressed in QCD by means of the correlator of quark vector currents

$$\Pi_V(Q^2) = \frac{(4\pi)^2}{3q^2} i \int dx e^{iqx} \langle 0 | T[J_\mu(x) J^\mu(0)] | 0 \rangle$$

in terms of discontinuity of its imaginary part

$$R_V(s) = \frac{1}{\pi} \text{Im} \Pi_V(-s - i\epsilon),$$

so that

$$D(Q^2) = Q^2 \int_0^\infty \frac{R_V(\sigma)}{(\sigma + Q^2)^2} d\sigma.$$

APT analysis of $D(Q^2)$ and $R_V(s)$

QCD PT gives us

$$D(Q^2) = 1 + \sum_{m>0} \frac{d_m}{\pi^m} (\alpha_s(Q^2))^m .$$

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In **APT (E)** we obtain

$$\mathcal{D}_N(Q^2) = 1 + \sum_{m>0}^N \frac{d_m}{\pi^m} \mathcal{A}_m^{\text{glob}}(Q^2)$$

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In APT (E) we obtain

$$\mathcal{D}_N(Q^2) = 1 + \sum_{m>0}^N \frac{d_m}{\pi^m} \mathcal{A}_m^{\text{glob}}(Q^2)$$

and in APT (M)

$$\mathcal{R}_{V;N}(s) = 1 + \sum_{m>0}^N \frac{d_m}{\pi^m} \mathcal{R}_m^{\text{glob}}(s)$$

Model for perturbative coefficients

Coefficients d_m of the PT series:

Model	d_1	d_2	d_3	d_4	d_5
pQCD with $N_f = 4$	1	1.52	2.59		—

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$c = 3.467, \beta = 1.325$	1	1.50	2.62		

We use model $\tilde{d}_n^{\text{mod}} = \frac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1} \Gamma(n)$

with parameters β and c estimated by known \tilde{d}_n

that possesses the **Lipatov** asymptotics $\tilde{d}_n^{\text{mod}} \sim b^n n!$ at $n \gg 1$.

Model for perturbative coefficients

Coefficients d_m of the PT series:

Model	d_1	d_2	d_3	d_4	d_5
pQCD with $N_f = 4$	1	1.52	2.59	27.4	—
$c = 3.467, \beta = 1.325$	1	1.50	2.62	27.8	

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$c = 3.456, \beta = 1.325$	1	1.49	2.60	27.5	

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“INNA” model	1	1.44	[3, 9]	[20, 48]	[674, 2786]

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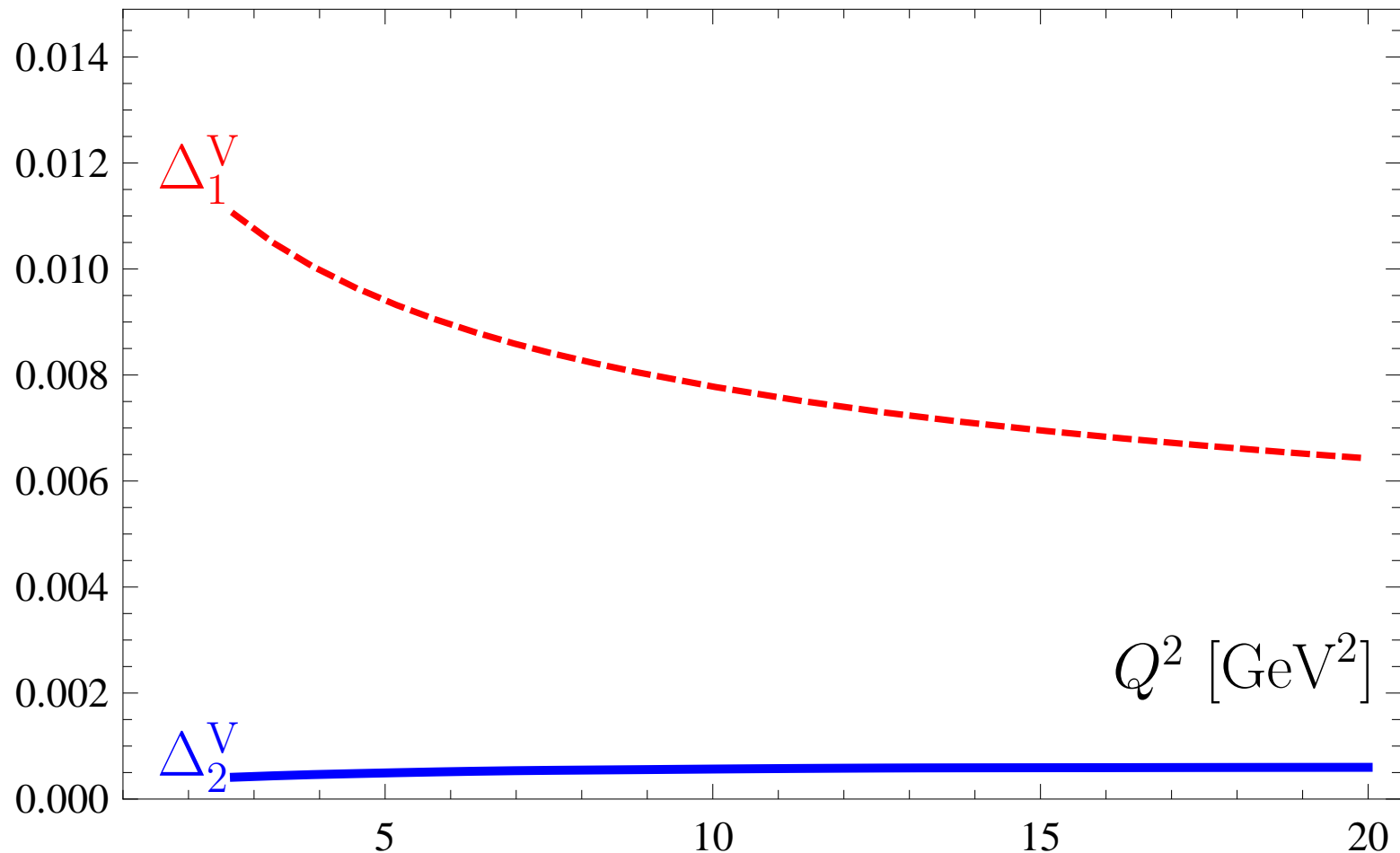
with parameters β and c estimated by known \tilde{d}_n

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Two-loop APT(E) for $\mathcal{D}(Q^2)$: Truncation errors

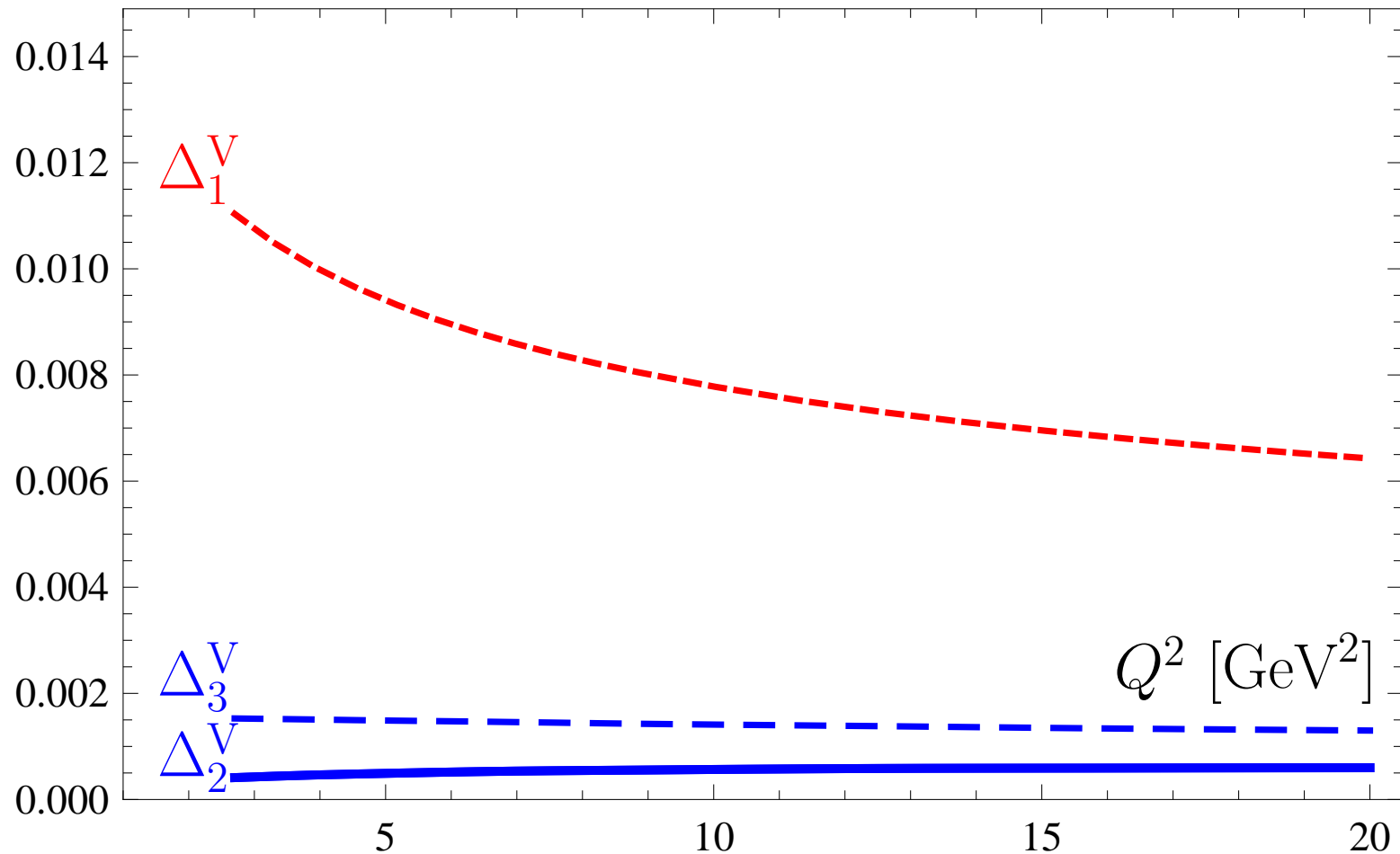
We define relative errors of series truncation at N th term:

$$\Delta_N^V[L] = 1 - \mathcal{D}_N[L]/\mathcal{D}_\infty[L]$$



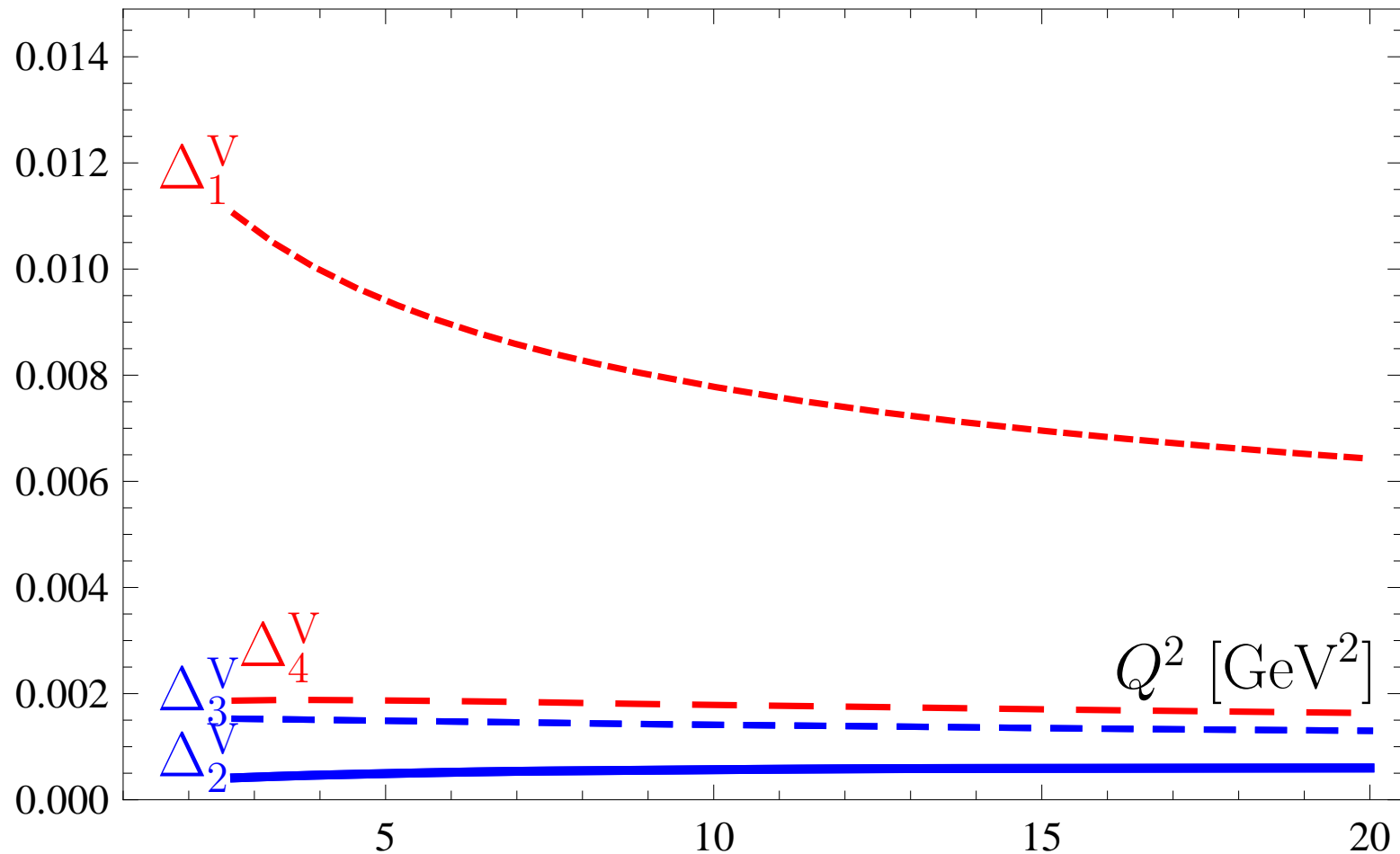
Two-loop APT(E) for $\mathcal{D}(Q^2)$: Truncation errors

Conclusion: The best accuracy (better than 0.1%) is achieved for **N²LO** approximation.



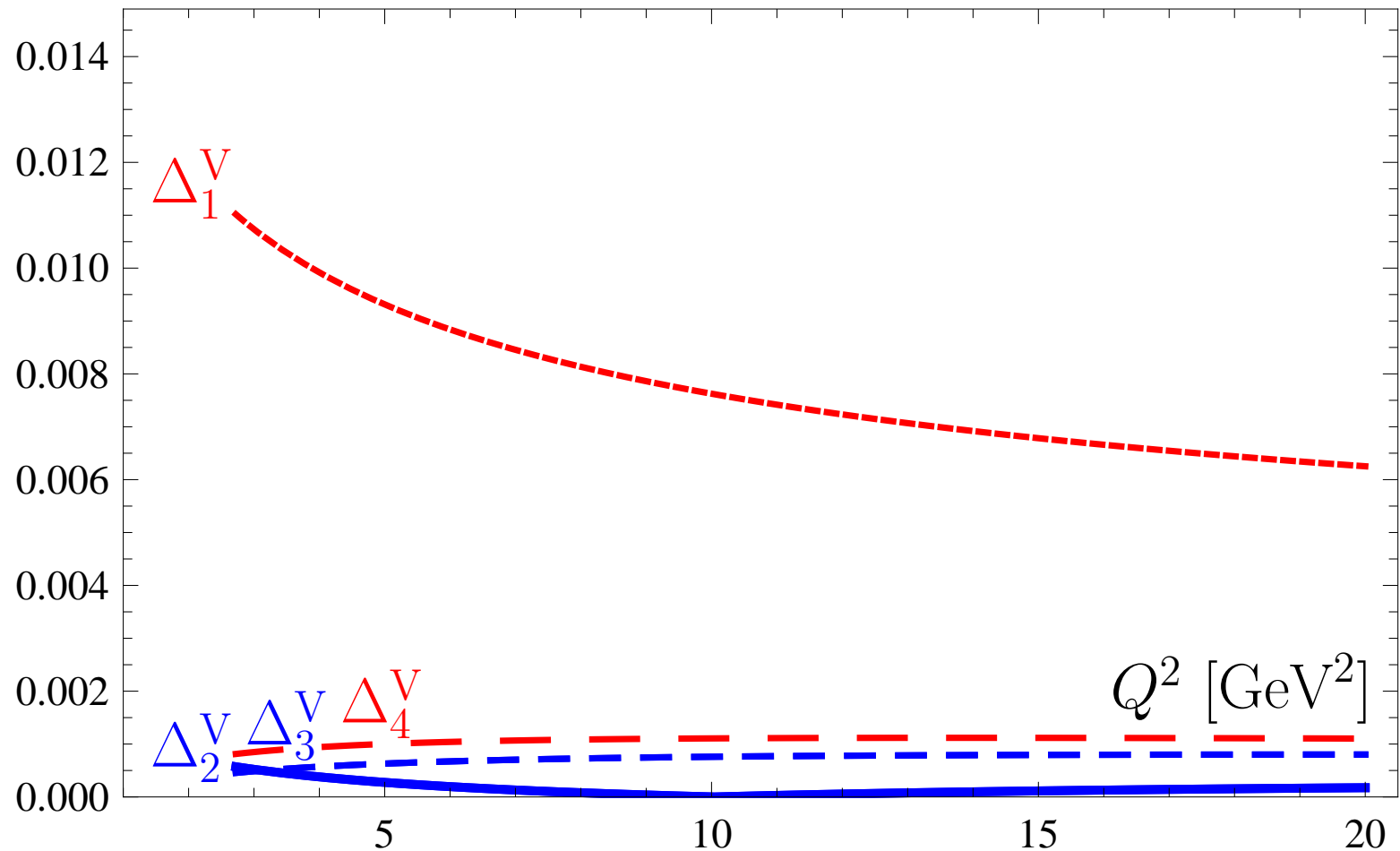
Two-loop APT(E) for $\mathcal{D}(Q^2)$: Truncation errors

Conclusion: If we add more terms N³LO — truncation error increases.



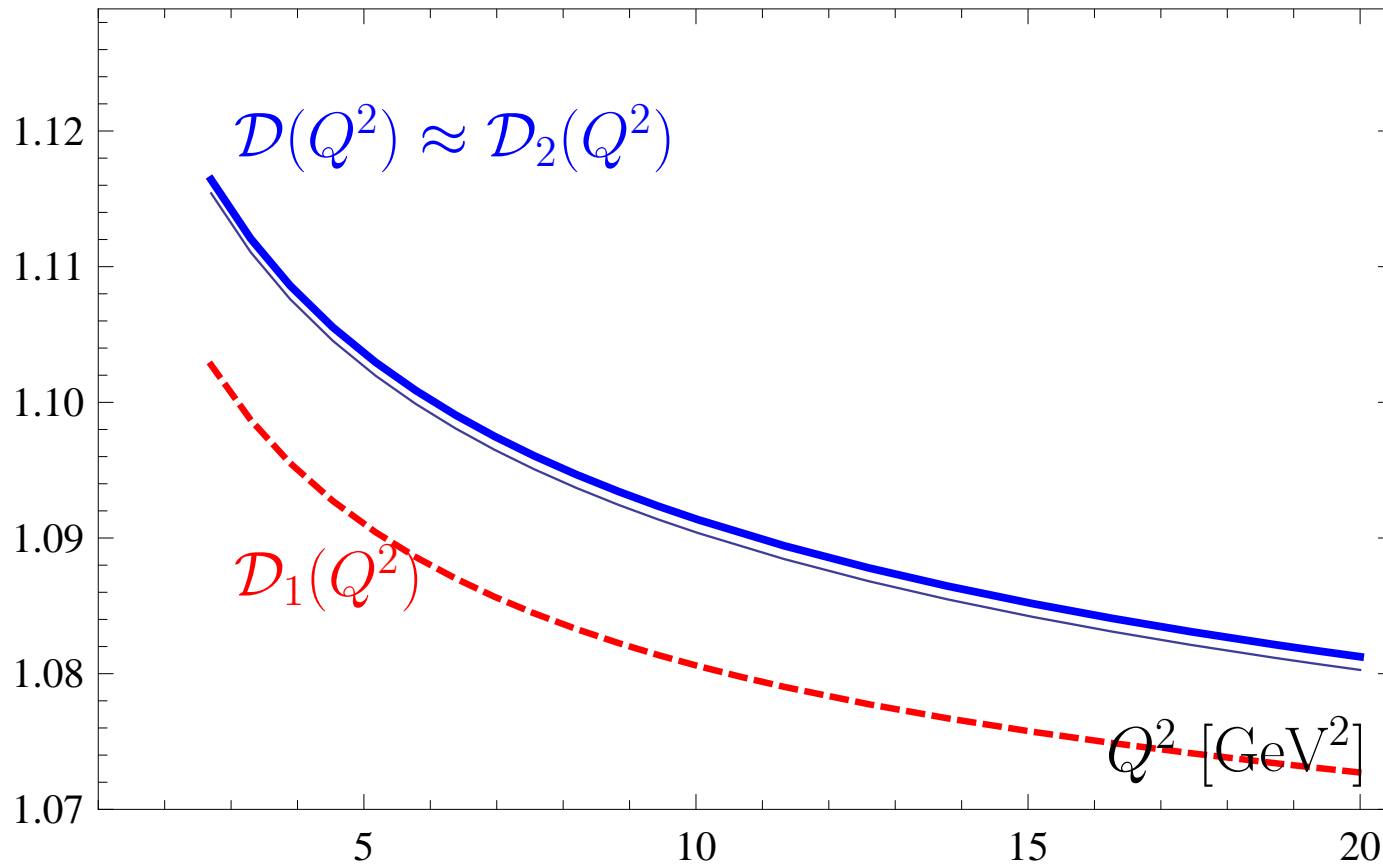
Three-loop APT(E) for $\mathcal{D}(Q^2)$: Truncation error

Conclusion: If we add more terms N^3LO — truncation error increases.



Two-loop APT(E) for $\mathcal{D}(Q^2)$: Truncation errors

Conclusion: The best accuracy (better than 0.1%) is achieved for **N²LO** approximation.



APT(E) for $\mathcal{D}(Q^2)$: Errors of modelling $P(t)$

We use model $d_n^{\text{mod}} = \frac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1} \Gamma(n)$

with parameters $\beta = 1.325$ and $c = 3.456$ estimated by known \tilde{d}_n and with use of **Lipatov** asymptotics.

We apply it to resum **APT** series and obtain $\mathcal{D}(Q^2)$.

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We deform our model for d_n by using coefficients $\beta_{\text{NNA}} = 1.322$ and $c_{\text{NNA}} = 3.885$

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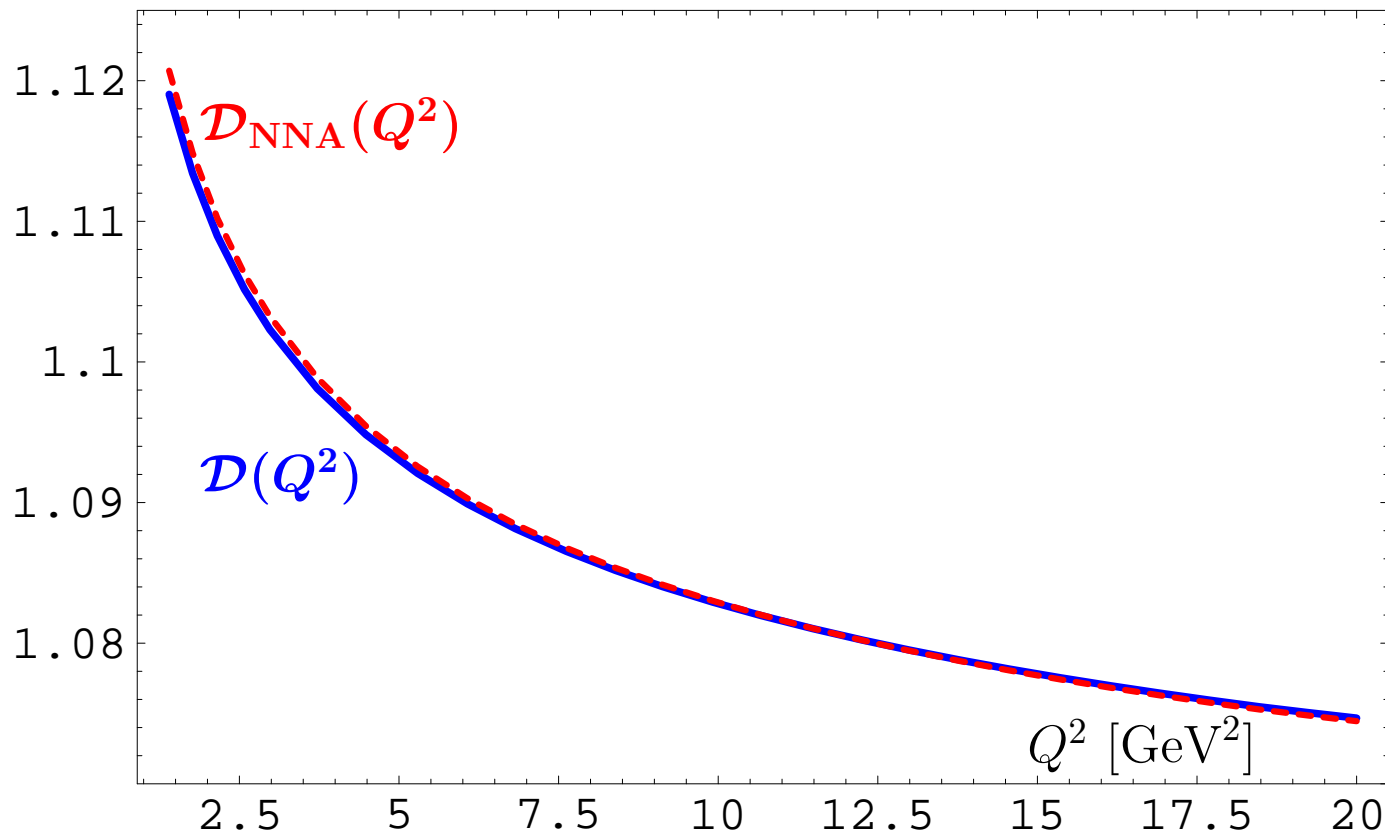
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We apply it to resum **APT** series and obtain $\mathcal{D}_{\text{NNA}}(Q^2)$.

APT(E) for $\mathcal{D}(Q^2)$: Errors of modelling $P(t)$

Conclusion: The result of resummation is stable to the variations of higher-order coefficients: deviation is of the order of 0.1%.



Application to Higgs boson decay

Higgs boson decay into $b\bar{b}$ -pair

This decay can be expressed in QCD by means of the correlator of quark scalar currents $J_S(x) = :\bar{b}(x)b(x):$:

$$\Pi(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0 | T [J_S(x) J_S(0)] | 0 \rangle$$

in terms of discontinuity of its imaginary part

$$R_S(s) = \text{Im} \Pi(-s - i\epsilon) / (2\pi s),$$

so that

$$\Gamma_{H \rightarrow b\bar{b}}(M_H) = \frac{G_F}{4\sqrt{2}\pi} M_H m_b^2(M_H) R_S(s = M_H^2).$$

FAPT(M) analysis of R_S

Running mass $m(Q^2)$ is described by the RG equation

$$m^2(Q^2) = \hat{m}^2 \alpha_s^{\nu_0}(Q^2) \left[1 + \frac{c_1 b_0 \alpha_s(Q^2)}{4\pi^2} \right]^{\nu_1}$$

with RG-invariant mass \hat{m}^2 (for b -quark $\hat{m}_b \approx 8.53$ GeV) and $\nu_0 = 1.04$, $\nu_1 = 1.86$. This gives us

$$\tilde{D}_S(Q^2) = 3\hat{m}_b^2 \left[\alpha_s^{\nu_0}(Q^2) + \sum_{m>0} \frac{d_m}{\pi^m} \alpha_s^{m+\nu_0}(Q^2) \right]$$

In 1-loop FAPT(M) we obtain

$$\tilde{\mathcal{R}}_S^{(1);N}[L] = 3\hat{m}_b^2 \left[\mathfrak{A}_{\nu_0}^{(1);glob}[L] + \sum_{m>0}^N \frac{d_m}{\pi^m} \mathfrak{A}_{m+\nu_0}^{(1);glob}[L] \right]$$

FAPT(M) analysis of R_S

Running mass $m(Q^2)$ is described by the RG equation

$$m^2(Q^2) = \hat{m}^2 \alpha_s^{\nu_0}(Q^2) \left[1 + \frac{c_1 b_0 \alpha_s(Q^2)}{4\pi^2} \right]^{\nu_1}$$

with RG-invariant mass \hat{m}^2 (for b -quark $\hat{m}_b \approx 8.53$ GeV) and $\nu_0 = 1.04$, $\nu_1 = 1.86$. This gives us

$$\tilde{D}_S(Q^2) = 3\hat{m}_b^2 \left[\alpha_s^{\nu_0}(Q^2) + \sum_{m>0} \frac{d_m}{\pi^m} \alpha_s^{m+\nu_0}(Q^2) \right]$$

In 2-loop FAPT(M) we obtain

$$\tilde{\mathcal{R}}_S^{(2);N}[L] = 3\hat{m}_b^2 \left[\mathfrak{B}_{\nu_0, \nu_1}^{(2);glob}[L] + \sum_{m>0}^N \frac{d_m}{\pi^m} \mathfrak{B}_{m+\nu_0, \nu_1}^{(2);glob}[L] \right]$$

Model for perturbative coefficients

Coefficients of our series, $\tilde{d}_m = d_m/d_1$, with $d_1 = 17/3$:

Model	\tilde{d}_1	\tilde{d}_2	\tilde{d}_3	\tilde{d}_4	\tilde{d}_5
pQCD	1	7.42	62.3		—
$c = 2.5, \beta = -0.48$	1	7.42	62.3	662	—
	—	—			

We use model $\tilde{d}_n^{\text{mod}} = \frac{c^{n-1}(\beta \Gamma(n) + \Gamma(n+1))}{\beta + 1}$

with parameters β and c estimated by known \tilde{d}_n and with use of **Lipatov** asymptotics.

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“PMS” model	—	—	64.8	547	7782

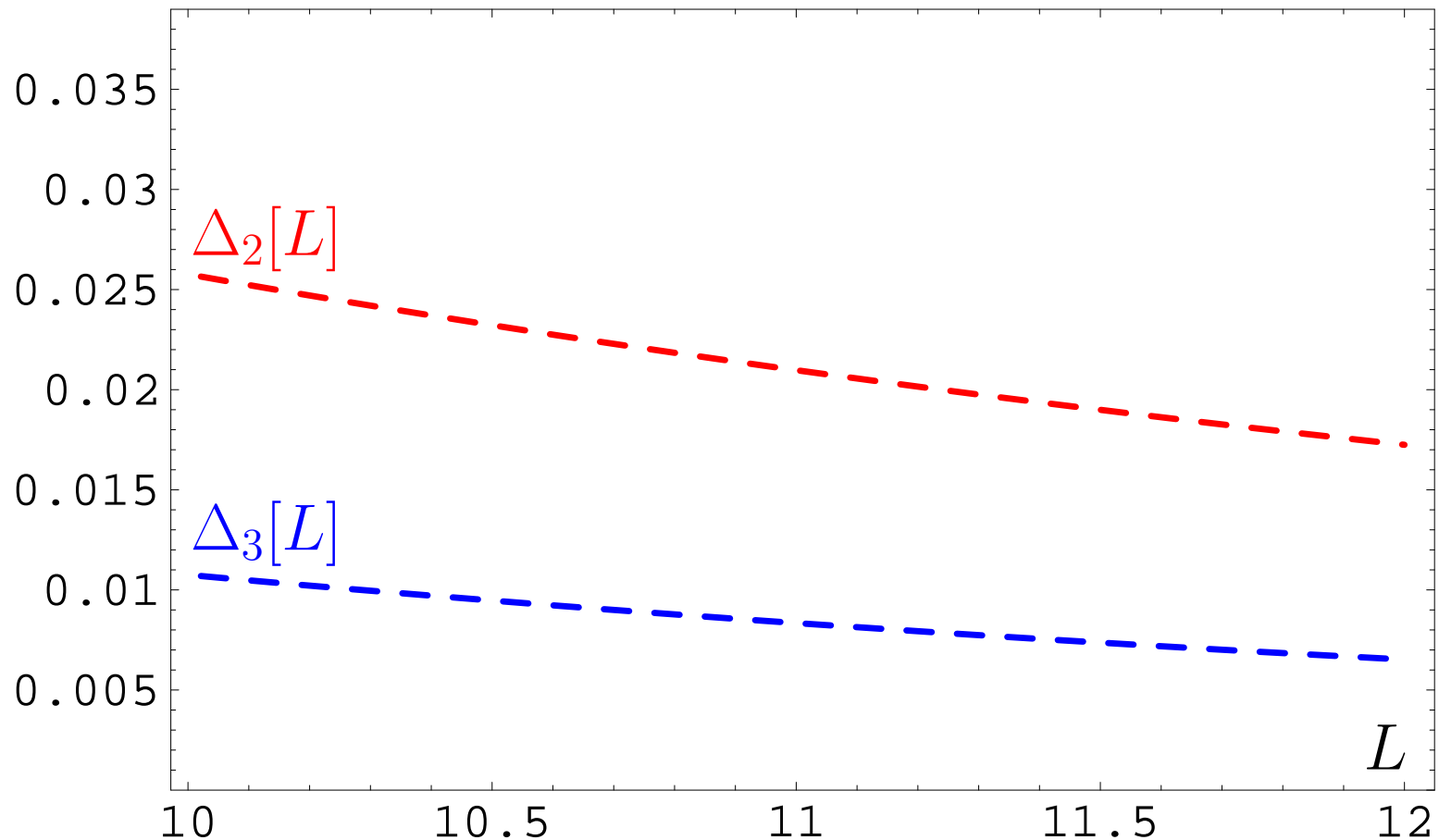
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FAPT(M) for $\Gamma_{H \rightarrow \bar{b}b}(m_H)$: Truncation errors

We define relative errors of series truncation at N th term:

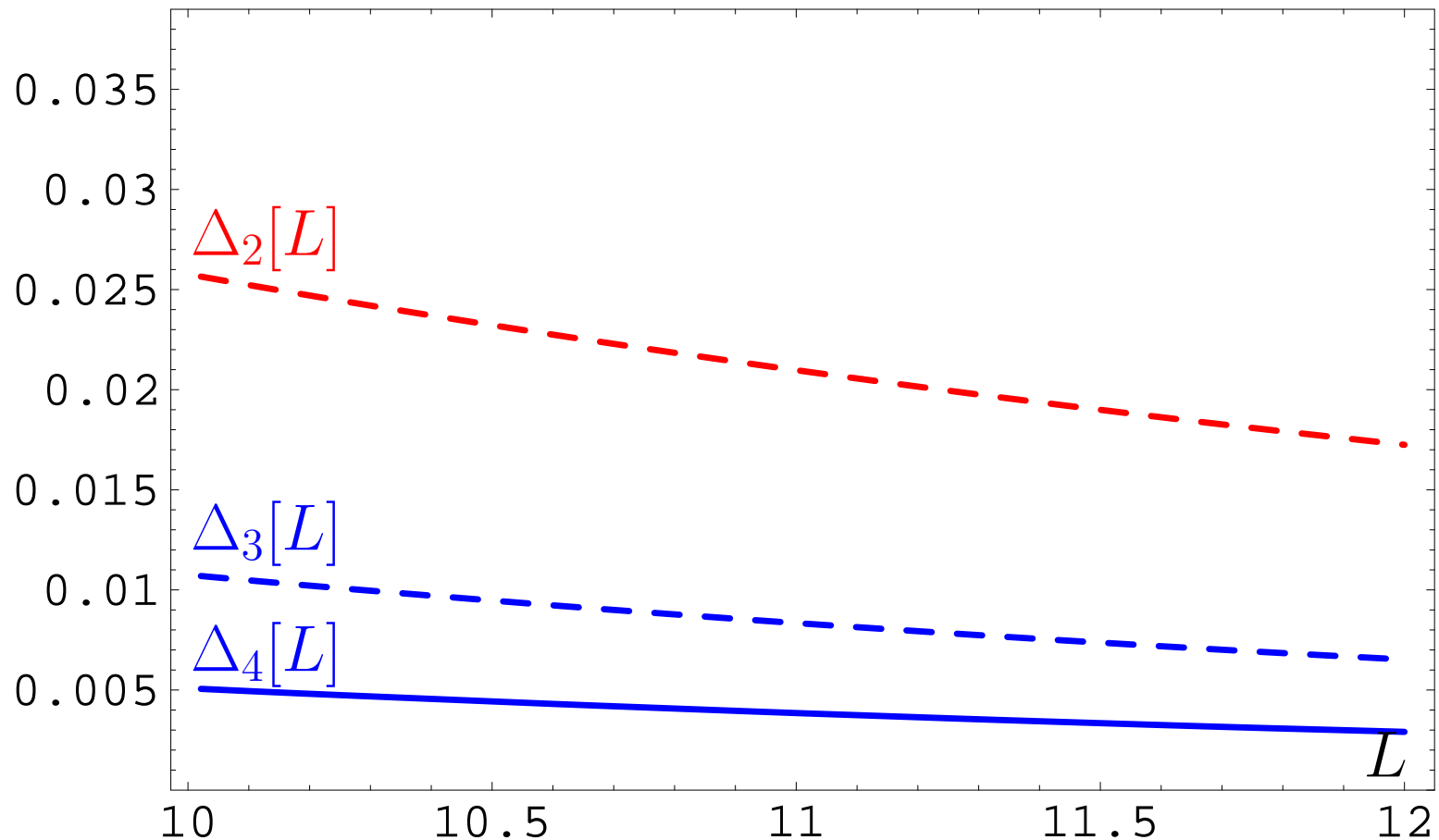
$$\Delta_N[L] = 1 - \tilde{\mathcal{R}}_s^{(2;N)}[L] / \tilde{\mathcal{R}}_s^{(2;\infty)}[L]$$



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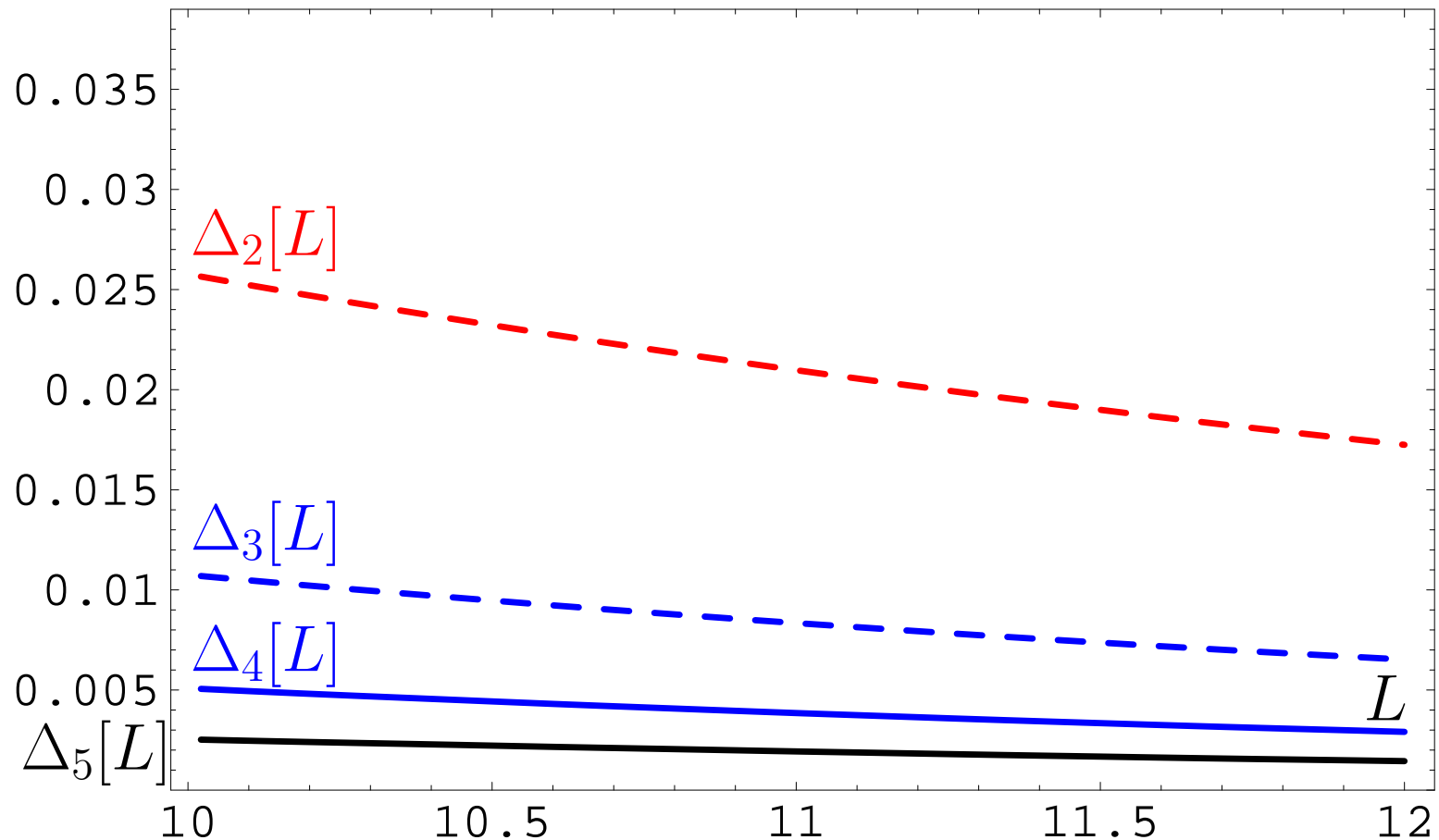
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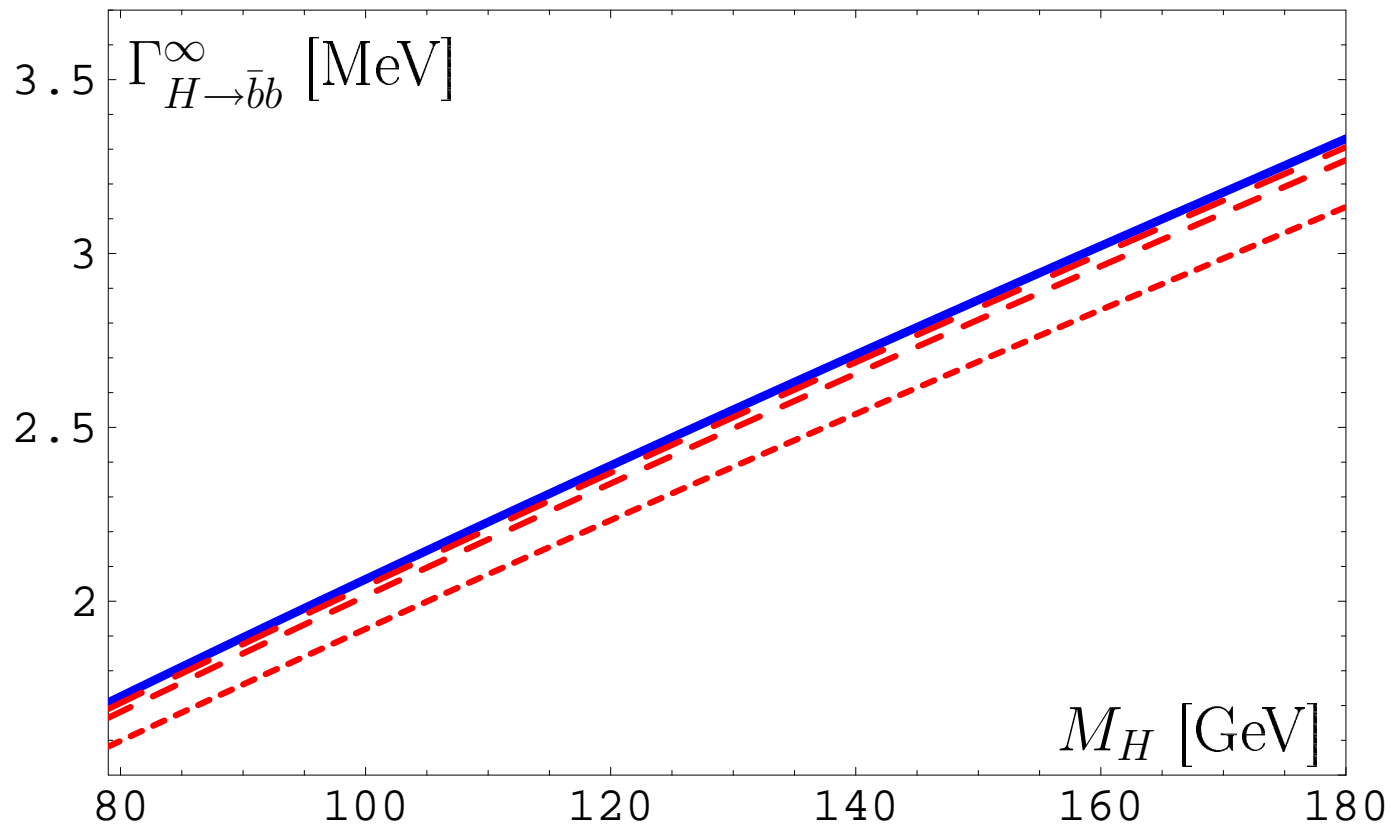
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FAPT(M) for $\Gamma_{H \rightarrow \bar{b}b}(m_H)$: Truncation errors

Conclusion: If we need accuracy better than 0.5% — only then we need to calculate the 5-th correction.

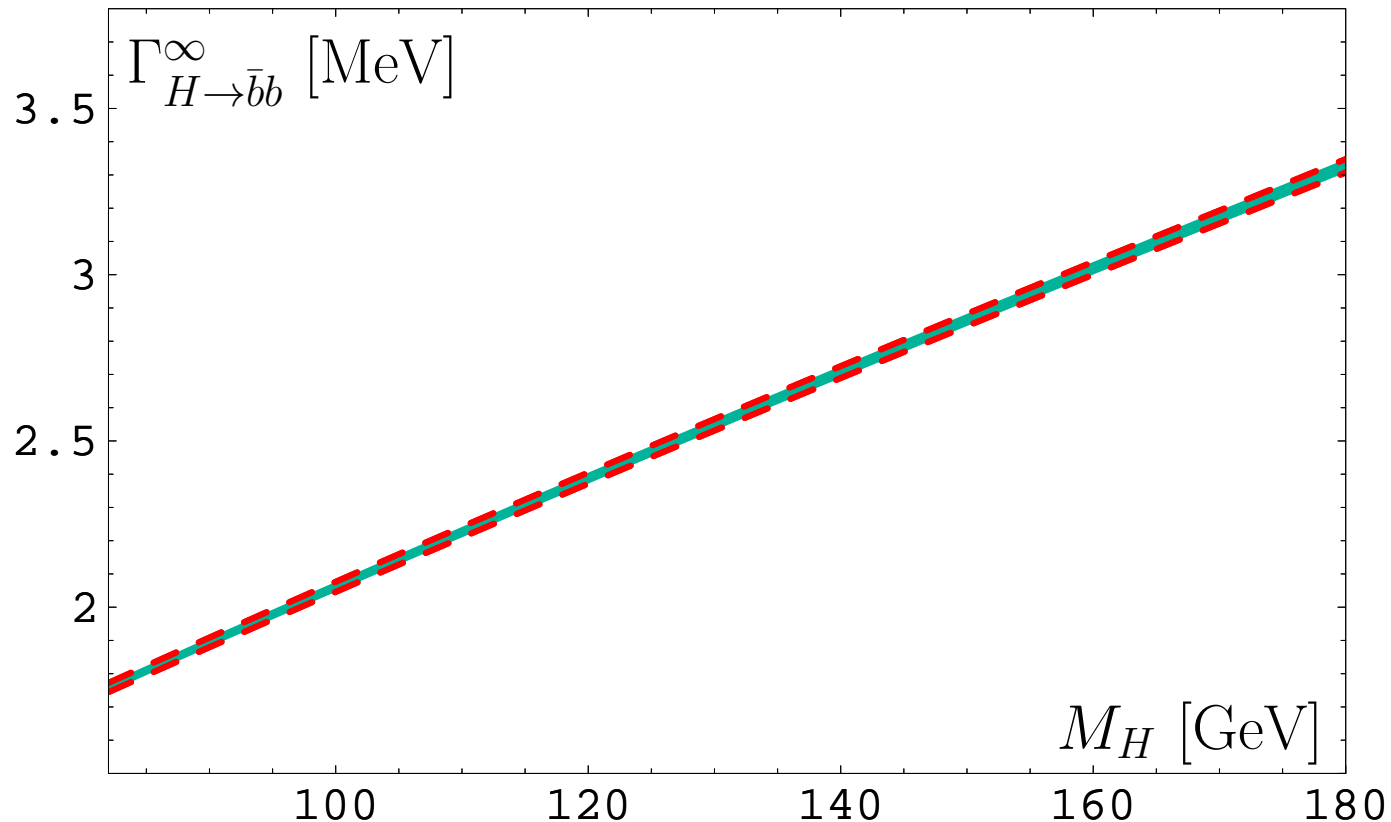
But profit will be tiny — instead of 0.5% one'll obtain 0.3%!



FAPT(M) for $\Gamma_{H \rightarrow \bar{b}b}(m_H)$: Truncation errors

Conclusion: If we need accuracy of the order 0.5% — then we need to take into account up to the 4-th correction.

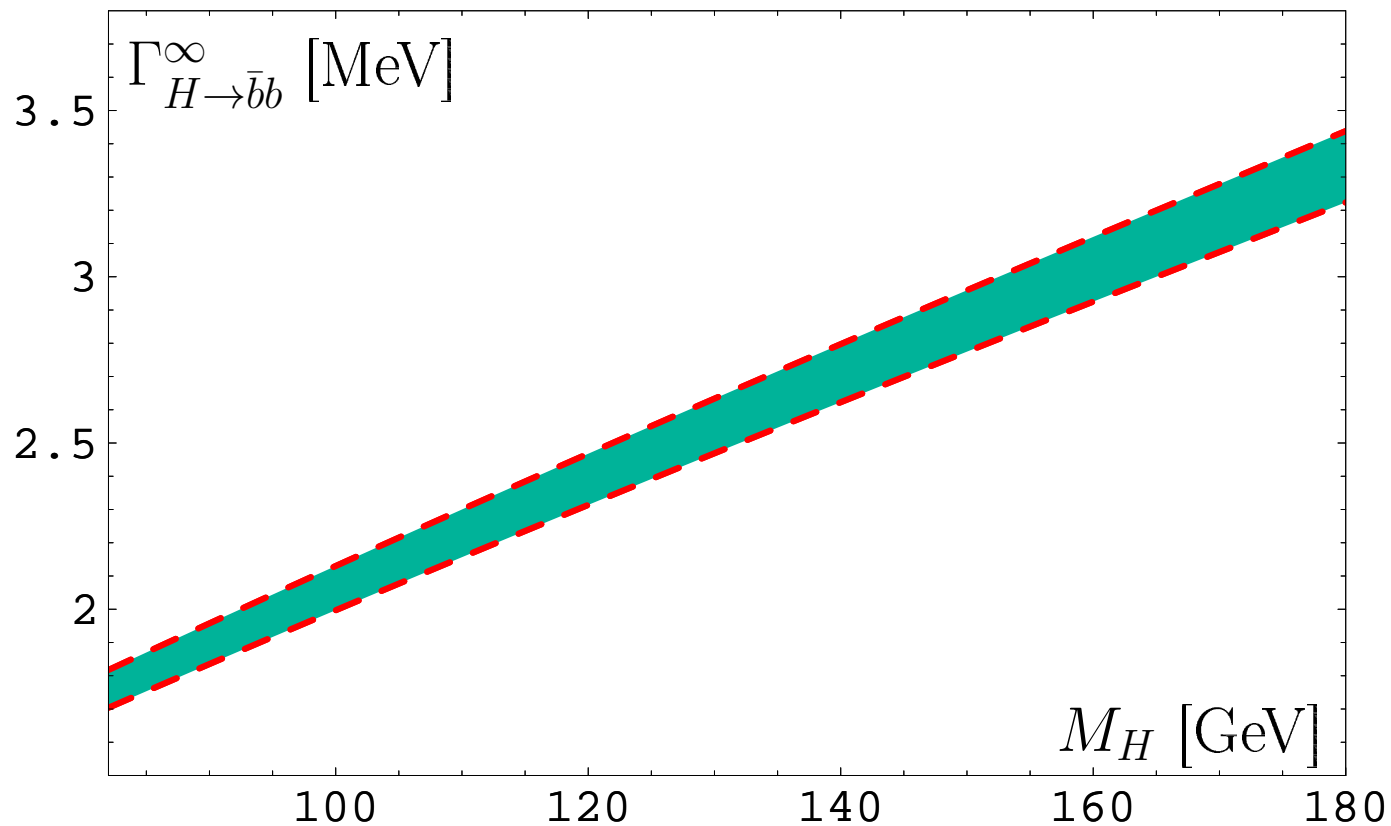
Note: uncertainty due to $P(t)$ -modelling is small $\lesssim 0.6\%$.



FAPT(M) for $\Gamma_{H \rightarrow \bar{b}b}(m_H)$: Truncation errors

Conclusion: If we need accuracy of the order 1% — then we need to take into account up to the 3-rd correction — in agreement with Kataev&Kim [0902.1442].

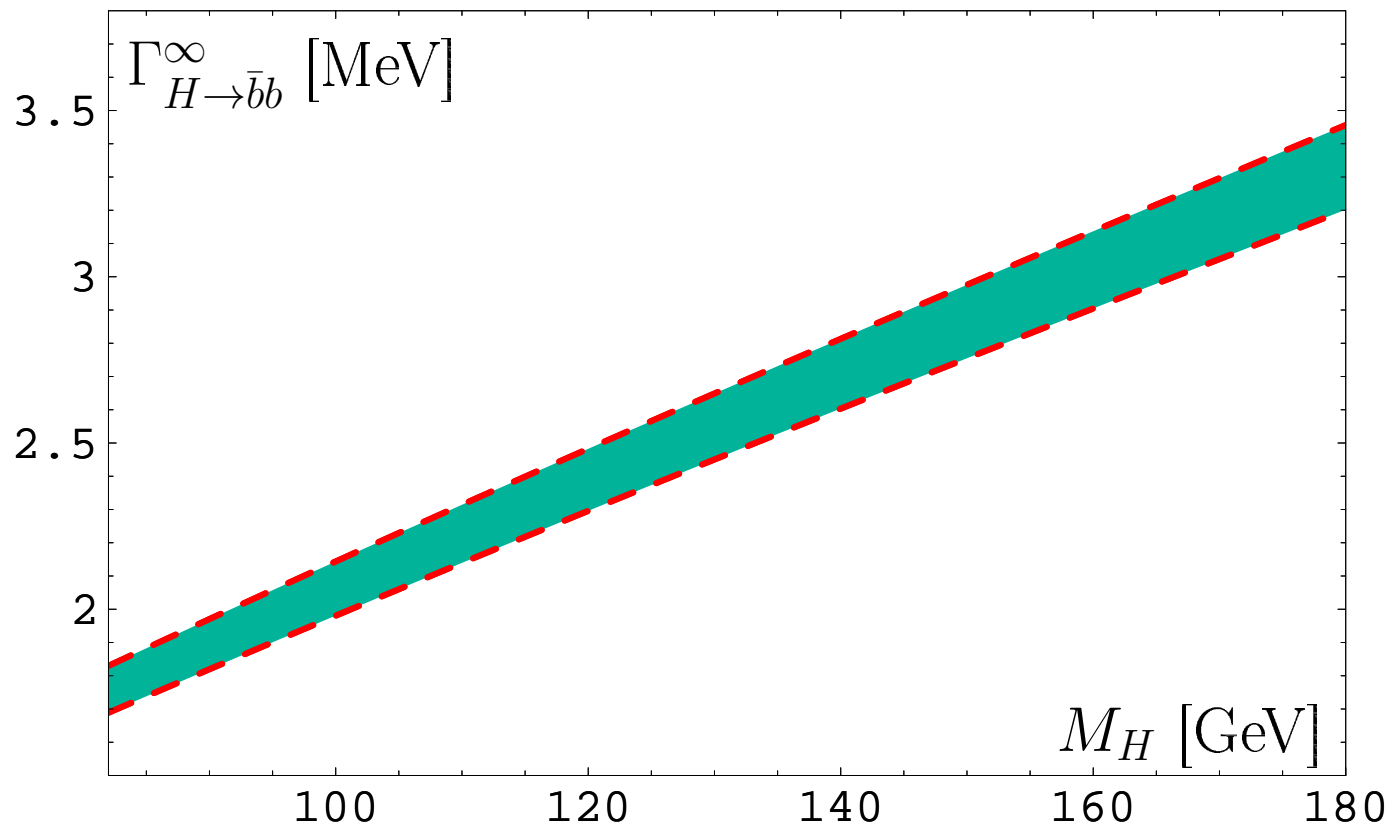
Note: RG-invariant mass uncertainty $\sim 2\%$.



FAPT(M) for $\Gamma_{H \rightarrow \bar{b}b}(m_H)$: Truncation errors

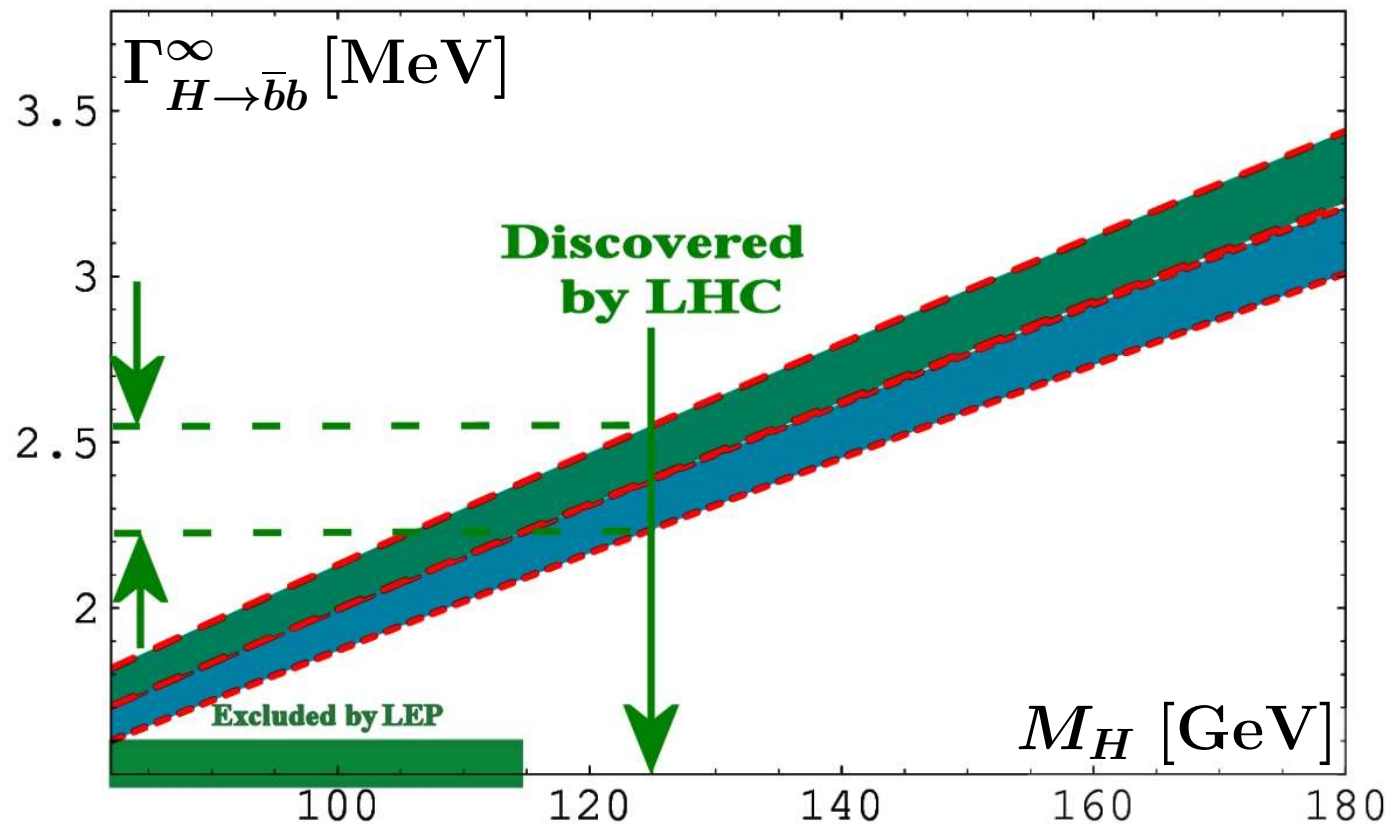
Conclusion: If we need accuracy of the order 1% — then we need to take into account up to the 3-rd correction — in agreement with Kataev&Kim [0902.1442].

Note: overall uncertainty $\sim 3\%$.



Resummation for $\Gamma_{H \rightarrow \bar{b}b}(m_H)$: Loop orders

Comparison of 1- (upper strip) and 2-loop (lower strip) results. We observe a 5% reduction of the two-loop estimate. After LHC discovery we can put our estimate for the decay width: $\Gamma_{H \rightarrow \bar{b}b} = 2.4 \pm 0.1 \text{ MeV}$.



Low Q^2 Evolution of Non-Singlet Parton Distributions

DGLAP Evolution of Non-Singlet PDFs

DGLAP Evolution for PDFs Moments:

$$\mu^2 \frac{d}{d\mu^2} M_N(\mu^2) = -\frac{\gamma_0(N) \alpha_s(\mu^2)}{8\pi} M_N(\mu^2)$$

- Standard PT QCD solution ($\nu(N) = \gamma_0(N)/(2b_0)$):

$$E_N^{\text{PT}}(\mu^2, \mu_0^2) = \frac{M_N^{\text{PT}}(\mu^2)}{M_N(\mu_0^2)} = \left[\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right]^{\nu(N)}$$

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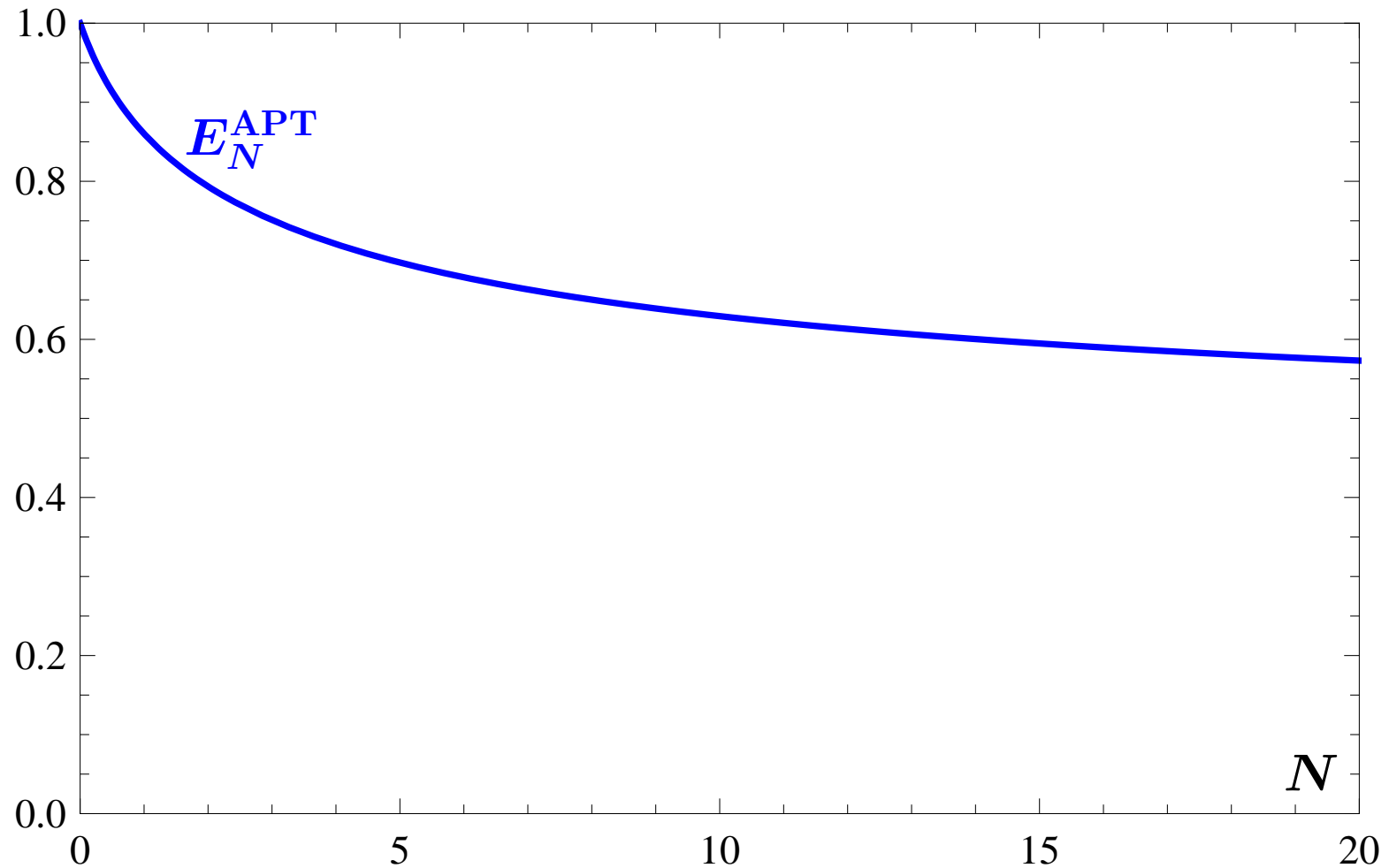
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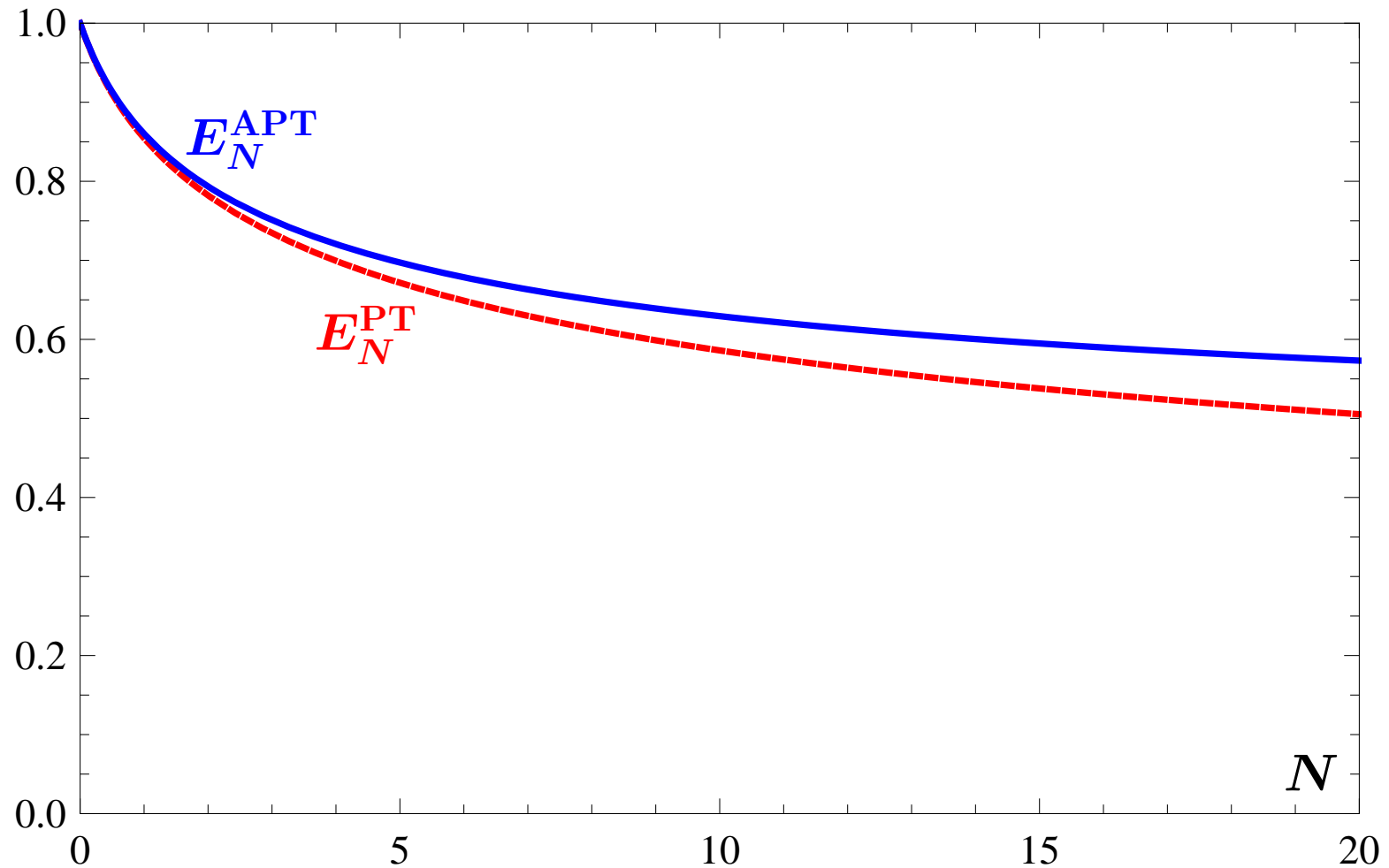
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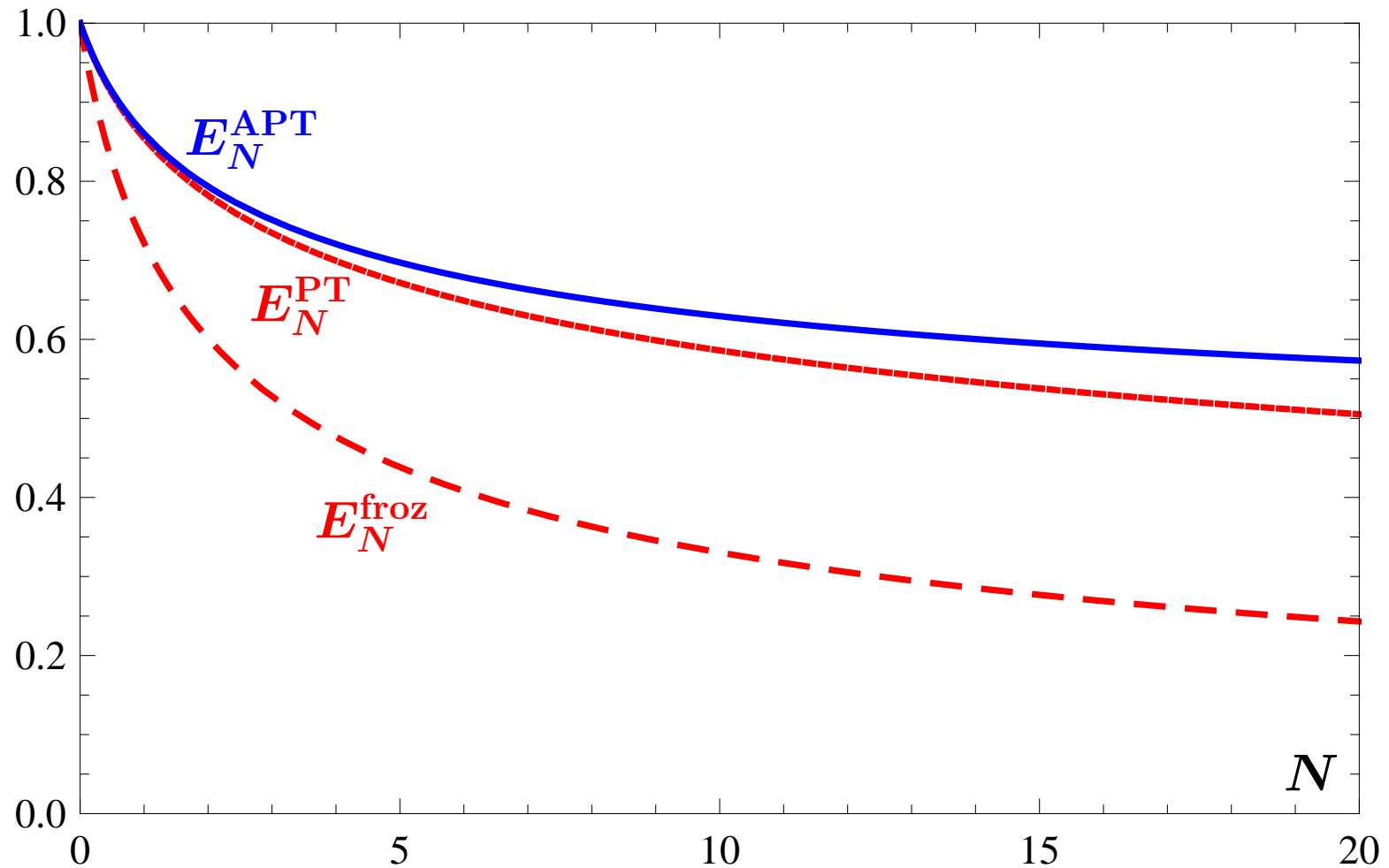
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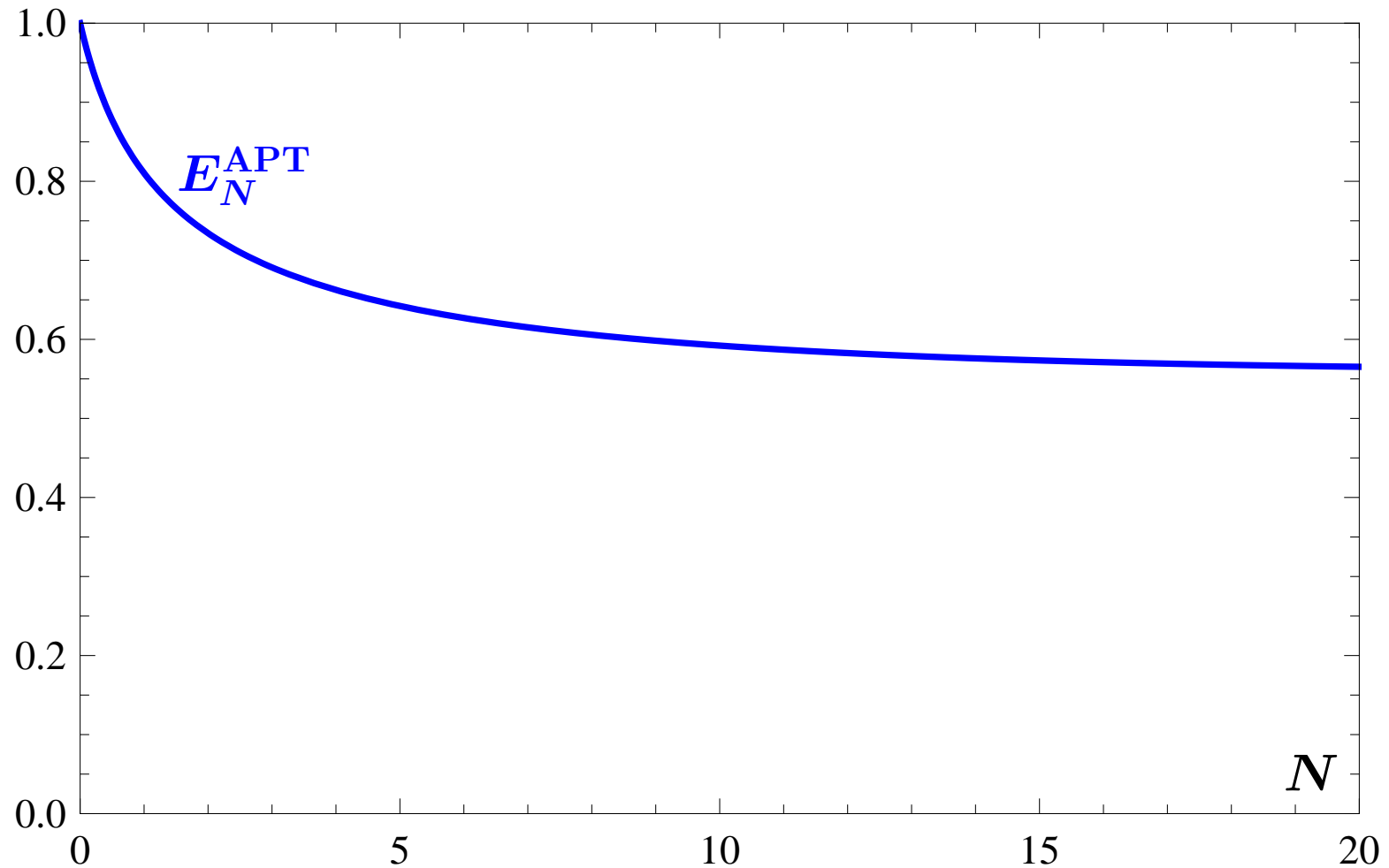
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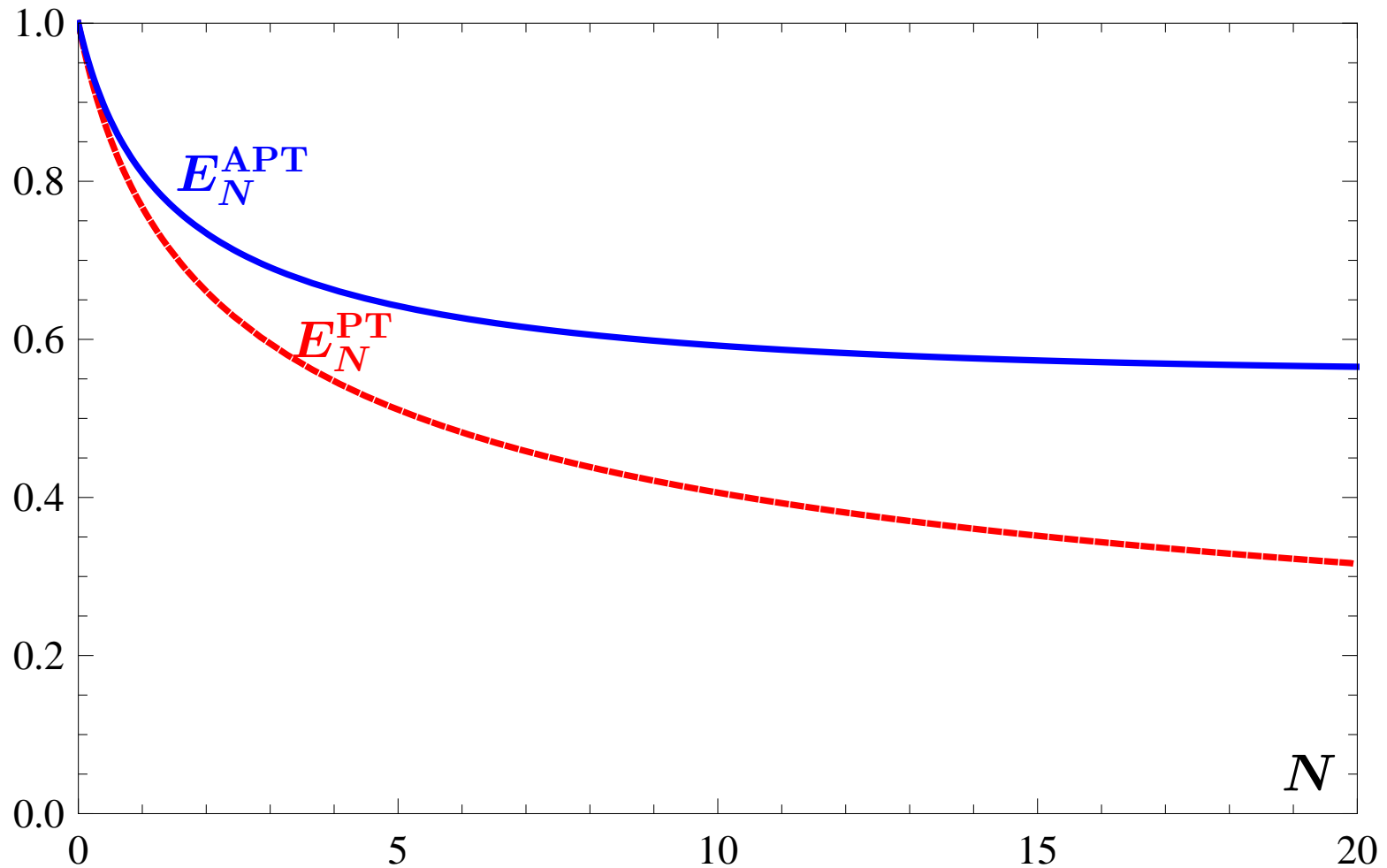
DGLAP Evolution of Non-Singlet PDF Moment

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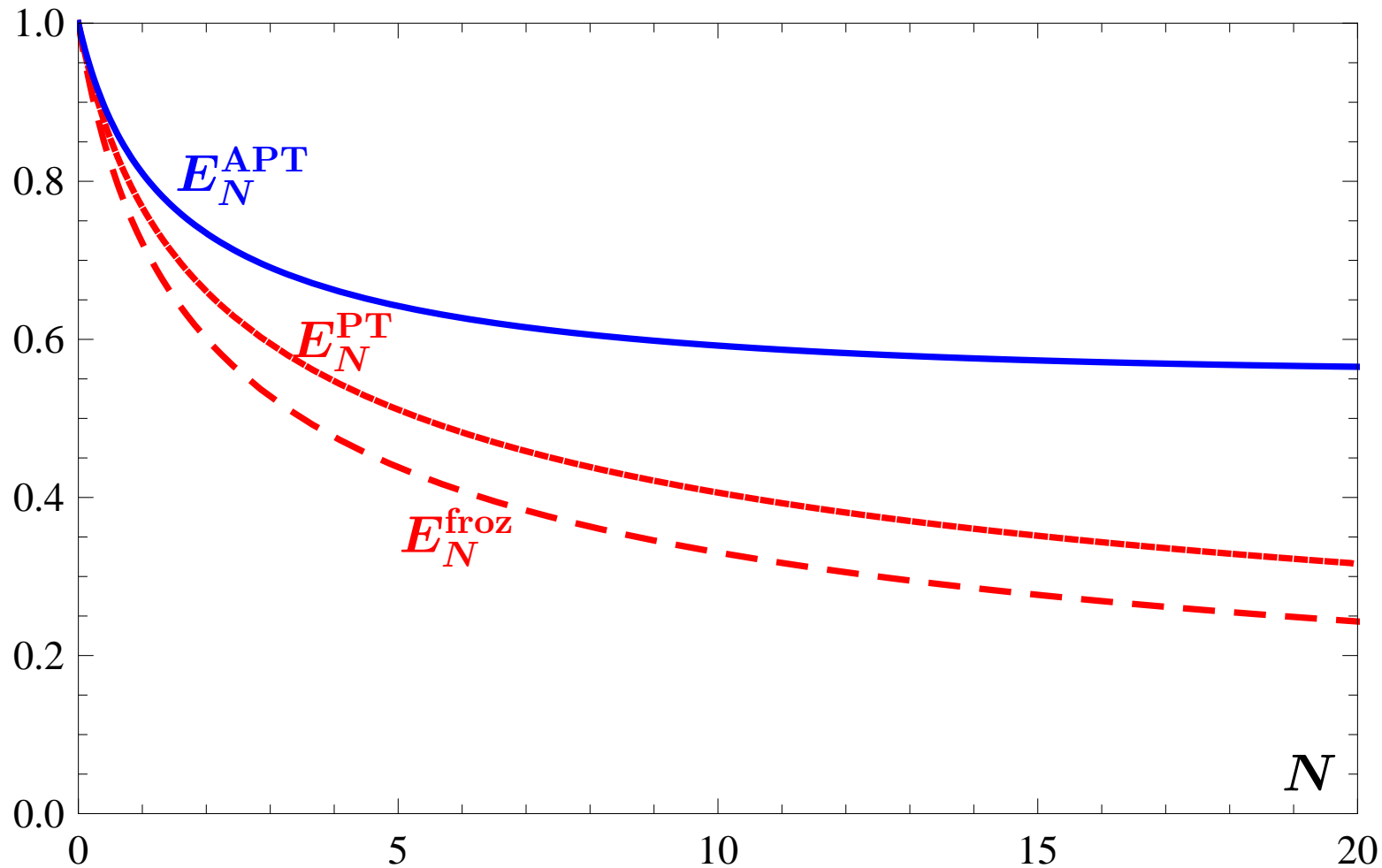
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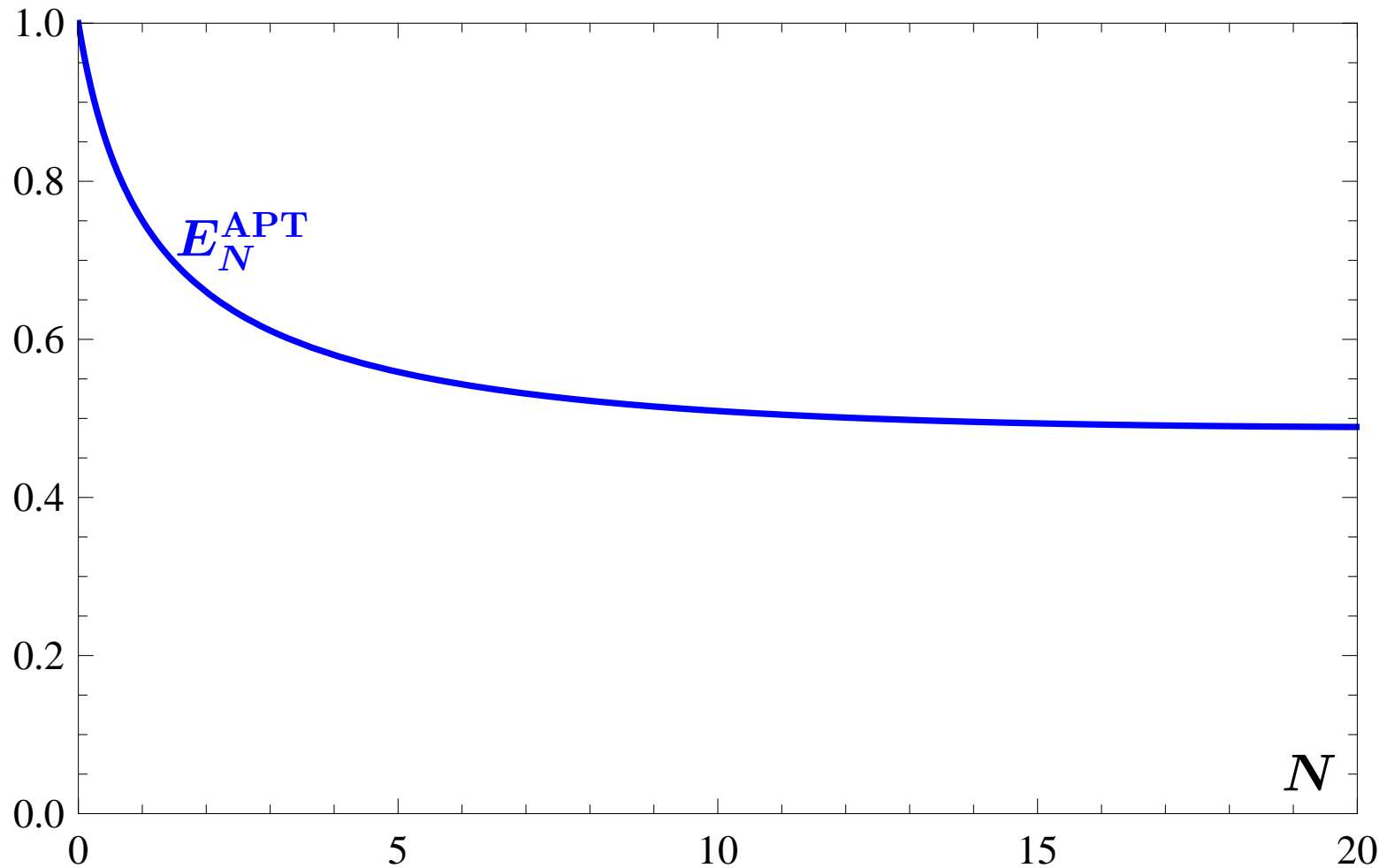
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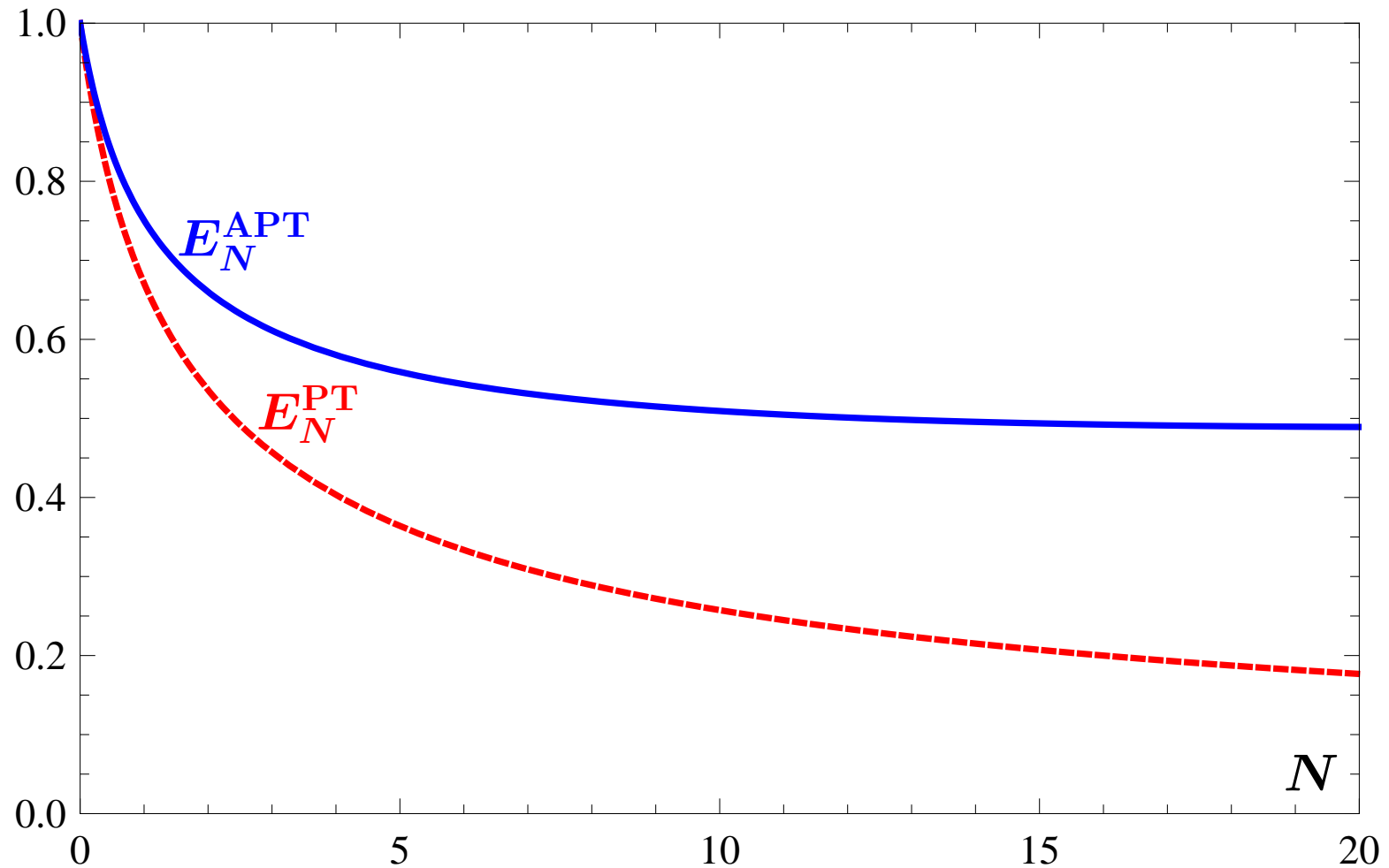
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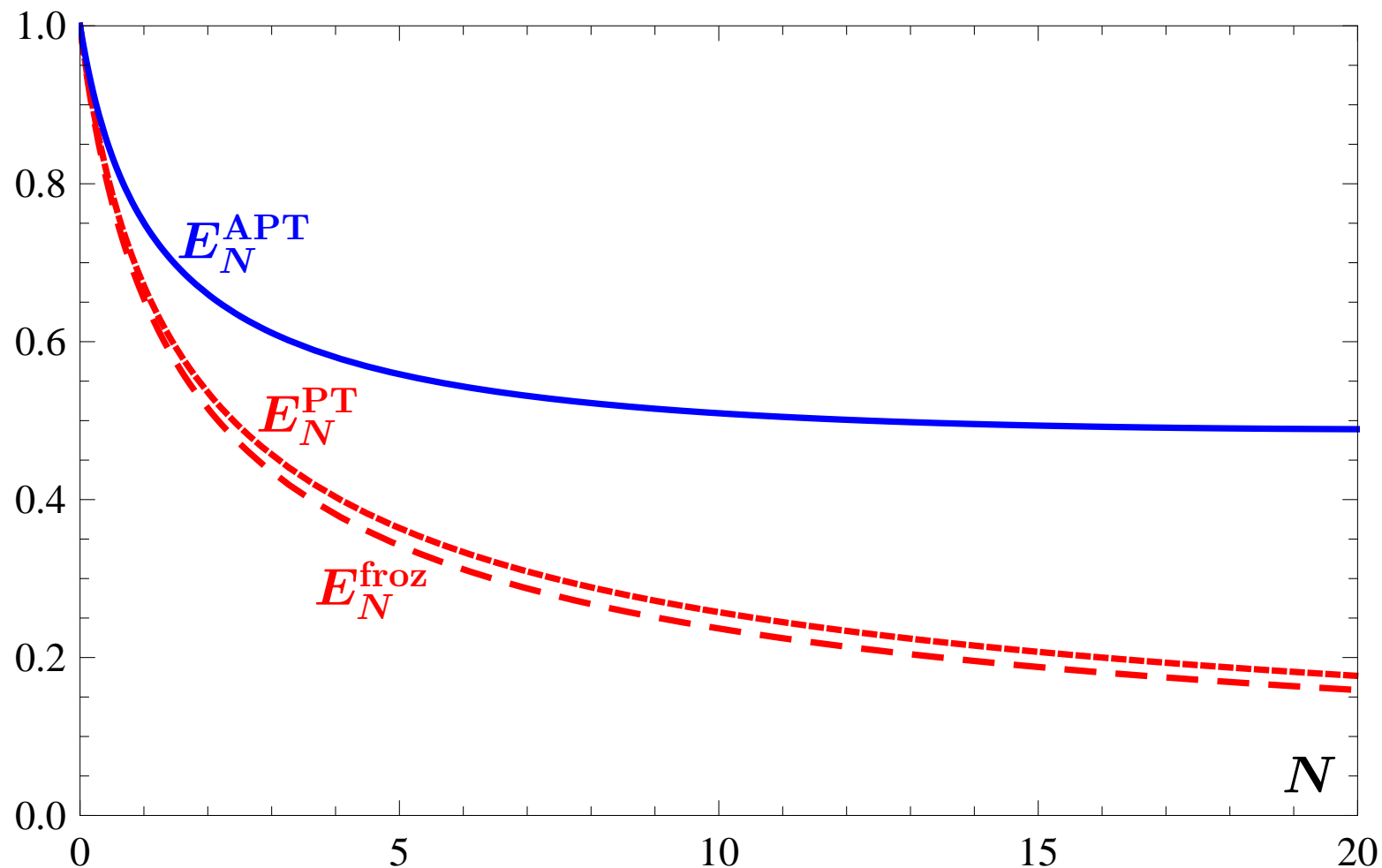
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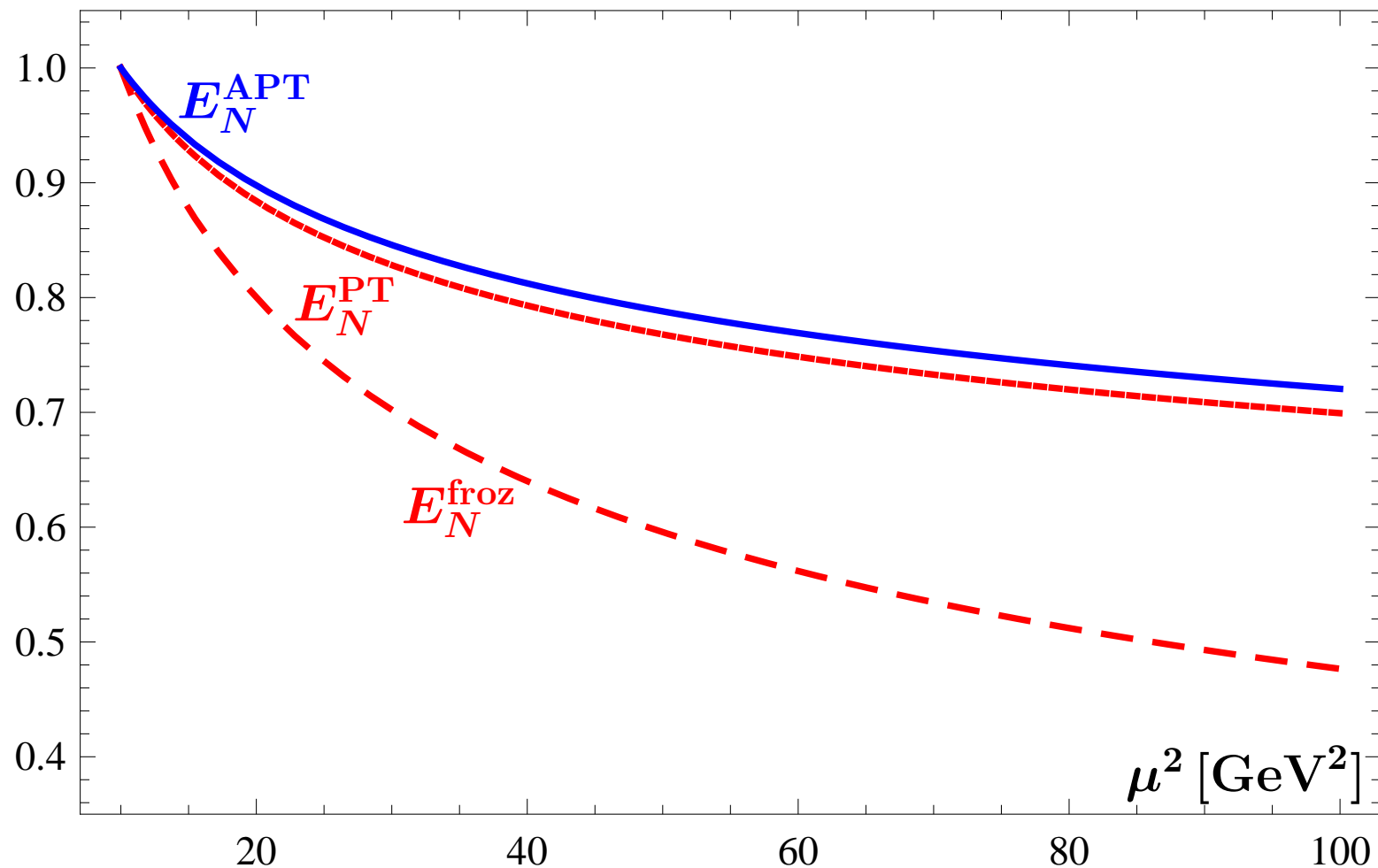
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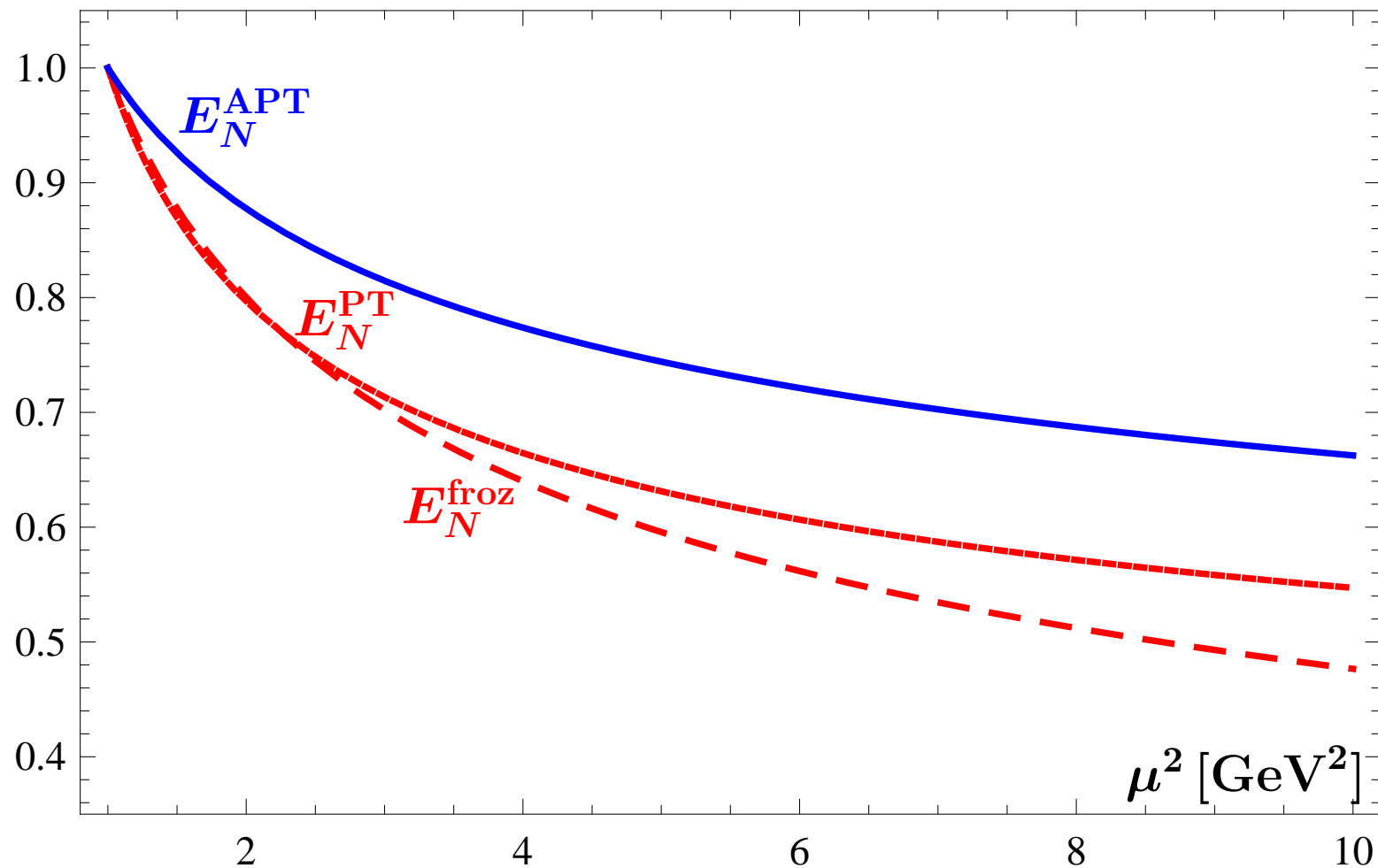
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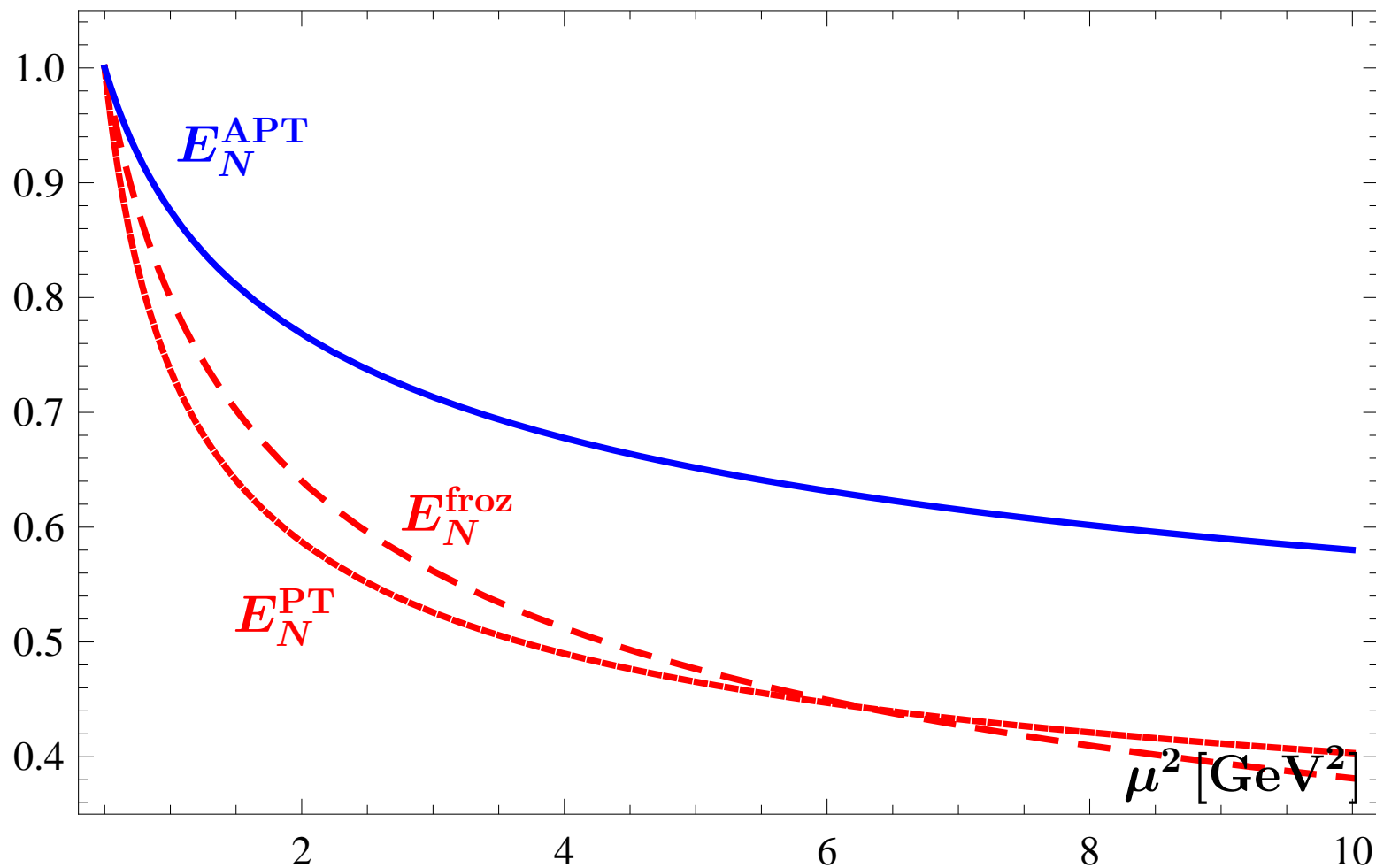
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