New methods for Feynman integrals: The method of Mellin–Barnes representation

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Evaluating Feynman integrals (STMP 211, Springer 2004)

Feynman Integrals Calculus (Springer 2006)

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$$F_\Gamma(a_1, a_2, \ldots) = \int \ldots \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots}$$

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Dimensional regularization: $d = 4 - 2\epsilon\,;$ $\mathrm{d}^4 k \to \mathrm{d}^d k$
$k = (k_0, \vec{k}) = (k_0, k_1, k_2, k_3)$
$k_1, k_2, \ldots$ are loop momenta;
$p_1, p_2, \ldots$ are momenta of the lines; they are linear combinations of $k_1, k_2, \ldots$ and external momenta $q_1, q_2, \ldots$

The propagator as a building block

$$\frac{1}{k^2 - m^2 + i0} = \lim_{\delta \to 0} \frac{1}{k^2 - m^2 + i\delta} \; ,$$

$$k^2 = k_0^2 - \vec{k}^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2$$

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HQET, NRQCD,... $\rightarrow$ other types of propagators, e.g.

$$\frac{1}{v \cdot k \pm i0} \; , \qquad v = (1, \vec{0})$$

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Dimensional regularization

[G. 't Hooft & M. Veltman'72]

[C.G. Bollini & J.J. Giambiagi'72; P. Breitenlohner & D. Maison'77]

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Analytic part: formally,

$$\mathrm{d}^4 k = \mathrm{d}k_0 \mathrm{d}\vec{k} \quad \rightarrow \quad \mathrm{d}^d k \,, \qquad d = 4 - 2\epsilon$$

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Informally, use alpha parameters

$$\frac{1}{(-k^2+m^2-i0)^a} = \frac{\mathrm{e}^{\mathrm{i}\pi a}}{\Gamma(a)} \int_0^\infty \alpha^{a-1} \mathrm{e}^{\mathrm{i}(k^2-m^2)\alpha} \mathrm{d}\alpha$$

$$\frac{1}{(-v\cdot k-i0)^a} = \frac{\mathrm{e}^{\mathrm{i}\pi a}}{\Gamma(a)} \int_0^\infty \alpha^{a-1} \mathrm{e}^{\mathrm{i}(v\cdot k)\alpha} \mathrm{d}\alpha$$

Dimensional regularization:
when deriving alpha representations, apply this rule with $d = 4 - 2\epsilon$

$$\int \mathrm{d}^4 k \, \mathrm{e}^{\mathrm{i}(\alpha k^2 - 2q\cdot k)} = -\mathrm{i}\pi^2 \alpha^{-2} \mathrm{e}^{-\mathrm{i}q^2/\alpha}$$

$\longrightarrow$

$$\int \mathrm{d}^d k \, \mathrm{e}^{\mathrm{i}(\alpha k^2 - 2q\cdot k)} = \mathrm{e}^{\mathrm{i}\pi(1-d/2)/2} \pi^{d/2} \alpha^{-d/2} \mathrm{e}^{-\mathrm{i}q^2/\alpha}$$

Graph $\Gamma \to$ dimensionally regularized Feynman integral

$$F_\Gamma(a_1 \ldots, a_L; d) \;=\; \frac{\mathrm{e}^{\mathrm{i}\pi(a+h(1-d/2))/2}\pi^{hd/2}}{\prod_l \Gamma(a_l)}$$

$$\times \int_0^\infty \mathrm{d}\alpha_1 \ldots \int_0^\infty \mathrm{d}\alpha_L \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-d/2} \mathrm{e}^{\mathrm{i}\mathcal{V}/\mathcal{U} - \mathrm{i}\sum m_l^2 \alpha_l} \;,$$

where $a = \sum a_i$

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where $a = \sum a_i$
For a Feynman integral with $1/(m^2-k^2-i0)^{a_l}$ propagators,

$$\mathcal{U} \;=\; \sum_{\text{trees } T} \; \prod_{l \notin T} \alpha_l \;,$$
$$\mathcal{V} \;=\; \sum_{2-\text{trees } T} \; \prod_{l \notin T} \alpha_l \left(q^T\right)^2 \;.$$

Alpha representation $\rightarrow$

- Mathematical proofs (for Feynman integrals at Euclidean external momenta, $(\sum q_i)^2 < 0$) Analysis of convergence.

  [K. Hepp'66; P. Breitenlohner & D. Maison'77; E. Speer'68,'77]

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- A tool to evaluate Feynman integrals analytically.
- A tool to evaluate Feynman integrals numerically.
  Modern sector decompositions
  [T. Binoth & G. Heinrich'00; C. Bogner & S. Weinzierl'07; A.V. Smirnov & M.N. Tentyukov'08; A.V. Smirnov: talk today]

One can deal with dimensionally regularized Feynman integrals as with usual integrals, e.g. use linearity. They are even better ;-)

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Use integration by parts (IBP) and neglect surface terms

[K. G. Chetyrkin & F. V. Tkachov'81]

$$\int \cdots \int \left[\left(q_i \cdot \frac{\partial}{\partial k_j} \right) \frac{1}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots} \right] \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots =$$

$$\int \cdots \int \left[\frac{\partial}{\partial k_j} \cdot k_i \frac{1}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots} \right] \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots =$$

An old straightforward analytical strategy:

to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

The standard modern strategy:

to derive, without calculation, and then apply IBP identities between the given family of Feynman integrals as recurrence relations.

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The whole problem of evaluation$\rightarrow$

- constructing a reduction procedure
- evaluating master integrals

Solving reduction problems algorithmically:

- Laporta's algorithm

  [S. Laporta & E. Remiddi'96; S. Laporta'00; T. Gehrmann & E. Remiddi'01]

  Two public versions:

  AIR [C. Anastasiou & A. Lazopoulos'04]

  FIRE [A.V. Smirnov'08; talk yesterday]

  Private versions

  [T. Gehrmann & E. Remiddi, M. Czakon, Y. Schröder, A. Pak, C. Sturm, P. Marquard & D. Seidel, V. Velizhanin, . . .]

- Baikov's method [Baikov'96-09]
- Gröbner bases [O.V. Tarasov'98, A.V. Smirnov & V.A. Smirnov'05]
- Lee's approach [R.N. Lee'08; talk yesterday]

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- Mellin–Barnes representation

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Evaluating individual Feynman integrals:
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Systematic evaluation of dimensionally regularized Feynman integrals (in particular, systematic resolution of the singularities in ϵ) [V.A. Smirnov'99, J.B. Tausk'99]

The basic formula:

$$\frac{1}{(X+Y)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{Y^z}{X^{\lambda+z}}\Gamma(\lambda+z)\Gamma(-z)\;.$$

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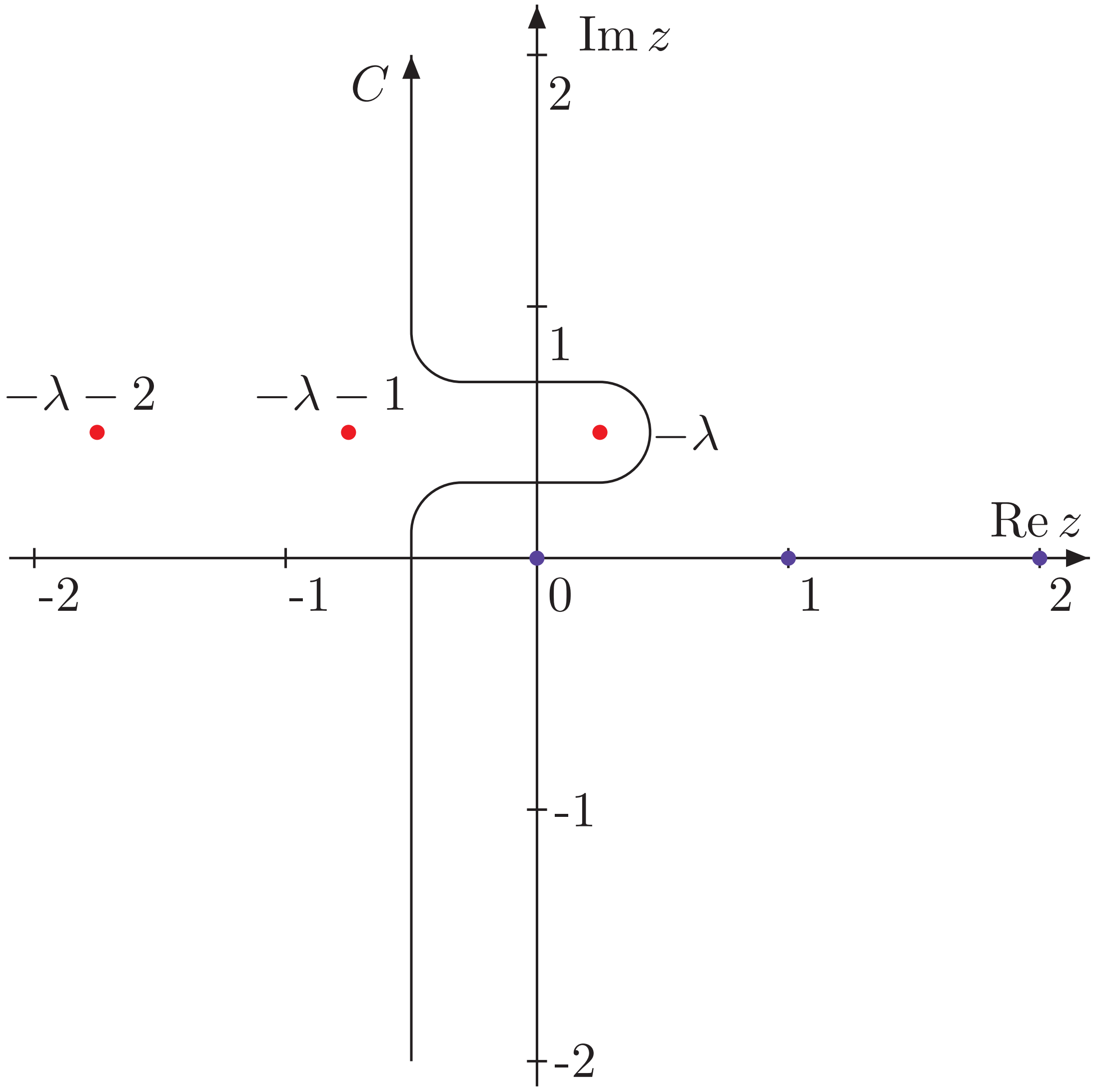
The poles with a $\Gamma(\ldots+z)$ dependence are to the left of the contour and the poles with a $\Gamma(\ldots-z)$ dependence are to the right



Im z
C
2
1
$-\lambda-2$
$-\lambda-1$
$-\lambda$
Re z
-2
-1
0
1
2
-1
-2

- Derive an MB representation

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- Expand in a Laurent series in ϵ
- Evaluate expanded MB integrals

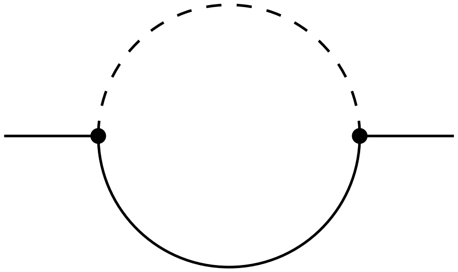
The simplest possibility:

$$\frac{1}{(m^2-k^2)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{(m^2)^z}{(-k^2)^{\lambda+z}}\Gamma(\lambda+z)\Gamma(-z)$$

The simplest possibility:

$$\frac{1}{(m^2-k^2)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{(m^2)^z}{(-k^2)^{\lambda+z}} \Gamma(\lambda+z)\Gamma(-z)$$

Example 1



$$F_\Gamma(q^2, m^2; a_1, a_2, d) = \int \frac{d^d k}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}}$$

$$F_\Gamma = \frac{1}{\Gamma(a_1)} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z (m^2)^z \Gamma(a_1+z)\Gamma(-z)$$

$$\times \int \frac{\mathrm{d}^d k}{(-k^2)^{a_1+z}(-(q-k)^2)^{a_2}}$$

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$$\int \frac{\mathrm{d}^d k}{(-k^2)^{a_1+z}[-(q-k)^2]^{a_2}} = \mathrm{i}\pi^{d/2} \frac{G(a_1+z, a_2)}{(-q^2)^{a_1+a_2+\epsilon-2+z}} \,,$$

$$G(a_1,a_2) = \frac{\Gamma(a_1+a_2+\epsilon-2)\Gamma(2-\epsilon-a_1)\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(4-a_1-a_2-2\epsilon)}$$

$$
\begin{aligned}
F_\Gamma(q^2, m^2; a_1, a_2, d) \quad &= \quad \frac{\mathrm{i}\pi^{d/2}\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)(-q^2)^{a_1+a_2+\epsilon-2}} \\
&\times \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(a_1+a_2+\epsilon-2+z) \\
&\times \frac{\Gamma(2-\epsilon-a_1-z)\Gamma(-z)}{\Gamma(4-2\epsilon-a_1-a_2-z)}
\end{aligned}
$$

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$$\times \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(a_1+a_2+\epsilon-2+z)$$
$$\times \frac{\Gamma(2-\epsilon-a_1-z)\Gamma(-z)}{\Gamma(4-2\epsilon-a_1-a_2-z)}$$

Unambiguous prescriptions for contours:
the poles with a $\Gamma(\ldots+z)$ dependence are to the left and the poles with a $\Gamma(\ldots-z)$ dependence are to the right of a contour

$$F_\Gamma(q^2, m^2; 1, 1, d) \;=\; \frac{\mathrm{i}\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon}$$

$$\times \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)}$$

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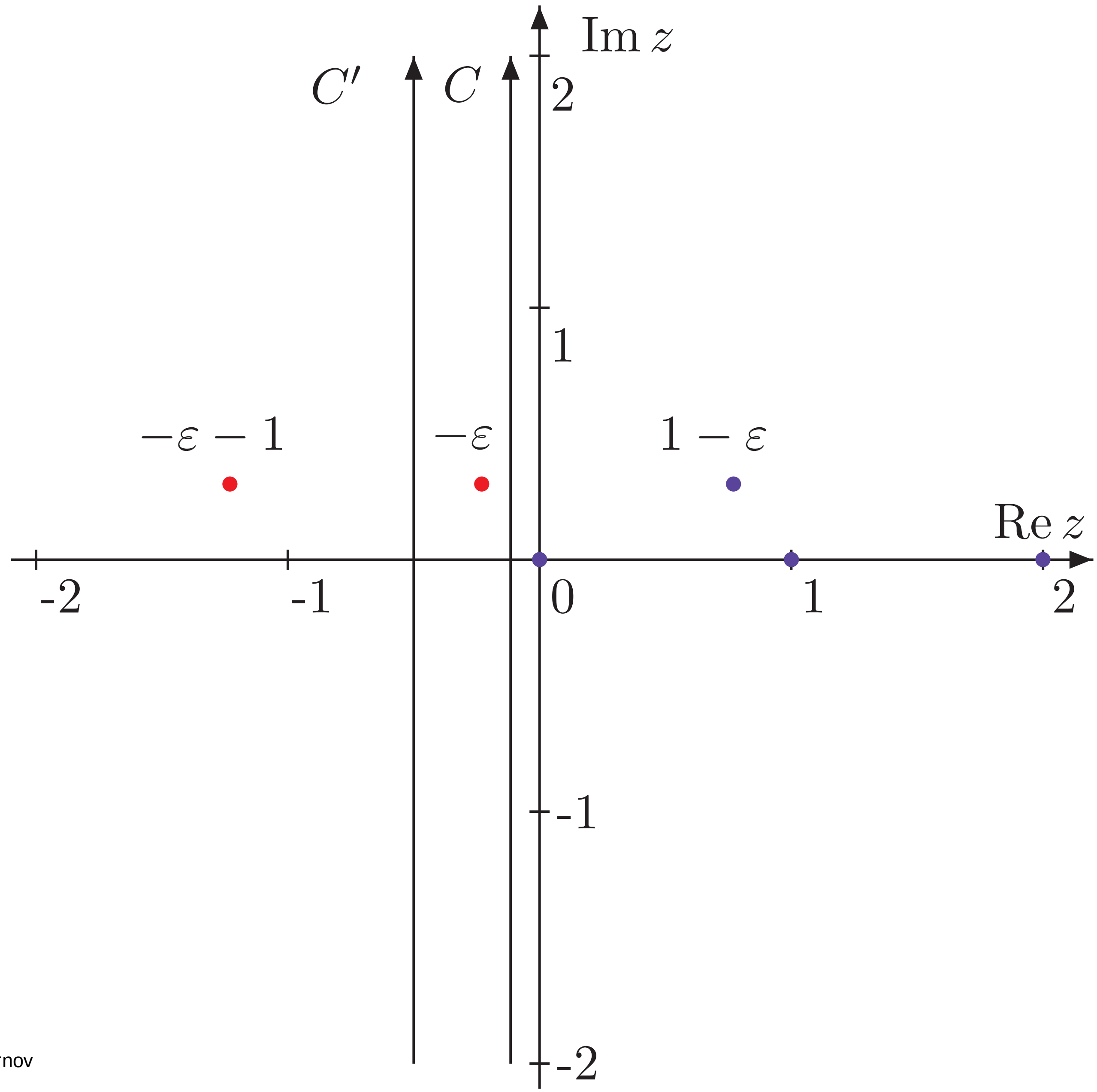
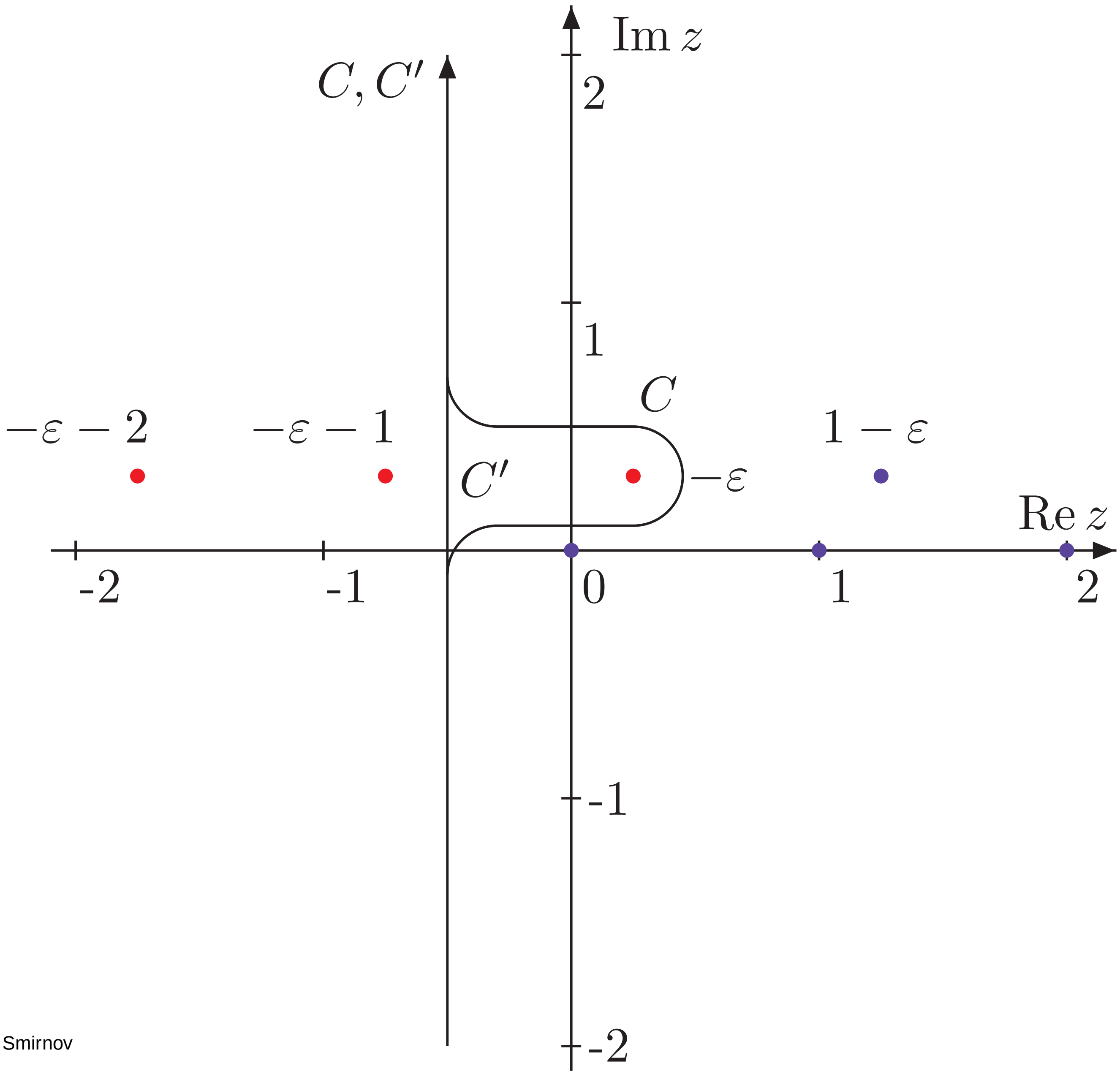
$\Gamma(\epsilon+z)\Gamma(-z) \rightarrow$ a singularity in ϵ



Im z
C′
C
2
1
−ε − 1
−ε
1 − ε
Re z
-2
-1
0
1
2
-1
-2



Im z
C, C'
2
1
C
$-\varepsilon - 2$
$-\varepsilon - 1$
$1 - \varepsilon$
C'
$-\varepsilon$
Re z
-2
-1
0
1
2
-1
-2

Take a residue at $z=-\epsilon$:

$$\mathrm{i}\pi^2 \frac{\Gamma(\epsilon)}{(m^2)^\epsilon(1-\epsilon)}$$

and shift the contour:

$$\frac{\mathrm{i}\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon}\frac{1}{2\pi\mathrm{i}}\int_{C'}\mathrm{d}z\left(\frac{m^2}{-q^2}\right)^z\frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)}$$

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$\Gamma(\epsilon{\color{red}+z})\Gamma({\color{blue}-z}) \to \Gamma(\epsilon)$

Take a residue at $z = -\epsilon$:

$$\mathrm{i}\pi^2 \frac{\Gamma(\epsilon)}{(m^2)^\epsilon (1-\epsilon)}$$

and shift the contour:

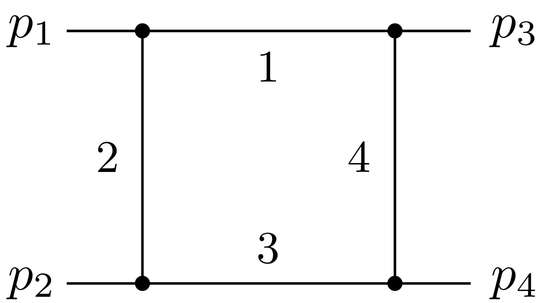
$$\frac{\mathrm{i}\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon} \frac{1}{2\pi\mathrm{i}} \int_{C'} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)}$$

$$\Gamma(\epsilon+z)\Gamma(-z) \to \Gamma(\epsilon)$$

NB:

$$\Gamma(\epsilon+z)\Gamma(1-\epsilon-z) = -\Gamma(1+\epsilon+z)\Gamma(-\epsilon-z)$$

Example 2. The massless on-shell box diagram, i.e. with $p_i^2 = 0, \; i = 1, 2, 3, 4$



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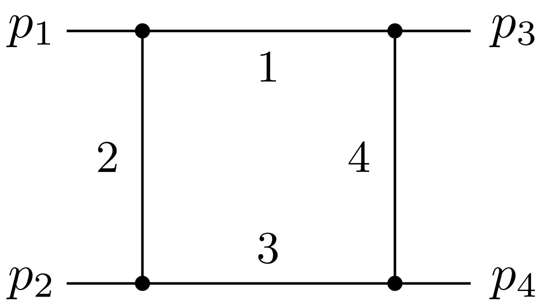

$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d)$$

$$= \int \frac{\mathrm{d}^d k}{(-k^2)^{a_1}[-(k+p_1)^2]^{a_2}[-(k+p_1+p_2)^2]^{a_3}[-(k-p_3)^2]^{a_4}} \; ,$$

where $s = (p_1+p_2)^2$ and $t = (p_1+p_3)^2$

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \; , \quad \mathcal{V} = t\alpha_1\alpha_3 + s\alpha_2\alpha_4 \; .$$

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$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) = \mathrm{i}\pi^{d/2}\frac{\Gamma(a+\epsilon-2)}{\prod \Gamma(a_l)}$$

$$\times \int_0^\infty \ldots \int_0^\infty \frac{\delta\left(\sum_{l=1}^4 \alpha_l - 1\right)}{(-t\alpha_1\alpha_3 - s\alpha_2\alpha_4)^{a+\epsilon-2}} \prod_l \alpha_l^{a_l-1} \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_4 \; ,$$

$a = a_1 + \ldots + a_4$.

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \;, \quad \mathcal{V} = t\alpha_1\alpha_3 + s\alpha_2\alpha_4 \;.$$

$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) = \mathrm{i}\pi^{d/2}\frac{\Gamma(a+\epsilon-2)}{\prod\Gamma(a_l)}$$

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Introduce new variables by $\alpha_1 = \eta_1\xi_1,\ \alpha_2 = \eta_1(1-\xi_1),\ \alpha_3 = \eta_2\xi_2,\ \alpha_4 = \eta_2(1-\xi_2)$, with the Jacobian $\eta_1\eta_2$

$$
\begin{aligned}
& F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) \\
= \; & \mathrm{i}\pi^{d/2} \frac{\Gamma(a+\epsilon-2)\Gamma(2-\epsilon-a_1-a_2)\Gamma(2-\epsilon-a_3-a_4)}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)} \\
\times & \int_0^1 \int_0^1 \frac{\xi_1^{a_1-1}(1-\xi_1)^{a_2-1}\xi_2^{a_3-1}(1-\xi_2)^{a_4-1}}{[-s\xi_1\xi_2 - t(1-\xi_1)(1-\xi_2) - \mathrm{i}0]^{a+\epsilon-2}} \, \mathrm{d}\xi_1 \mathrm{d}\xi_2
\end{aligned}
$$

$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d)$$

$$= \mathrm{i}\pi^{d/2} \frac{\Gamma(a+\epsilon-2)\Gamma(2-\epsilon-a_1-a_2)\Gamma(2-\epsilon-a_3-a_4)}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)}$$

$$\times \int_0^1 \int_0^1 \frac{\xi_1^{a_1-1}(1-\xi_1)^{a_2-1}\xi_2^{a_3-1}(1-\xi_2)^{a_4-1}}{[-s\xi_1\xi_2 - t(1-\xi_1)(1-\xi_2) - \mathrm{i}0]^{a+\epsilon-2}} \, \mathrm{d}\xi_1 \mathrm{d}\xi_2$$

Apply the basic formula to separate $-s\xi_1\xi_2$ and $-t(1-\xi_1)(1-\xi_2)$ in the denominator

$$
F_\Gamma(s,t;a_1,a_2,a_3,a_4,d)
$$

$$
= \mathrm{i}\pi^{d/2} \frac{\Gamma(a+\epsilon-2)\Gamma(2-\epsilon-a_1-a_2)\Gamma(2-\epsilon-a_3-a_4)}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)}
$$

$$
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$$

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Change the order of integration over z and ξ-parameters, evaluate parametric integrals in terms of gamma functions

$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) = \frac{\mathrm{i}\pi^{d/2}}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)(-s)^{a+\epsilon-2}}$$

$$\times\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z\left(\frac{t}{s}\right)^z\Gamma(a+\epsilon-2+z)\Gamma(a_2+z)\Gamma(a_4+z)\Gamma(-z)$$

$$\times\Gamma(2-a_1-a_2-a_4-\epsilon-z)\Gamma(2-a_2-a_3-a_4-\epsilon-z)$$

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$$F_\Gamma(s,t;1,1,1,1,d) = \frac{\mathrm{i}\pi^{d/2}}{\Gamma(-2\epsilon)(-s)^{2+\epsilon}}$$

$$\times \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{t}{s}\right)^z \Gamma(2+\epsilon+z)\Gamma(1+z)^2\Gamma(-1-\epsilon-z)^2\Gamma(-z)$$

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Take minus residue at $z=-1-\epsilon$ and shift the contour

Example 3.

$$f(\epsilon) = \frac{1}{2\pi \mathrm{i}} \int_{C_1} \int_{C_2} \Gamma(\epsilon + z_1 + z_2)\Gamma(-z_1)\Gamma(-z_2) x^{z_1} y^{z_2} \mathrm{d}z_1 \mathrm{d}z_2$$

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Shift contours and take residues

$$
\begin{aligned}
f(\epsilon) &= \frac{1}{2\pi \mathrm{i}} \int_{C_1} \int_{C_2'} \Gamma(\epsilon + z_1 + z_2)\Gamma(-z_1)\Gamma(-z_2) x^{z_1} y^{z_2} \mathrm{d}z_1 \mathrm{d}z_2 \\
&+ \frac{1}{2\pi \mathrm{i}} \int_{C_1} \Gamma(\epsilon + z_1)\Gamma(-z_1) x^{z_1} y^{\epsilon - z_1} \mathrm{d}z_1 \ ,
\end{aligned}
$$

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$$\frac{1}{2\pi \mathrm{i}} \int_{C_1} \Gamma(\epsilon + z_1)\Gamma(-z_1) x^{z_1} y^{\epsilon - z_1} \mathrm{d}z_1$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{C_1'} \Gamma(\epsilon + z_1)\Gamma(-z_1) x^{z_1} y^{\epsilon - z_1} \mathrm{d}z_1 + \Gamma(\epsilon) x^{-\epsilon} y^{2\epsilon}$$

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Two strategies: Strategy A and Strategy B

- Strategy A [V.A. Smirnov'99]

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Analysis of the integrand. Think of integrations over z-variables in various orders.

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The general rule: $\Gamma(a+z)\Gamma(b-z)$, where a and b depend on the rest of the variables, generates a pole of the type $\Gamma(a+b)$

Identifying key gamma functions (responsible for the generation of poles in ϵ).

Let $\Gamma(A_i)$ with $A_i = a_i + b_i\epsilon + \sum_j c_{ij} z_j$
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Re$A_i > 0 \;\rightarrow -1 <$ Re $A_i < 0$
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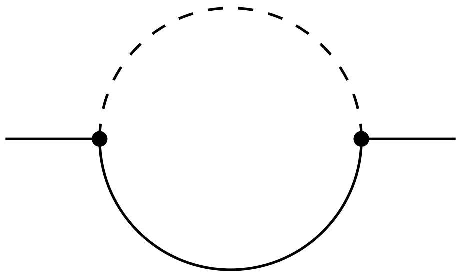
Taking residues and shifting contours.

For each resulting residue, which involves one integration less, apply a similar procedure, etc.

- Strategy B [J.B. Tausk'99, Anastasiou'05, Czakon'05].

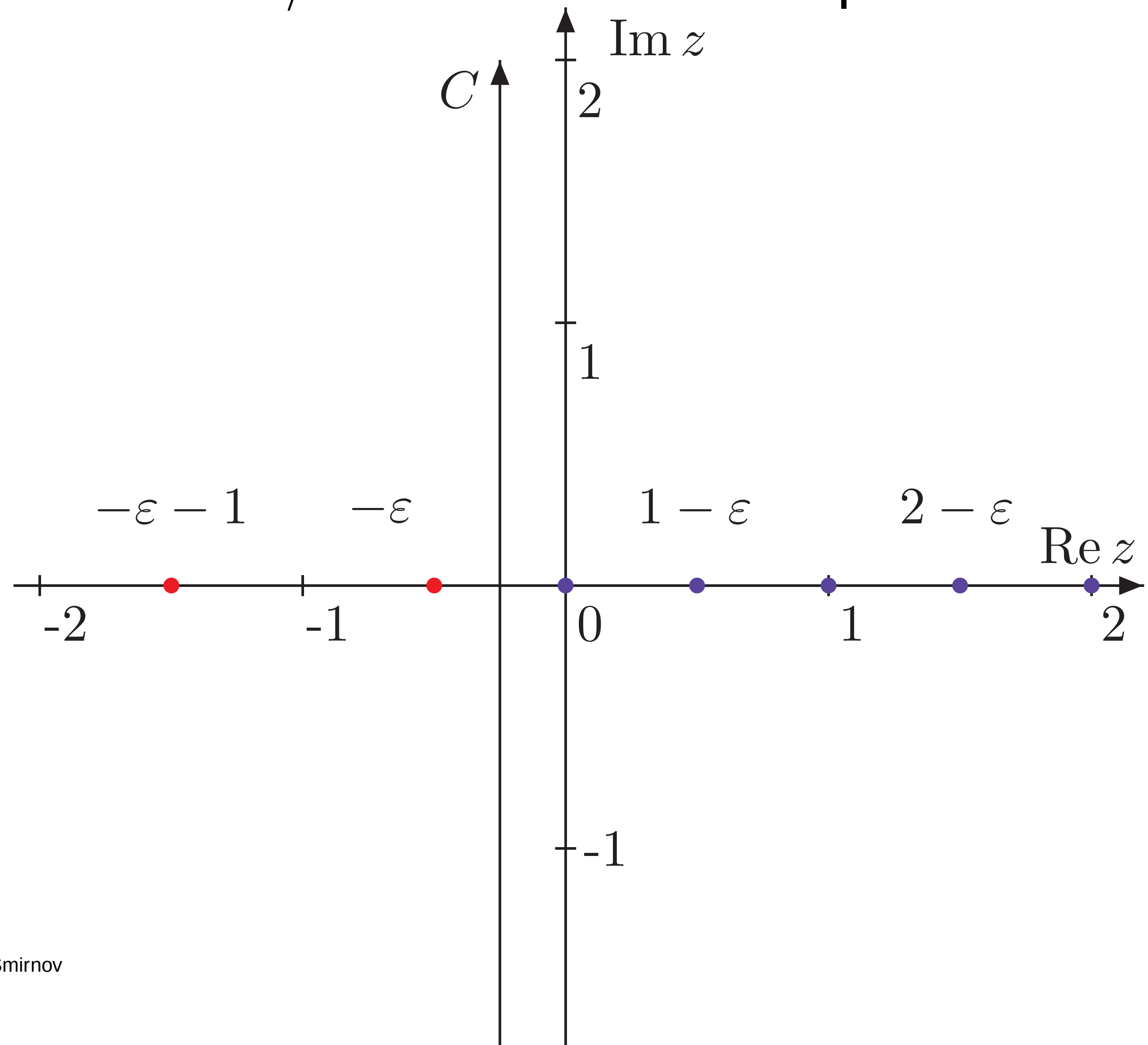
Example 1 (again)



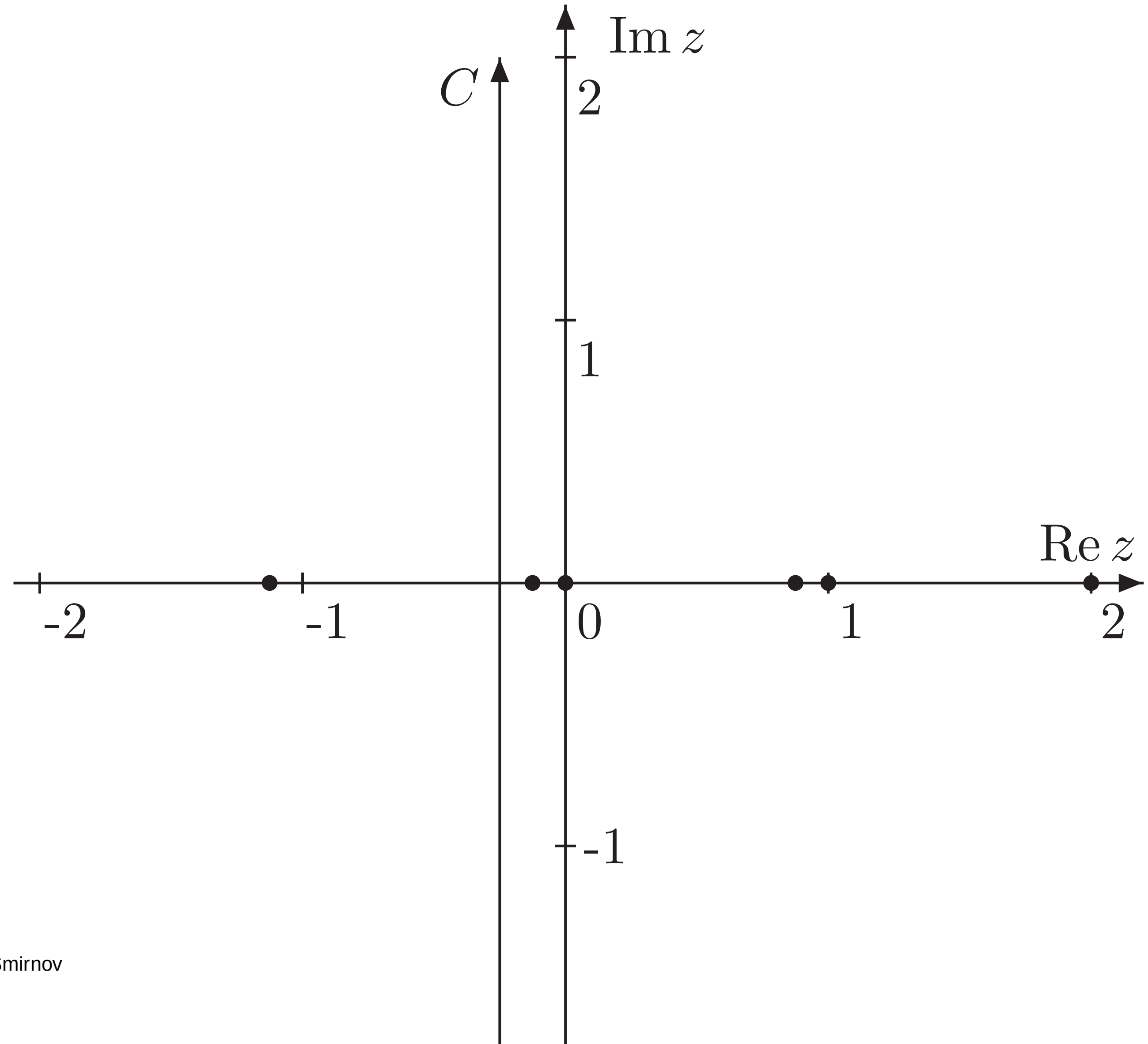
$$F_\Gamma(q^2, m^2; 1, 1, d) = \frac{i\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon}$$

$$\times \frac{1}{2\pi i}\int_C \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)}$$

Take ϵ real. Choose ϵ and a straight contour, e.g. $\epsilon = 1/2$, Re$z = -1/4$. The contour is kept fixed. Tend ϵ to zero.



Whenever a pole of some gamma function is crossed add a residue and tend ϵ to zero further



- Strategy B [J.B. Tausk'99, Anastasiou'05, Czakon'05].

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Choose a domain of ϵ and Rez_i,...Rew_i in such a way that *all* the integrations over the MB variables can be performed over straight lines parallel to imaginary axis.

Let $\epsilon \to 0$. Whenever a pole of some gamma function is crossed, take into account the corresponding residue.

Strategy B [J.B. Tausk'99, Anastasiou'05, Czakon'05].

Choose a domain of ϵ and Rez_i,. . . Rew_i in such a way that *all* the integrations over the MB variables can be performed over straight lines parallel to imaginary axis.

Let $\epsilon \to 0$. Whenever a pole of some gamma function is crossed, take into account the corresponding residue.

For every resulting residue, which involves one integration less, apply a similar procedure, etc.

Two algorithmic descriptions [C. Anastasiou'05, M. Czakon'05]

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The Czakon’s version MB.m implemented in `Mathematica` is public.
http://projects.hepforge.org/mbtools/

In[2]:= `<< MB/MB.m`

```
MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08
```

```
(*   The integrand of the MB integral for the one-
  loop propagator diagram with m1=m and m2=0 *)
```

In[3]:=
```
MB[a1_, a2_] := (-1)^(a1 + a2) / QQ^(a1 + a2 + ep - 2) / Gamma[a1] / Gamma[a2]
   Gamma[2 - ep - a2] Gamma[a1 + a2 + ep - 2 + z] Gamma[2 - ep - a1 - z]
   Gamma[-z] / Gamma[4 - 2 ep - a1 - a2 - z] mm^z / QQ^z;
```

```
(*  Notation: mm=m^2, QQ=-q^2;
I Pi^(d/2) is pulled out  *)
```

```
(*  The diagram with a1=1 and a2=1  *)
```

In[4]:= `P1 = MB[1, 1]`

Out[4]= $\frac{\text{mm}^{z}\,\text{QQ}^{-\text{ep}-z}\,\text{Gamma}[1-\text{ep}]\,\text{Gamma}[1-\text{ep}-z]\,\text{Gamma}[-z]\,\text{Gamma}[\text{ep}+z]}{\text{Gamma}[2-2\,\text{ep}-z]}$

In[5]:= `P1Rules = MBoptimizedRules[P1, ep → 0, {}, {ep}]`

MBrules::norules : no rules could be found to regulate this integral

Out[5]= $\left\{\left\{\text{ep}\to\frac{3}{4}\right\},\left\{z\to-\frac{1}{2}\right\}\right\}$

In[6]:= `P1cont = MBcontinue[P1, ep → 0, P1Rules]`

```
Level 1

Taking +residue in z = -ep

Level 2

Integral {1}

2 integral(s) found
```

Out[6]= $\left\{\left\{\text{MBint}\left[\frac{\text{mm}^{-\text{ep}}\,\text{Gamma}[1-\text{ep}]\,\text{Gamma}[\text{ep}]}{\text{Gamma}[2-\text{ep}]},\{\{\text{ep}\to 0\},\{\}\}\right]\right\},\right.$
$\left.\text{MBint}\left[\frac{\text{mm}^{z}\,\text{QQ}^{-\text{ep}-z}\,\text{Gamma}[1-\text{ep}]\,\text{Gamma}[1-\text{ep}-z]\,\text{Gamma}[-z]\,\text{Gamma}[\text{ep}+z]}{\text{Gamma}[2-2\,\text{ep}-z]},\left\{\{\text{ep}\to 0\},\left\{z\to-\frac{1}{2}\right\}\right\}\right]\right\}$

In[7]:= `P1select = MBpreselect[P1cont, {ep, 0, 0}]`

Out[7]= $\left\{\text{MBint}\left[\frac{\text{mm}^{-\text{ep}}\,\text{Gamma}[1-\text{ep}]\,\text{Gamma}[\text{ep}]}{\text{Gamma}[2-\text{ep}]},\{\{\text{ep}\to 0\},\{\}\}\right],\right.$
$\left.\text{MBint}\left[\frac{\text{mm}^{z}\,\text{QQ}^{-\text{ep}-z}\,\text{Gamma}[1-\text{ep}]\,\text{Gamma}[1-\text{ep}-z]\,\text{Gamma}[-z]\,\text{Gamma}[\text{ep}+z]}{\text{Gamma}[2-2\,\text{ep}-z]},\left\{\{\text{ep}\to 0\},\left\{z\to-\frac{1}{2}\right\}\right\}\right]\right\}$

In[8]:= `P1exp = MBexpand[P1select, E^(EulerGamma ep), {ep, 0, 0}]`

Out[8]= $\left\{\text{MBint}\left[1+\frac{1}{\text{ep}}-\text{Log}[\text{mm}],\{\{\text{ep}\to 0\},\{\}\}\right],\right.$
$\left.\text{MBint}\left[\frac{\text{mm}^{z}\,\text{QQ}^{-z}\,\text{Gamma}[1-z]\,\text{Gamma}[-z]\,\text{Gamma}[z]}{\text{Gamma}[2-z]},\left\{\{\text{ep}\to 0\},\left\{z\to-\frac{1}{2}\right\}\right\}\right]\right\}$

In[9]:= **P1exp[[1]]**

Out[9]= $\text{MBint}\left[1+\frac{1}{\text{ep}}-\text{Log}[\text{mm}],\ \{\{\text{ep}\to 0\},\ \{\}\}\right]$

In[10]:= **P1exp[[2]]**

Out[10]= $\text{MBint}\left[\frac{\text{mm}^z\,\text{QQ}^{-z}\,\text{Gamma}[1-z]\,\text{Gamma}[-z]\,\text{Gamma}[z]}{\text{Gamma}[2-z]},\ \left\{\{\text{ep}\to 0\},\ \left\{z\to -\frac{1}{2}\right\}\right\}\right]$

In[11]:= **int1 = P1exp[[2]] [[1]] //. {Gamma[2 - z] → (1 - z) Gamma[1 - z], Gamma[1 - z] → -z Gamma[-z], Gamma[z] → Gamma[1 + z] / z}**

Out[11]= $\frac{\text{mm}^z\,\text{QQ}^{-z}\,\text{Gamma}[-z]\,\text{Gamma}[1+z]}{(1-z)\,z}$

(* The MB integral can be evaluated by closing the integration contour to the right in the complex z-plane. *)

(* First two residues *)

In[12]:= **res12 = -Residue[int1, {z, 0}] - Residue[int1, {z, 1}]**

Out[12]= $1+\text{Log}[\text{mm}]-\text{Log}[\text{QQ}]-\frac{\text{mm}-\text{mm}\,\text{Log}[\text{mm}]+\text{mm}\,\text{Log}[\text{QQ}]}{\text{QQ}}$

(* Now we take residues at z=2,3,... *)

In[13]:= **int1 /. {Gamma[-z] Gamma[1 + z] → -π Csc[π z]}**

Out[13]= $-\frac{\text{mm}^z\,\pi\,\text{QQ}^{-z}\,\text{Csc}[\pi z]}{(1-z)\,z}$

(* Now we take (minus) the residue at z=n,
To do this we shift the variable and then take a residue at z=0 *)

In[14]:= **% /. z → z + n**

Out[14]= $-\frac{\text{mm}^{n+z}\,\pi\,\text{QQ}^{-n-z}\,\text{Csc}[\pi\,(n+z)]}{(1-n-z)\,(n+z)}$

In[15]:= **% /. {Csc[π (n + z)] → (-1)^n Csc[π z] }**

Out[15]= $-\frac{(-1)^n\,\text{mm}^{n+z}\,\pi\,\text{QQ}^{-n-z}\,\text{Csc}[\pi z]}{(1-n-z)\,(n+z)}$

In[16]:= **-Residue[%, {z, 0}]**

Out[16]= $-\frac{(-1)^n\,\text{mm}^n\,\text{QQ}^{-n}}{(-1+n)\,n}$

(* Sum up contributions of the residues at z=2,3,... *)

In[17]:= **Sum[%, {n, 2, Infinity}]**

Out[17]= $-\frac{-\text{mm}+\text{mm}\,\text{Log}\left[1+\frac{\text{mm}}{\text{QQ}}\right]+\text{QQ}\,\text{Log}\left[1+\frac{\text{mm}}{\text{QQ}}\right]}{\text{QQ}}$

(* We add the above contributions from the residues at z=0 and z=1 as well as B1exp[[1]] *)

In[18]:=
```
Simplify[% + res12 + P1exp[[1]][[1]]]
```

Out[18]=
$$\frac{1}{ep\,QQ}\left(QQ + 2\,ep\,QQ + ep\,mm\,Log[mm] - ep\,mm\,Log[QQ] - ep\,QQ\,Log[QQ] - ep\,(mm + QQ)\,Log\left[\frac{mm + QQ}{QQ}\right]\right)$$

In[19]:=
$$\%\ /.\ Log\left[\frac{mm + QQ}{QQ}\right] \rightarrow Log[mm + QQ] - Log[QQ]$$

Out[19]=
$$\frac{1}{ep\,QQ}$$
$$(QQ + 2\,ep\,QQ + ep\,mm\,Log[mm] - ep\,mm\,Log[QQ] - ep\,QQ\,Log[QQ] - ep\,(mm + QQ)\,(-Log[QQ] + Log[mm + QQ]))$$

In[20]:=
```
res = % /. Log[mm + QQ] → Log[(mm + QQ) / mm] + Log[mm]
```

Out[20]=
$$\frac{1}{ep\,QQ}\left(QQ + 2\,ep\,QQ + ep\,mm\,Log[mm] - ep\,mm\,Log[QQ] -\right.$$
$$\left.ep\,QQ\,Log[QQ] - ep\,(mm + QQ)\,\left(Log[mm] - Log[QQ] + Log\left[\frac{mm + QQ}{mm}\right]\right)\right)$$

```
(*  result            *)
```

In[21]:=
$$resSimpl = 2 + \frac{1}{ep} - Log[mm] - \left(1 + \frac{mm}{QQ}\right)\,Log\left[1 + \frac{QQ}{mm}\right];$$

```
(*   check   *)
```

In[22]:=
```
Simplify[res - resSimpl]
```

Out[22]= 0

In[2]:= `<< MB/MB.m`

```
MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08
```

(* The integrand of the MB integral for the one-loop massless box diagram with p1^2= p2^2=p3^2=p4^2=0 *)

In[3]:= **Box1[a1_, a2_, a3_, a4_] :=**

$$\left(S^{2-a1-a2-a3-a4-ep-z}\,T^z\,\text{Gamma}[a1+a2+a3+a4-2+ep+z]\,\text{Gamma}[a2+z]\,\text{Gamma}[a4+z]\,\text{Gamma}[2-a1-a2-a4-ep-z]\,\text{Gamma}[2-a2-a3-a4-ep-z]\,\text{Gamma}[-z]\right)\Big/\left(\text{Gamma}[a1]\,\text{Gamma}[a2]\,\text{Gamma}[a3]\,\text{Gamma}[a4]\,\text{Gamma}[4-a1-a2-a3-a4-2\,ep]\right);$$

(* Notation:
s=(p1+p2)^2=-S, t=(p1+p2)^2=-T;
I Pi^(d/2) is pulled out, as always *)

(* The box with the powers of the propagators equal to one *)

In[4]:= **Box1[1, 1, 1, 1]**

Out[4]=
$$\frac{S^{-2-ep-z}\,T^z\,\text{Gamma}[-1-ep-z]^2\,\text{Gamma}[-z]\,\text{Gamma}[1+z]^2\,\text{Gamma}[2+ep+z]}{\text{Gamma}[-2\,ep]}$$

In[5]:= **B1 = % /. {S → 1, T → x}**

Out[5]=
$$\frac{x^z\,\text{Gamma}[-1-ep-z]^2\,\text{Gamma}[-z]\,\text{Gamma}[1+z]^2\,\text{Gamma}[2+ep+z]}{\text{Gamma}[-2\,ep]}$$

In[6]:= **B1Rules = MBoptimizedRules[B1, ep → 0, {}, {ep}]**

MBrules::norules : no rules could be found to regulate this integral

Out[6]=
$$\left\{\{ep\to -1\},\ \left\{z\to -\frac{1}{2}\right\}\right\}$$

In[7]:= **B1cont = MBcontinue[B1, ep → 0, B1Rules]**

```
Level 1

Taking -residue in z = -1 - ep

Level 2

Integral {1}

2 integral(s) found
```

Out[7]=
$$\left\{\left\{\text{MBint}\left[-\frac{\text{EulerGamma}\,x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]}{\text{Gamma}[-2\,ep]}-\frac{x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{Log}[x]}{\text{Gamma}[-2\,ep]}-\frac{2\,x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{PolyGamma}[0,-ep]}{\text{Gamma}[-2\,ep]}+\frac{x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{PolyGamma}[0,1+ep]}{\text{Gamma}[-2\,ep]},\ \{\{ep\to 0\},\{\}\}\right]\right\},\right.$$
$$\left.\text{MBint}\left[\frac{x^z\,\text{Gamma}[-1-ep-z]^2\,\text{Gamma}[-z]\,\text{Gamma}[1+z]^2\,\text{Gamma}[2+ep+z]}{\text{Gamma}[-2\,ep]},\ \left\{\{ep\to 0\},\ \left\{z\to -\frac{1}{2}\right\}\right\}\right]\right\}$$

In[8]:= **B1select = MBpreselect[B1cont, {ep, 0, 0}]**

Out[8]= $\left\{\text{MBint}\left[-\frac{\text{EulerGamma}\,x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]}{\text{Gamma}[-2\,ep]}-\frac{x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{Log}[x]}{\text{Gamma}[-2\,ep]}-\frac{2\,x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{PolyGamma}[0,-ep]}{\text{Gamma}[-2\,ep]}+\frac{x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{PolyGamma}[0,1+ep]}{\text{Gamma}[-2\,ep]},\{\{ep\to 0\},\{\}\}\right]\right\}$

In[9]:= **B1select = MBpreselect[B1cont, {ep, 0, 1}]**

Out[9]= $\left\{\text{MBint}\left[-\frac{\text{EulerGamma}\,x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]}{\text{Gamma}[-2\,ep]}-\frac{x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{Log}[x]}{\text{Gamma}[-2\,ep]}-\frac{2\,x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{PolyGamma}[0,-ep]}{\text{Gamma}[-2\,ep]}+\frac{x^{-1-ep}\,\text{Gamma}[-ep]^2\,\text{Gamma}[1+ep]\,\text{PolyGamma}[0,1+ep]}{\text{Gamma}[-2\,ep]},\{\{ep\to 0\},\{\}\}\right],\ \text{MBint}\left[\frac{x^z\,\text{Gamma}[-1-ep-z]^2\,\text{Gamma}[-z]\,\text{Gamma}[1+z]^2\,\text{Gamma}[2+ep+z]}{\text{Gamma}[-2\,ep]},\left\{\{ep\to 0\},\left\{z\to -\frac{1}{2}\right\}\right\}\right]\right\}$

In[10]:= **B1exp = MBexpand[B1select, E^(EulerGamma ep), {ep, 0, 1}]**

Out[10]= $\left\{\text{MBint}\left[\frac{4}{ep^2\,x}-\frac{4\,\pi^2}{3\,x}-\frac{2\,\text{Log}[x]}{ep\,x}+\frac{7\,ep\,\pi^2\,\text{Log}[x]}{6\,x}+\frac{ep\,\text{Log}[x]^3}{3\,x}+\frac{17\,ep\,\text{PolyGamma}[2,1]}{3\,x},\{\{ep\to 0\},\{\}\}\right],\ \text{MBint}\left[-2\,ep\,x^z\,\text{Gamma}[-1-z]^2\,\text{Gamma}[-z]\,\text{Gamma}[1+z]^2\,\text{Gamma}[2+z],\left\{\{ep\to 0\},\left\{z\to -\frac{1}{2}\right\}\right\}\right]\right\}$

In[11]:= **res1 = B1exp[[1]][[1]]**

Out[11]= $\frac{4}{ep^2\,x}-\frac{4\,\pi^2}{3\,x}-\frac{2\,\text{Log}[x]}{ep\,x}+\frac{7\,ep\,\pi^2\,\text{Log}[x]}{6\,x}+\frac{ep\,\text{Log}[x]^3}{3\,x}+\frac{17\,ep\,\text{PolyGamma}[2,1]}{3\,x}$

In[12]:= **B1exp[[2]]**

Out[12]= $\text{MBint}\left[-2\,ep\,x^z\,\text{Gamma}[-1-z]^2\,\text{Gamma}[-z]\,\text{Gamma}[1+z]^2\,\text{Gamma}[2+z],\left\{\{ep\to 0\},\left\{z\to -\frac{1}{2}\right\}\right\}\right]$

In[13]:= **int1 = B1exp[[2]][[1]]**

Out[13]= $-2\,ep\,x^z\,\text{Gamma}[-1-z]^2\,\text{Gamma}[-z]\,\text{Gamma}[1+z]^2\,\text{Gamma}[2+z]$

In[14]:= **Simplify[% //. { Gamma[-1 - z] → Gamma[-z] / (-1 - z), Gamma[2 + z] → Gamma[1 + z] (1 + z) }]**

Out[14]= $-\frac{2\,ep\,x^z\,\text{Gamma}[-z]^3\,\text{Gamma}[1+z]^3}{1+z}$

In[15]:= **% /. {Gamma[-z]^3 Gamma[1 + z]^3 → -π^3 Csc[π z]^3}**

Out[15]= $\frac{2\,ep\,\pi^3\,x^z\,\text{Csc}[\pi\,z]^3}{1+z}$

(* Now we take residues at z=0,1,2,.,.. *)

In[16]:= **% /. z → z + n**

Out[16]= $\frac{2\,ep\,\pi^3\,x^{n+z}\,\text{Csc}[\pi\,(n+z)]^3}{1+n+z}$

In[17]:= `% /. {Csc[π (n + z)] → (-1)^n Csc[π z] }`

Out[17]= $\frac{2\,(-1)^{3\,n}\,\mathrm{ep}\,\pi^3\,x^{n+z}\,\mathrm{Csc}[\pi\,z]^3}{1+n+z}$

In[18]:= `% /. { (-1)^(3 n) → (-1)^n}`

Out[18]= $\frac{2\,(-1)^{n}\,\mathrm{ep}\,\pi^3\,x^{n+z}\,\mathrm{Csc}[\pi\,z]^3}{1+n+z}$

In[19]:= `-Residue[%, {z, 0}]`

Out[19]= $-\frac{1}{(1+n)^3}(-1)^n\,\mathrm{ep}\,x^n\left(2+\pi^2+2\,n\,\pi^2+n^2\,\pi^2-2\,\mathrm{Log}[x]-2\,n\,\mathrm{Log}[x]+\mathrm{Log}[x]^2+2\,n\,\mathrm{Log}[x]^2+n^2\,\mathrm{Log}[x]^2\right)$

In[20]:= `Apart[% /. n → n - 1, n]`

Out[20]= $\frac{2\,(-1)^n\,\mathrm{ep}\,x^{-1+n}}{n^3}-\frac{2\,(-1)^n\,\mathrm{ep}\,x^{-1+n}\,\mathrm{Log}[x]}{n^2}+\frac{(-1)^n\,\mathrm{ep}\,x^{-1+n}\left(\pi^2+\mathrm{Log}[x]^2\right)}{n}$

In[21]:= `res2 = Sum[%, {n, 1, Infinity}]`

Out[21]= $\frac{1}{x}\left(-\mathrm{ep}\,\pi^2\,\mathrm{Log}[1+x]-\mathrm{ep}\,\mathrm{Log}[x]^2\,\mathrm{Log}[1+x]-2\,\mathrm{ep}\,\mathrm{Log}[x]\,\mathrm{PolyLog}[2,-x]+2\,\mathrm{ep}\,\mathrm{PolyLog}[3,-x]\right)$

`(*  Numerical check   *)`

In[22]:= `% / ep /. {x → 0.76, ep → 0.3}`

Out[22]= -9.70976

In[23]:= `NIntegrate[int1 / ep /. {x → 0.76, ep → 0.3, z → -0.5 + I * y1}, {y1, -Infinity, Infinity}] / 2 / Pi`

Out[23]= $-9.70976+2.67167\times10^{-13}\,i$

In[24]:= `res1 + res2`

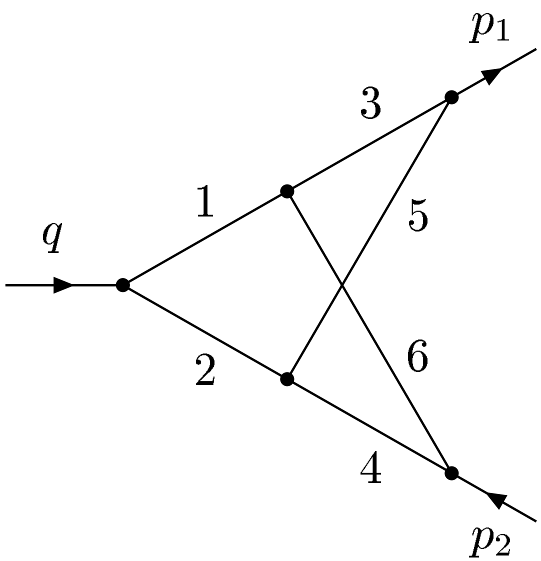
Out[24]= $\frac{4}{\mathrm{ep}^2\,x}-\frac{4\,\pi^2}{3\,x}-\frac{2\,\mathrm{Log}[x]}{\mathrm{ep}\,x}+\frac{7\,\mathrm{ep}\,\pi^2\,\mathrm{Log}[x]}{6\,x}+\frac{\mathrm{ep}\,\mathrm{Log}[x]^3}{3\,x}+\frac{17\,\mathrm{ep}\,\mathrm{PolyGamma}[2,1]}{3\,x}+\frac{1}{x}\left(-\mathrm{ep}\,\pi^2\,\mathrm{Log}[1+x]-\mathrm{ep}\,\mathrm{Log}[x]^2\,\mathrm{Log}[1+x]-2\,\mathrm{ep}\,\mathrm{Log}[x]\,\mathrm{PolyLog}[2,-x]+2\,\mathrm{ep}\,\mathrm{PolyLog}[3,-x]\right)$

`(*  This is our result (up to I Pi^(d/2) ) *)`

In[25]:= `(% /. x → T / S ) S^(-2-ep)`

Out[25]= $S^{-2-\mathrm{ep}}\left(\frac{4\,S}{\mathrm{ep}^2\,T}-\frac{4\,\pi^2\,S}{3\,T}-\frac{2\,S\,\mathrm{Log}\left[\frac{T}{S}\right]}{\mathrm{ep}\,T}+\frac{7\,\mathrm{ep}\,\pi^2\,S\,\mathrm{Log}\left[\frac{T}{S}\right]}{6\,T}+\frac{\mathrm{ep}\,S\,\mathrm{Log}\left[\frac{T}{S}\right]^3}{3\,T}+\frac{17\,\mathrm{ep}\,S\,\mathrm{PolyGamma}[2,1]}{3\,T}+\frac{1}{T}S\left(-\mathrm{ep}\,\pi^2\,\mathrm{Log}\left[1+\frac{T}{S}\right]-\mathrm{ep}\,\mathrm{Log}\left[\frac{T}{S}\right]^2\mathrm{Log}\left[1+\frac{T}{S}\right]-2\,\mathrm{ep}\,\mathrm{Log}\left[\frac{T}{S}\right]\mathrm{PolyLog}\left[2,-\frac{T}{S}\right]+2\,\mathrm{ep}\,\mathrm{PolyLog}\left[3,-\frac{T}{S}\right]\right)\right)$

Non-planar two-loop massless vertex diagram with
$p_1^2 = p_2^2 = 0$, $Q^2 = -(p_1 - p_2)^2 = 2p_1 \cdot p_2$



$$F_\Gamma(Q^2; a_1, \ldots, a_6, d) = \int\int \frac{\mathrm{d}^d k \, \mathrm{d}^d l}{[(k+l)^2 - 2p_1 \cdot (k+l)]^{a_1}}$$

$$\times \frac{1}{[(k+l)^2 - 2p_2 \cdot (k+l)]^{a_2} (k^2 - 2p_1 \cdot k)^{a_3} (l^2 - 2p_2 \cdot l)^{a_4} (k^2)^{a_5} (l^2)^{a_6}}$$

$$\frac{1}{(k^2 - 2p_1 \cdot k)^{a_3} (k^2)^{a_5}} = \frac{(-1)^{a_3+a_5} \Gamma(a_3 + a_5)}{\Gamma(a_3)\Gamma(a_5)} \times \int_0^1 \frac{\mathrm{d}\xi_1 \, \xi_1^{a_3-1} (1 - \xi_1)^{a_5-1}}{[-(k - \xi_1 p_1)^2 - \mathrm{i}0]^{a_3+a_5}}$$

and, similarly, for the second pair, with the replacements

$$\xi_1 \to \xi_2, \; p_1 \to p_2, \; k \to l, \; a_3 \to a_4, \; a_5 \to a_6$$

Change the integration variable $l \to r = k + l$ and integrate over k by means of our massless one-loop formula

$$\int \frac{\mathrm{d}k}{[-(k-\xi_1 p_1)^2]^{a_3+a_5}[-(r-\xi_2 p_2-k)^2]^{a_4+a_6}}$$

$$= \mathrm{i}\pi^{d/2} \frac{G(a_3+a_5, a_4+a_6)}{[-(r-\xi_1 p_1-\xi_2 p_2)^2]^{a_3+a_4+a_5+a_6+\epsilon-2}}$$

Apply Feynman parametric formula to the propagators 1 and 2 and the propagator arising from the previous integration, with a resulting integral over r evaluated in terms of gamma functions:

$$\int \frac{\mathrm{d}^d r}{[-(r^2-Q^2 A(\xi_1,\xi_2,\xi_3,\xi_4))]^{a+\epsilon-2}}$$

$$= \mathrm{i}\pi^{d/2} \frac{\Gamma(a+2\epsilon-4)}{\Gamma(a+\epsilon-2)} \frac{1}{(Q^2)^{a+2\epsilon-4} A(\xi_1,\xi_2,\xi_3,\xi_4)^{a+2\epsilon-}}$$

where $a = a_1 + \ldots + a_6$ and

$$A(\xi_1,\xi_2,\xi_3,\xi_4) = \xi_3\xi_4 + (1-\xi_3-\xi_4)[\xi_2\xi_3(1-\xi_1)+\xi_1\xi_4(1-\xi_2)]$$

Gonsalves'83:

$$F_\Gamma(Q^2;a_1,\ldots,a_6,d) \; = \; \frac{(-1)^a\left(\mathrm{i}\pi^{d/2}\right)^2\Gamma(2-\epsilon-a_{35})\Gamma(2-\epsilon-a_{46})}{(Q^2)^{a+2\epsilon-4}\prod\Gamma(a_l)\;\Gamma(4-2\epsilon-a_{3456})}$$

$$\times\Gamma(a+2\epsilon-4)\int_0^1 \mathrm{d}\xi_1\ldots\int_0^1 \mathrm{d}\xi_4\,\xi_1^{a_3-1}(1-\xi_1)^{a_5-1}\xi_2^{a_4-1}(1-\xi_2)^{a_6-1}$$

$$\times\xi_3^{a_1-1}\xi_4^{a_2-1}(1-\xi_3-\xi_4)_+^{a_{3456}+\epsilon-3}A(\xi_1,\xi_2,\xi_3,\xi_4)^{4-2\epsilon-a}$$

$$\frac{\Gamma(a+2\epsilon-4)}{[\eta\xi(1-\xi)+(1-\eta)(\xi\xi_2(1-\xi_1)+(1-\xi)\xi_1(1-\xi_2))]^{a+2\epsilon-4}}$$

$$= \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_1\,\Gamma(-z_1)\eta^{z_1}\xi^{z_1}(1-\xi)^{z_1}}{(1-\eta)^{a+2\epsilon-4+z_1}}$$

$$\times \frac{\Gamma(a+2\epsilon-4+z_1)}{[\xi\xi_2(1-\xi_1)+(1-\xi)\xi_1(1-\xi_2)]^{a+2\epsilon-4+z_1}}$$

The last line $\to$

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_2\,\Gamma(a+2\epsilon-4+z_1+z_2)\Gamma(-z_2)\xi^{z_2}\xi_2^{z_2}(1-\xi_1)^{z_2}}{(1-\xi)^{a+2\epsilon-4+z_1+z_2}\xi_1^{a+2\epsilon-4+z_1+z_2}(1-\xi_2)^{a+2\epsilon-4+z_1+z_2}}$$

$$F_\Gamma(Q^2; a_1, \ldots, a_6, d) = \frac{(-1)^a \left(\mathrm{i}\pi^{d/2}\right)^2 \Gamma(2-\epsilon-a_{35})}{(Q^2)^{a+2\epsilon-4}\Gamma(6-3\epsilon-a)\prod \Gamma(a_l)}$$

$$\times \frac{\Gamma(2-\epsilon-a_{46})}{\Gamma(4-2\epsilon-a_{3456})} \frac{1}{(2\pi\mathrm{i})^2} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z_1 \mathrm{d}z_2 \Gamma(a+2\epsilon-4+z_1+z_2)$$

$$\times \Gamma(-z_1)\Gamma(-z_2)\Gamma(a_4+z_2)\Gamma(a_5+z_2)\Gamma(a_1+z_1+z_2)$$

$$\times \frac{\Gamma(2-\epsilon-a_{12}-z_1)\Gamma(4-2\epsilon+a_2-a-z_2)}{\Gamma(4-2\epsilon-a_{1235}-z_1)\Gamma(4-2\epsilon-a_{1246}-z_1)}$$

$$\times \Gamma(4-2\epsilon+a_3-a-z_1-z_2)\Gamma(4-2\epsilon+a_6-a-z_1-z_2)\ ,$$

where $a_{3456} = a_3 + a_4 + a_5 + a_6$, etc.

The first Barnes lemma

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z)$$

$$= \frac{\Gamma(\lambda_1+\lambda_3)\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_3)\Gamma(\lambda_2+\lambda_4)}{\Gamma(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}$$

Multiple corollaries, e.g.,

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma(\lambda_3-z)$$

$$= \Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)\left[\psi(\lambda_1-\lambda_2)-\psi(\lambda_1+\lambda_3)\right]$$

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z}{z} \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z)$$

$$= \frac{\Gamma(2-\lambda_1-\lambda_3)\Gamma(1-\lambda_2-\lambda_3)\Gamma(\lambda_1+\lambda_3-1)\Gamma(\lambda_2+\lambda_3)}{\Gamma(1-\lambda_1)\Gamma(1-\lambda_2)}$$

$$\times \left[\Gamma(1-\lambda_1)\Gamma(1-\lambda_2) - \Gamma(2-\lambda_1-\lambda_2-\lambda_3)\Gamma(\lambda_3)\right]$$

The second Barnes lemma

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \, \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3+z)\Gamma(\lambda_4-z)\Gamma(\lambda_5-z)}{\Gamma(\lambda_6+z)}$$

$$= \frac{\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_4)\Gamma(\lambda_3+\lambda_4)\Gamma(\lambda_1+\lambda_5)}{\Gamma(\lambda_1+\lambda_2+\lambda_4+\lambda_5)\Gamma(\lambda_1+\lambda_3+\lambda_4+\lambda_5)}$$

$$\times \frac{\Gamma(\lambda_2+\lambda_5)\Gamma(\lambda_3+\lambda_5)}{\Gamma(\lambda_2+\lambda_3+\lambda_4+\lambda_5)} \,, \qquad \lambda_6 = \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5$$

```
In[2]:= << MB/MB.m
MB 1.1
by Michal Czakon
more info in hep-ph/0511200
last modified 06 Mar 08

        (*   2fold MB representation for the non-planar vertex massless diagram.
             The factor QQ^(4-a1-a2-a3-a4-a5-a6-2 ep) is omitted.
             QQ=-(p1-p2)^2.
           The factor (I Pi^(d/2))^2 is also omitted as usually.  *)

In[3]:= NPMB[a1_, a2_, a3_, a4_, a5_, a6_] := ((-1)^(a1 + a2 + a3 + a4 + a5 + a6) /
          (Gamma[a1] Gamma[a2] Gamma[a3] Gamma[a4] Gamma[a5] Gamma[a6])
        Gamma[2 - ep - a3 - a5] Gamma[2 - ep - a4 - a6] / Gamma[4 - 2 ep - a3 - a4 - a5 - a6] /
          Gamma[6 - 3 ep - a1 - a2 - a3 - a4 - a5 - a6]
        Gamma[a1 + a2 + a3 + a4 + a5 + a6 + 2 ep - 4 + z1 + z2] Gamma[-z1] Gamma[-z2]
        Gamma[2 - ep - a1 - a2 - z1] Gamma[a4 + z2] Gamma[a1 + z1 + z2]
        Gamma[4 - 2 ep - a1 - a3 - a4 - a5 - a6 - z2] Gamma[4 - 2 ep - a1 - a2 - a4 - a5 - a6 - z1 - z2]
        Gamma[a5 + z2] Gamma[4 - 2 ep - a1 - a2 - a3 - a4 - a5 - z1 - z2] /
          Gamma[4 - 2 ep - a1 - a2 - a4 - a6 - z1] / Gamma[4 - 2 ep - a1 - a2 - a3 - a5 - z1]);

        (*   The diagram with all powers of the propagators equal
         to one. We shall evaluate it in expansion in ep up to ep^0. *)

In[4]:= V2 = NPMB[1, 1, 1, 1, 1, 1]

Out[4]= (Gamma[-ep]^2 Gamma[-ep - z1] Gamma[-z1] Gamma[-1 - 2 ep - z2] Gamma[-1 - 2 ep - z1 - z2]^2
          Gamma[-z2] Gamma[1 + z2]^2 Gamma[1 + z1 + z2] Gamma[2 + 2 ep + z1 + z2]) /
        (Gamma[-3 ep] Gamma[-2 ep] Gamma[-2 ep - z1]^2)

In[5]:= V2rules = MBoptimizedRules[V2, ep -> 0, {}, {ep}]

Out[5]= {{ep -> -5/8}, {z1 -> -1/4, z2 -> -1/4}}

In[6]:= V2cont = MBcontinue[V2, ep -> 0, V2rules];

Level 1
Taking -residue in z2 = -1 - 2 ep
Taking -residue in z2 = -1 - 2 ep - z1
Level 2
Integral {1}
Taking +residue in z1 = 2 ep
Integral {2}
Level 3
Integral {1, 1}
4 integral(s) found

In[7]:= V2select = MBpreselect[MBmerge[V2cont], {ep, 0, 0}]
```

In[8]:= **V2exp = Simplify[MBexpand[V2select, Exp[2 ep EulerGamma], {ep, 0, 0}]]**

Out[8]= $\{\text{MBint}\left[\frac{1}{ep^4} - \frac{\pi^2}{2\,ep^2} - \frac{41\,\pi^4}{40} + \frac{55\,\text{PolyGamma}[2,1]}{3\,ep}, \{\{ep \to 0\}, \{\}\}\right],$

$\text{MBint}\left[\frac{1}{4\,ep^2}\text{Gamma}[-z1]^2\,\text{Gamma}[z1]\,\text{Gamma}[1+z1]\,\left(12 + 12\,ep\,\text{EulerGamma} + 6\,ep^2\,\text{EulerGamma}^2 - 7\,ep^2\,\pi^2 + 6\,ep^2\,\text{PolyGamma}[0,-z1]^2 + 12\,ep^2\,\text{PolyGamma}[0,z1]^2 - 12\,ep^2\,\text{PolyGamma}[0,1+z1]^2 - 24\,ep\,\text{PolyGamma}[0,z1]\,(1 + ep\,\text{EulerGamma} + ep\,\text{PolyGamma}[0,1+z1]) + 12\,ep\,\text{PolyGamma}[0,-z1]\,(3 + 3\,ep\,\text{EulerGamma} - 2\,ep\,\text{PolyGamma}[0,z1] + 4\,ep\,\text{PolyGamma}[0,1+z1]) - 66\,ep^2\,\text{PolyGamma}[1,-z1] + 12\,ep^2\,\text{PolyGamma}[1,z1] - 12\,ep^2\,\text{PolyGamma}[1,1+z1]\right),$

$\left\{\{ep \to 0\}, \left\{z1 \to -\frac{1}{4}\right\}\right\}\right], \text{MBint}\left[6\,\text{Gamma}[-1-z2]\,\text{Gamma}[-1-z1-z2]^2\,\text{Gamma}[-z2]\,\text{Gamma}[1+z2]^2\,\text{Gamma}[1+z1+z2]\,\text{Gamma}[2+z1+z2], \left\{\{ep \to 0\}, \left\{z1 \to -\frac{1}{4}, z2 \to -\frac{1}{4}\right\}\right\}\right]\}$

In[9]:= **Length[V2exp]**

Out[9]= 3

In[10]:= **res1 = V2exp[[1]][[1]]**

Out[10]= $\frac{1}{ep^4} - \frac{\pi^2}{2\,ep^2} - \frac{41\,\pi^4}{40} + \frac{55\,\text{PolyGamma}[2,1]}{3\,ep}$

In[11]:= **V2exp[[3]]**

Out[11]= $\text{MBint}\left[6\,\text{Gamma}[-1-z2]\,\text{Gamma}[-1-z1-z2]^2\,\text{Gamma}[-z2]\,\text{Gamma}[1+z2]^2\,\text{Gamma}[1+z1+z2]\,\text{Gamma}[2+z1+z2], \left\{\{ep \to 0\}, \left\{z1 \to -\frac{1}{4}, z2 \to -\frac{1}{4}\right\}\right\}\right]$

In[12]:= **Barnes1[V2exp[[3]], z1]**

Out[12]= $\text{MBint}\left[\pi^2\,\text{Gamma}[-1-z2]\,\text{Gamma}[-z2]\,\text{Gamma}[1+z2]^2, \left\{\{ep \to 0\}, \left\{z2 \to -\frac{1}{4}\right\}\right\}\right]$

In[13]:= **res2 = Barnes1[%, z2][[1]]**

Out[13]= $-\frac{\pi^4}{6}$

In[14]:= **V2exp[[2]]**

Out[14]= $\text{MBint}\left[\frac{1}{4\,ep^2}\text{Gamma}[-z1]^2\,\text{Gamma}[z1]\,\text{Gamma}[1+z1]\,\left(12 + 12\,ep\,\text{EulerGamma} + 6\,ep^2\,\text{EulerGamma}^2 - 7\,ep^2\,\pi^2 + 6\,ep^2\,\text{PolyGamma}[0,-z1]^2 + 12\,ep^2\,\text{PolyGamma}[0,z1]^2 - 12\,ep^2\,\text{PolyGamma}[0,1+z1]^2 - 24\,ep\,\text{PolyGamma}[0,z1]\,(1 + ep\,\text{EulerGamma} + ep\,\text{PolyGamma}[0,1+z1]) + 12\,ep\,\text{PolyGamma}[0,-z1]\,(3 + 3\,ep\,\text{EulerGamma} - 2\,ep\,\text{PolyGamma}[0,z1] + 4\,ep\,\text{PolyGamma}[0,1+z1]) - 66\,ep^2\,\text{PolyGamma}[1,-z1] + 12\,ep^2\,\text{PolyGamma}[1,z1] - 12\,ep^2\,\text{PolyGamma}[1,1+z1]\right), \left\{\{ep \to 0\}, \left\{z1 \to -\frac{1}{4}\right\}\right\}\right]$

In[15]:= **CoeffEps[X_, n_] := (X /. X[[1]] → Simplify[Coefficient[X[[1]], ep, n]]);**

In[16]:= **CoeffEps[V2exp[[2]], -2]**

Out[16]= $\text{MBint}\left[3\,\text{Gamma}[-z1]^2\,\text{Gamma}[z1]\,\text{Gamma}[1+z1], \left\{\{ep \to 0\}, \left\{z1 \to -\frac{1}{4}\right\}\right\}\right]$

In[17]:= **res32 = Barnes1[CoeffEps[V2exp[[2]], -2], z1][[1]]**

Out[17]= $-\frac{\pi^2}{2}$

In[18]:= **CoeffEps[V2exp[[2]], -1]**

Out[18]= MBint[3 Gamma[-z1]^2 Gamma[z1] Gamma[1 + z1]
(EulerGamma + 3 PolyGamma[0, -z1] - 2 PolyGamma[0, z1]), {{ep → 0}, {z1 → $-\frac{1}{4}$}}]

In[19]:= **res31 = 9 Zeta[3];**

In[20]:= **res31 // N**

Out[20]= 10.8185

In[21]:= **NIntegrate[CoeffEps[V2exp[[2]], -1][[1]] / (2 Pi) /. {z1 → $-\frac{1}{4}$ + I * y1},
{y1, -Infinity, Infinity}]**

Out[21]= $10.8185 + 2.13163 \times 10^{-14}$ i

In[22]:= **CoeffEps[V2exp[[2]], 0]**

Out[22]= MBint[$\frac{1}{4}$ Gamma[-z1]^2 Gamma[z1] Gamma[1 + z1]
(6 EulerGamma^2 - 7 π^2 + 6 PolyGamma[0, -z1]^2 + 12 PolyGamma[0, z1]^2 -
12 PolyGamma[0, 1 + z1]^2 - 24 PolyGamma[0, z1] (EulerGamma + PolyGamma[0, 1 + z1]) +
12 PolyGamma[0, -z1] (3 EulerGamma - 2 PolyGamma[0, z1] + 4 PolyGamma[0, 1 + z1]) -
66 PolyGamma[1, -z1] + 12 PolyGamma[1, z1] - 12 PolyGamma[1, 1 + z1]), {{ep → 0}, {z1 → $-\frac{1}{4}$}}]

In[23]:= **res30 = $\frac{7\pi^4}{10}$;**

In[24]:= **res30 // N**

Out[24]= 68.1864

In[25]:= **NIntegrate[CoeffEps[V2exp[[2]], 0][[1]] / (2 Pi) /. {z1 → $-\frac{1}{4}$ + I * y1},
{y1, -Infinity, Infinity}]**

Out[25]= 68.1864 + 0. i

(* result *)

In[26]:= **FullSimplify[res1 + res2 + res32 / ep^2 + res31 / ep + res30]**

Out[26]= $\frac{1}{ep^4} - \frac{\pi^2}{ep^2} - \frac{59\pi^4}{120} - \frac{83\,Zeta[3]}{3\,ep}$

In[3]:= **SetDirectory["c:/diskE/job2008/Zurich"];**

In[4]:= **<< MB/MB.m**

MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08

(* Example 4a *)

In[5]:= **F = Gamma[3 / 2 + ep + z] Gamma[-1 - 2 ep - z] Gamma[4 ep + z] Gamma[-z] Gamma[1 / 2 - ep - z] / Gamma[1 - 2 ep - z]**

Out[5]= $$\frac{1}{\text{Gamma}[1-2\,\text{ep}-z]}\text{Gamma}[-1-2\,\text{ep}-z]\,\text{Gamma}\left[\frac{1}{2}-\text{ep}-z\right]\text{Gamma}[-z]\,\text{Gamma}\left[\frac{3}{2}+\text{ep}+z\right]\text{Gamma}[4\,\text{ep}+z]$$

(* Strategy #2 *)

In[6]:= **Frules = MBoptimizedRules[F, ep → 0, {}, {ep}]**

MBrules::norules : no rules could be found to regulate this integral

Out[6]= { }

(* Strategy #1 *)

(* The two residues *)

In[7]:= **-Residue[F, {z, -1 - 2 ep}] + Residue[F, {z, -4 ep}]**

Out[7]= $$\text{Gamma}\left[\frac{1}{2}-\text{ep}\right]\text{Gamma}\left[\frac{3}{2}+\text{ep}\right]\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}[1+2\,\text{ep}]+\frac{\text{Gamma}\left[\frac{3}{2}-3\,\text{ep}\right]\text{Gamma}[4\,\text{ep}]\,\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}\left[\frac{1}{2}+3\,\text{ep}\right]}{\text{Gamma}[1+2\,\text{ep}]}$$

(* plus an integral with the first poles of Gamma[4 ep+z]and Gamma[-1-2 ep-z] of the opposite nature *)

(* Strategy #2: introduce an auxiliary analytic regularization *)

In[8]:= **F = Gamma[3 / 2 + ep + z] Gamma[-1 - 2 ep - z + y] Gamma[4 ep + z] Gamma[-z] Gamma[1 / 2 - ep - z] / Gamma[1 - 2 ep - z]**

Out[8]= $$\frac{1}{\text{Gamma}[1-2\,\text{ep}-z]}\text{Gamma}\left[\frac{1}{2}-\text{ep}-z\right]\text{Gamma}[-1-2\,\text{ep}+y-z]\,\text{Gamma}[-z]\,\text{Gamma}\left[\frac{3}{2}+\text{ep}+z\right]\text{Gamma}[4\,\text{ep}+z]$$

In[9]:= **Step1rules = MBoptimizedRules[F, y → 0, {}, {ep, y}]**

MBrules::norules : no rules could be found to regulate this integral

Out[9]= $$\left\{\left\{\text{ep}\to\frac{1}{2},\ y\to\frac{7}{4}\right\},\ \left\{z\to-\frac{5}{4}\right\}\right\}$$

In[10]:= **con1 = MBcontinue[F, y → 0, Step1rules]**

Level 1

Taking -residue in z = -1 - 2 ep + y

Level 2

Integral {1}

2 integral(s) found

Out[10]= $\left\{\left\{\text{MBint}\left[\frac{\text{Gamma}\left[\frac{3}{2}+\text{ep}-\text{y}\right]\,\text{Gamma}[1+2\,\text{ep}-\text{y}]\,\text{Gamma}\left[\frac{1}{2}-\text{ep}+\text{y}\right]\,\text{Gamma}[-1+2\,\text{ep}+\text{y}]}{\text{Gamma}[2-\text{y}]},\left\{\left\{\text{ep}\to\frac{1}{2},\,\text{y}\to 0\right\},\,\{\}\right\}\right]\right\},\,\text{MBint}\left[\frac{1}{\text{Gamma}[1-2\,\text{ep}-\text{z}]}\text{Gamma}\left[\frac{1}{2}-\text{ep}-\text{z}\right]\,\text{Gamma}[-1-2\,\text{ep}+\text{y}-\text{z}]\,\text{Gamma}[-\text{z}]\,\text{Gamma}\left[\frac{3}{2}+\text{ep}+\text{z}\right]\,\text{Gamma}[4\,\text{ep}+\text{z}],\left\{\left\{\text{ep}\to\frac{1}{2},\,\text{y}\to 0\right\},\,\left\{\text{z}\to-\frac{5}{4}\right\}\right\}\right]\right\}$

In[11]:= **exp1 = MBexpand[con1, 1, {y, 0, 0}]**

Out[11]= $\left\{\text{MBint}\left[\text{Gamma}\left[\frac{1}{2}-\text{ep}\right]\,\text{Gamma}\left[\frac{3}{2}+\text{ep}\right]\,\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}[1+2\,\text{ep}],\,\left\{\left\{\text{ep}\to\frac{1}{2},\,\text{y}\to 0\right\},\,\{\}\right\}\right],\,\text{MBint}\left[\frac{1}{\text{Gamma}[1-2\,\text{ep}-\text{z}]}\text{Gamma}[-1-2\,\text{ep}-\text{z}]\,\text{Gamma}\left[\frac{1}{2}-\text{ep}-\text{z}\right]\,\text{Gamma}[-\text{z}]\,\text{Gamma}\left[\frac{3}{2}+\text{ep}+\text{z}\right]\,\text{Gamma}[4\,\text{ep}+\text{z}],\,\left\{\left\{\text{ep}\to\frac{1}{2},\,\text{y}\to 0\right\},\,\left\{\text{z}\to-\frac{5}{4}\right\}\right\}\right]\right\}$

In[12]:= **con2 = Table[MBcontinue[exp1[[i, 1]], ep → 0, exp1[[i, 2]]], {i, Length[exp1]}]**

```
Level 1

1 integral(s) found

Level 1

Taking +residue in z = -1 - 2 ep

Taking +residue in z = -4 ep

Taking +residue in z = -1 - 4 ep

Level 2

Integral {1}

Integral {2}

Integral {3}

4 integral(s) found
```

Out[12]= $\left\{\left\{\text{MBint}\left[\text{Gamma}\left[\frac{1}{2}-\text{ep}\right]\text{Gamma}\left[\frac{3}{2}+\text{ep}\right]\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}[1+2\,\text{ep}],\{\{\text{ep}\to 0,\text{y}\to 0\},\{\}\}\right]\right\},\right.$

$\left\{\left\{\text{MBint}\left[-\text{Gamma}\left[\frac{1}{2}-\text{ep}\right]\text{Gamma}\left[\frac{3}{2}+\text{ep}\right]\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}[1+2\,\text{ep}],\{\{\text{ep}\to 0,\text{y}\to 0\},\{\}\}\right]\right\},\right.$

$\left\{\text{MBint}\left[\frac{\text{Gamma}\left[\frac{3}{2}-3\,\text{ep}\right]\text{Gamma}[4\,\text{ep}]\,\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}\left[\frac{1}{2}+3\,\text{ep}\right]}{\text{Gamma}[1+2\,\text{ep}]},\{\{\text{ep}\to 0,\text{y}\to 0\},\{\}\}\right]\right\},$

$\left\{\text{MBint}\left[-\frac{\text{Gamma}\left[\frac{1}{2}-3\,\text{ep}\right]\text{Gamma}[2\,\text{ep}]\,\text{Gamma}\left[\frac{3}{2}+3\,\text{ep}\right]\text{Gamma}[1+4\,\text{ep}]}{\text{Gamma}[2+2\,\text{ep}]},\{\{\text{ep}\to 0,\text{y}\to 0\},\{\}\}\right]\right\},$

$\text{MBint}\left[\frac{1}{\text{Gamma}[1-2\,\text{ep}-\text{z}]}\text{Gamma}[-1-2\,\text{ep}-\text{z}]\,\text{Gamma}\left[\frac{1}{2}-\text{ep}-\text{z}\right]\right.$

$\left.\left.\text{Gamma}[-\text{z}]\,\text{Gamma}\left[\frac{3}{2}+\text{ep}+\text{z}\right]\text{Gamma}[4\,\text{ep}+\text{z}],\left\{\{\text{ep}\to 0,\text{y}\to 0\},\left\{\text{z}\to -\frac{5}{4}\right\}\right\}\right]\right\}\right\}$

In[13]:= **MBmerge[%]**

Out[13]= $\left\{\text{MBint}\left[\frac{\text{Gamma}\left[\frac{3}{2}-3\,\text{ep}\right]\text{Gamma}[4\,\text{ep}]\,\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}\left[\frac{1}{2}+3\,\text{ep}\right]}{\text{Gamma}[1+2\,\text{ep}]}-\right.\right.$

$\left.\frac{\text{Gamma}\left[\frac{1}{2}-3\,\text{ep}\right]\text{Gamma}[2\,\text{ep}]\,\text{Gamma}\left[\frac{3}{2}+3\,\text{ep}\right]\text{Gamma}[1+4\,\text{ep}]}{\text{Gamma}[2+2\,\text{ep}]},\{\{\text{ep}\to 0,\text{y}\to 0\},\{\}\}\right],$

$\text{MBint}\left[\frac{1}{\text{Gamma}[1-2\,\text{ep}-\text{z}]}\text{Gamma}[-1-2\,\text{ep}-\text{z}]\,\text{Gamma}\left[\frac{1}{2}-\text{ep}-\text{z}\right]\text{Gamma}[-\text{z}]\right.$

$\left.\left.\text{Gamma}\left[\frac{3}{2}+\text{ep}+\text{z}\right]\text{Gamma}[4\,\text{ep}+\text{z}],\left\{\{\text{ep}\to 0,\text{y}\to 0\},\left\{\text{z}\to -\frac{5}{4}\right\}\right\}\right]\right\}$

In[14]:= **exp2 = MBexpand[%, Exp[2 ep EulerGamma], {ep, 0, 0}]**

In[15]:= **MBmerge[%]**

Out[15]= $\{\text{MBint}\Big[-\frac{1}{96\,\text{ep}^2}\pi\Big(6-6\,\text{ep}\Big(-6+2\,\text{EulerGamma}-3\,\text{PolyGamma}\big[0,\tfrac{1}{2}\big]+3\,\text{PolyGamma}\big[0,\tfrac{3}{2}\big]\Big)+\text{ep}^2\Big(12\,\text{EulerGamma}^2+35\,\pi^2-36\,\text{EulerGamma}\Big(2+\text{PolyGamma}\big[0,\tfrac{1}{2}\big]-\text{PolyGamma}\big[0,\tfrac{3}{2}\big]\Big)+3\Big(-44+9\,\text{PolyGamma}\big[0,\tfrac{1}{2}\big]^2+12\,\text{PolyGamma}\big[0,\tfrac{3}{2}\big]+9\,\text{PolyGamma}\big[0,\tfrac{3}{2}\big]^2-6\,\text{PolyGamma}\big[0,\tfrac{1}{2}\big]\Big(2+3\,\text{PolyGamma}\big[0,\tfrac{3}{2}\big]\Big)\Big)\Big)\Big),\{\{\text{ep}\to 0,\text{y}\to 0\},\{\}\}\Big],$

$\text{MBint}\Big[\frac{\text{Gamma}[-1-z]\,\text{Gamma}[\tfrac{1}{2}-z]\,\text{Gamma}[-z]\,\text{Gamma}[z]\,\text{Gamma}[\tfrac{3}{2}+z]}{\text{Gamma}[1-z]},\{\{\text{ep}\to 0,\text{y}\to 0\},\{z\to -\tfrac{5}{4}\}\}\Big]\}$

(*   Example 4b   *)

In[16]:= **F = Gamma[-1 / 2 + ep + z] Gamma[1 + ep + z] Gamma[3 / 2 - ep - z] Gamma[-z]**

Out[16]= $\text{Gamma}\big[\tfrac{3}{2}-\text{ep}-z\big]\,\text{Gamma}[-z]\,\text{Gamma}\big[-\tfrac{1}{2}+\text{ep}+z\big]\,\text{Gamma}[1+\text{ep}+z]$

(*   Strategy #1:
there are no poles. Expand the integrand in epsilon. However,
the contour cannot be a straight line.   *)

(*   Strategy #2   *)

In[17]:= **F = Gamma[-1 / 2 + ep + z] Gamma[1 + ep + z] Gamma[3 / 2 - ep - z] Gamma[-z]**

Out[17]= $\text{Gamma}\big[\tfrac{3}{2}-\text{ep}-z\big]\,\text{Gamma}[-z]\,\text{Gamma}\big[-\tfrac{1}{2}+\text{ep}+z\big]\,\text{Gamma}[1+\text{ep}+z]$

In[18]:= **Frules = MBoptimizedRules[F, ep → 0, {}, {ep}]**

MBresidues::contour : contour starts and/or ends on a pole of Gamma[1 + ep + z]

MBresidues::contour : contour starts and/or ends on a pole of Gamma[1 + ep + z]

MBresidues::contour : contour starts and/or ends on a pole of Gamma[1 + ep + z]

General::stop : Further output of MBresidues::contour will be suppressed during this calculation. ≫

Out[18]= $Aborted

(*The integral of Gamma[a+s] Gamma[b+s] Gamma[c-s] Gamma[dd-s]*)

In[19]:= **Mel40[a_, b_, c_, d_] := Gamma[a + c] Gamma[a + d] Gamma[b + c] Gamma[b + d] / Gamma[a + b + c + d];**

In[20]:= **Mel40[-1 / 2 + ep, 1 + ep, 3 / 2 - ep, 0]**

Out[20]= $\frac{3\sqrt{\pi}\,\text{Gamma}\big[-\tfrac{1}{2}+\text{ep}\big]\,\text{Gamma}[1+\text{ep}]}{4\,\text{Gamma}[2+\text{ep}]}$

- Strategy A in a slightly modified form

[A.V. Smirnov & V.A. Smirnov'09]

- Strategy A in a slightly modified form

[A.V. Smirnov & V.A. Smirnov'09]

Strategy B: straight contours in the beginning
Strategy A: straight contours in the end

- Strategy A in a slightly modified form

[A.V. Smirnov & V.A. Smirnov'09]

Strategy B: straight contours in the beginning
Strategy A: straight contours in the end

Set $\epsilon = 0$

- Strategy A in a slightly modified form

[A.V. Smirnov & V.A. Smirnov'09]

Strategy B: straight contours in the beginning
Strategy A: straight contours in the end

Set $\epsilon = 0$

Look for straight contours (i.e. Rez_i) for which gamma functions are changed in a minimal way

- Strategy A in a slightly modified form

[A.V. Smirnov & V.A. Smirnov'09]

Strategy B: straight contours in the beginning
Strategy A: straight contours in the end

Set $\epsilon = 0$

Look for straight contours (i.e. Rez_i) for which gamma functions are changed in a minimal way

Let $\prod \Gamma(A_i)$ with $A_i = a_i + b_i\epsilon + \sum_j c_{ij} z_j$

be the numerator of a multiple MB integral

Let $\sigma(x) = [(1-x)_+]$ where $[\ldots]$ is the integer part of a number and $x_+ = x$ for $x > 0$ and 0 otherwise.

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In other words, if $-n < x < -n+1$ then $\sigma(x) = n$ for $n > 0$ and $\sigma(x) = 0$ for $n \leq 0$.

Let $\sigma(x) = [(1-x)_+]$ where $[\ldots]$ is the integer part of a number and $x_+ = x$ for $x > 0$ and 0 otherwise.

In other words, if $-n < x < -n+1$ then $\sigma(x) = n$ for $n > 0$ and $\sigma(x) = 0$ for $n \leq 0$.

Choose contours, i.e. Rez_i, for which

$$\sum_i \sigma\left(\text{Re}A_i|_{\epsilon=0}\right) \equiv \sum_i \sigma\left(a_i + \sum_j c_{ij}\text{Re}z_j\right)$$

is minimal

After such a choice is done identify gamma functions which should be changed

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The second step in Strategy A is the same as in the old version:

take care of the distinguished gamma functions, i.e. take a residue and replace Γ by $\Gamma^{(1)}(A_i)$ (and, possibly, $\Gamma^{(1)}(A_i)$ by $\Gamma^{(2)}(A_i)$ etc.)

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Proceed iteratively: every residue is considered from the scratch, i.e. treated in the same way as the initial MB integral

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Proceed iteratively: every residue is considered from the scratch, i.e. treated in the same way as the initial MB integral

MBresolve.m
http://projects.hepforge.org/mbtools/

```
In[1]:= SetDirectory["c:/diskE/job2008/Zurich"];

        (* http://www-ttp.particle.uni-karlsruhe.de/~asmirnov  *)

In[2]:= << MB/MB.m;
        << MB/MBresolve.m
```

```
MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08

MBresolve 1.0

by Alexander Smirnov

last modified 22 Oct 08
```

```
        (*   The integrand of the MB integral for the one-
          loop propagator diagram with m1=m and m2=0 *)

In[4]:= MB[a1_, a2_] := (-1)^(a1 + a2) / QQ^(a1 + a2 + ep - 2) / Gamma[a1] / Gamma[a2]
           Gamma[2 - ep - a2] Gamma[a1 + a2 + ep - 2 + z] Gamma[2 - ep - a1 - z]
           Gamma[-z] / Gamma[4 - 2 ep - a1 - a2 - z] mm^z / QQ^z;

        (*  Notation: mm=m^2, QQ=-q^2;
        I Pi^(d/2) is pulled out  *)

        (*  The diagram with a1=1 and a2=1  *)

In[5]:= P1 = MB[1, 1]
```

Out[5]=
$$\frac{mm^{z}\, QQ^{-ep-z}\, \text{Gamma}[1-ep]\, \text{Gamma}[1-ep-z]\, \text{Gamma}[-z]\, \text{Gamma}[ep+z]}{\text{Gamma}[2-2\,ep-z]}$$

```
In[6]:= MBresolve[P1, ep]

CREATING RESIDUES LIST..........0.2812 seconds
EVALUATING RESIDUES..........0.1875 seconds
```

Out[6]=
$$\left\{\text{MBint}\left[\frac{QQ^{-ep}\,\text{Gamma}[1-ep]^2\,\text{Gamma}[ep]}{\text{Gamma}[2-2\,ep]}, \{\{ep\to 0\}, \{\}\}\right], \text{MBint}\left[\frac{mm^{z}\, QQ^{-ep-z}\, \text{Gamma}[1-ep]\, \text{Gamma}[1-ep-z]\, \text{Gamma}[-z]\, \text{Gamma}[ep+z]}{\text{Gamma}[2-2\,ep-z]}, \{\{ep\to 0\}, \{z\to 0.511912\}\}\right]\right\}$$

```
In[7]:= Box1[a1_, a2_, a3_, a4_] :=
          (S^(2-a1-a2-a3-a4-ep-z) T^z Gamma[a1 + a2 + a3 + a4 - 2 + ep + z] Gamma[a2 + z] Gamma[a4 + z]
             Gamma[2 - a1 - a2 - a4 - ep - z] Gamma[2 - a2 - a3 - a4 - ep - z] Gamma[-z] ) /
          (Gamma[a1] Gamma[a2] Gamma[a3] Gamma[a4] Gamma[4 - a1 - a2 - a3 - a4 - 2 ep]);

In[8]:= Box1[1, 1, 1, 1]
```

Out[8]=
$$\frac{S^{-2-ep-z}\, T^{z}\, \text{Gamma}[-1-ep-z]^2\, \text{Gamma}[-z]\, \text{Gamma}[1+z]^2\, \text{Gamma}[2+ep+z]}{\text{Gamma}[-2\,ep]}$$

```
In[9]:= B1 = % /. {S -> 1, T -> x}
```

Out[9]=
$$\frac{x^{z}\, \text{Gamma}[-1-ep-z]^2\, \text{Gamma}[-z]\, \text{Gamma}[1+z]^2\, \text{Gamma}[2+ep+z]}{\text{Gamma}[-2\,ep]}$$

In[10]:= **MBresolve[B1, ep]**

```
CREATING RESIDUES LIST..........0.4688 seconds
EVALUATING RESIDUES..........0.3125 seconds
```

Out[10]=
$$\left\{\text{MBint}\left[-\frac{\text{EulerGamma Gamma}[-\text{ep}]^2\,\text{Gamma}[1+\text{ep}]}{\text{x Gamma}[-2\,\text{ep}]}+\frac{\text{Gamma}[-\text{ep}]^2\,\text{Gamma}[1+\text{ep}]\,\text{Log}[\text{x}]}{\text{x Gamma}[-2\,\text{ep}]}-\frac{2\,\text{Gamma}[-\text{ep}]^2\,\text{Gamma}[1+\text{ep}]\,\text{PolyGamma}[0,-\text{ep}]}{\text{x Gamma}[-2\,\text{ep}]}+\frac{\text{Gamma}[-\text{ep}]^2\,\text{Gamma}[1+\text{ep}]\,\text{PolyGamma}[0,1+\text{ep}]}{\text{x Gamma}[-2\,\text{ep}]},\{\{\text{ep}\to 0\},\{\}\}\right],\right.$$
$$\left.\text{MBint}\left[\frac{\text{x}^{\text{z}}\,\text{Gamma}[-1-\text{ep}-\text{z}]^2\,\text{Gamma}[-\text{z}]\,\text{Gamma}[1+\text{z}]^2\,\text{Gamma}[2+\text{ep}+\text{z}]}{\text{Gamma}[-2\,\text{ep}]},\{\{\text{ep}\to 0\},\{\text{z}\to -1.81305\}\}\right]\right\}$$

In[11]:=
```
NPMB[a1_, a2_, a3_, a4_, a5_, a6_] := ((-1)^(a1 + a2 + a3 + a4 + a5 + a6) /
    (Gamma[a1] Gamma[a2] Gamma[a3] Gamma[a4] Gamma[a5] Gamma[a6])
   Gamma[2 - ep - a3 - a5] Gamma[2 - ep - a4 - a6] / Gamma[4 - 2 ep - a3 - a4 - a5 - a6] /
    Gamma[6 - 3 ep - a1 - a2 - a3 - a4 - a5 - a6]
   Gamma[a1 + a2 + a3 + a4 + a5 + a6 + 2 ep - 4 + z1 + z2] Gamma[-z1] Gamma[-z2]
   Gamma[2 - ep - a1 - a2 - z1] Gamma[a4 + z2] Gamma[a1 + z1 + z2]
   Gamma[4 - 2 ep - a1 - a3 - a4 - a5 - a6 - z2] Gamma[4 - 2 ep - a1 - a2 - a4 - a5 - a6 - z1 - z2]
   Gamma[a5 + z2] Gamma[4 - 2 ep - a1 - a2 - a3 - a4 - a5 - z1 - z2] /
     Gamma[4 - 2 ep - a1 - a2 - a4 - a6 - z1] / Gamma[4 - 2 ep - a1 - a2 - a3 - a5 - z1]);
```

In[12]:= **V2 = NPMB[1, 1, 1, 1, 1, 1]**

Out[12]=
$$\left(\text{Gamma}[-\text{ep}]^2\,\text{Gamma}[-\text{ep}-\text{z1}]\,\text{Gamma}[-\text{z1}]\,\text{Gamma}[-1-2\,\text{ep}-\text{z2}]\,\text{Gamma}[-1-2\,\text{ep}-\text{z1}-\text{z2}]^2\,\text{Gamma}[-\text{z2}]\,\text{Gamma}[1+\text{z2}]^2\,\text{Gamma}[1+\text{z1}+\text{z2}]\,\text{Gamma}[2+2\,\text{ep}+\text{z1}+\text{z2}]\right)\Big/\left(\text{Gamma}[-3\,\text{ep}]\,\text{Gamma}[-2\,\text{ep}]\,\text{Gamma}[-2\,\text{ep}-\text{z1}]^2\right)$$

```
In[13]:= MBresolve[V2, ep]

CREATING RESIDUES LIST..........0.625 seconds
EVALUATING RESIDUES..........0.25 seconds

Out[13]= {MBint[Gamma[-2 ep]^4 Gamma[-ep]^2 Gamma[1 + 2 ep]^2 / Gamma[-4 ep]^2, {{ep -> 0}, {}}],
  MBint[(Gamma[-2 ep] Gamma[-ep]^2 Gamma[1 + 2 ep] Gamma[-ep - z1] Gamma[-z1]^3 Gamma[1 + z1]
     Gamma[-2 ep + z1]) / (Gamma[-3 ep] Gamma[-2 ep - z1]^2), {{ep -> 0}, {z1 -> -0.859981}}],
  MBint[-1/Gamma[-3 ep] EulerGamma Gamma[-ep]^2 Gamma[-1 - 2 ep - z2] Gamma[-z2]
     Gamma[1 + ep + z2] Gamma[1 + 2 ep + z2] - 1/Gamma[-3 ep] Gamma[-ep]^2 Gamma[-1 - 2 ep - z2]
    Gamma[-z2] Gamma[1 + ep + z2] Gamma[1 + 2 ep + z2] PolyGamma[0, -2 ep] - 1/Gamma[-3 ep]
   2 Gamma[-ep]^2 Gamma[-1 - 2 ep - z2] Gamma[-z2] Gamma[1 + ep + z2] Gamma[1 + 2 ep + z2]
    PolyGamma[0, 1 + z2] + 1/Gamma[-3 ep] Gamma[-ep]^2 Gamma[-1 - 2 ep - z2]
    Gamma[-z2] Gamma[1 + ep + z2] Gamma[1 + 2 ep + z2] PolyGamma[0, 1 + ep + z2] +
   1/Gamma[-3 ep] Gamma[-ep]^2 Gamma[-1 - 2 ep - z2] Gamma[-z2] Gamma[1 + ep + z2]
    Gamma[1 + 2 ep + z2] PolyGamma[0, 1 + 2 ep + z2], {{ep -> 0}, {z2 -> -0.859981}}],
  MBint[(Gamma[-ep]^2 Gamma[-ep - z1] Gamma[-z1] Gamma[-1 - 2 ep - z2] Gamma[-1 - 2 ep - z1 - z2]^2
     Gamma[-z2] Gamma[1 + z2]^2 Gamma[1 + z1 + z2] Gamma[2 + 2 ep + z1 + z2]) /
   (Gamma[-3 ep] Gamma[-2 ep] Gamma[-2 ep - z1]^2),
  {{ep -> 0}, {z1 -> -0.72274, z2 -> -0.274294}}]}

In[14]:= Simplify[%]

Out[14]= {MBint[Gamma[-2 ep]^4 Gamma[-ep]^2 Gamma[1 + 2 ep]^2 / Gamma[-4 ep]^2, {{ep -> 0}, {}}],
  MBint[(Gamma[-2 ep] Gamma[-ep]^2 Gamma[1 + 2 ep] Gamma[-ep - z1] Gamma[-z1]^3 Gamma[1 + z1]
     Gamma[-2 ep + z1]) / (Gamma[-3 ep] Gamma[-2 ep - z1]^2), {{ep -> 0}, {z1 -> -0.859981}}],
  MBint[1/Gamma[-3 ep] Gamma[-ep]^2 Gamma[-1 - 2 ep - z2] Gamma[-z2] Gamma[1 + ep + z2]
   Gamma[1 + 2 ep + z2] (-EulerGamma - PolyGamma[0, -2 ep] - 2 PolyGamma[0, 1 + z2] +
     PolyGamma[0, 1 + ep + z2] + PolyGamma[0, 1 + 2 ep + z2]), {{ep -> 0}, {z2 -> -0.859981}}],
  MBint[(Gamma[-ep]^2 Gamma[-ep - z1] Gamma[-z1] Gamma[-1 - 2 ep - z2] Gamma[-1 - 2 ep - z1 - z2]^2
     Gamma[-z2] Gamma[1 + z2]^2 Gamma[1 + z1 + z2] Gamma[2 + 2 ep + z1 + z2]) /
   (Gamma[-3 ep] Gamma[-2 ep] Gamma[-2 ep - z1]^2),
  {{ep -> 0}, {z1 -> -0.72274, z2 -> -0.274294}}]}

(*      Example  4a      *)
```

In[15]:= **F = Gamma[3 / 2 + ep + z] Gamma[-1 - 2 ep - z] Gamma[4 ep + z] Gamma[-z] Gamma[1 / 2 - ep - z] / Gamma[1 - 2 ep - z]**

Out[15]= $\frac{1}{\text{Gamma}[1-2\,\text{ep}-z]}\ \text{Gamma}[-1-2\,\text{ep}-z]\ \text{Gamma}\left[\frac{1}{2}-\text{ep}-z\right]\ \text{Gamma}[-z]\ \text{Gamma}\left[\frac{3}{2}+\text{ep}+z\right]\ \text{Gamma}[4\,\text{ep}+z]$

In[16]:= **MBresolve[F, ep]**

```
CREATING RESIDUES LIST..........0.3281 seconds
EVALUATING RESIDUES..........0.3125 seconds
```

Out[16]= $\left\{\text{MBint}\left[\frac{\text{Gamma}\left[\frac{3}{2}-3\,\text{ep}\right]\text{Gamma}[4\,\text{ep}]\,\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}\left[\frac{1}{2}+3\,\text{ep}\right]}{\text{Gamma}[1+2\,\text{ep}]},\ \{\{\text{ep}\to 0\},\ \{\}\}\right],\right.$
$\text{MBint}\left[-\frac{\text{Gamma}\left[\frac{1}{2}-3\,\text{ep}\right]\text{Gamma}[2\,\text{ep}]\,\text{Gamma}\left[\frac{3}{2}+3\,\text{ep}\right]\text{Gamma}[1+4\,\text{ep}]}{\text{Gamma}[2+2\,\text{ep}]},\ \{\{\text{ep}\to 0\},\ \{\}\}\right],$
$\text{MBint}\left[\left(\text{Gamma}[-1-2\,\text{ep}-z]\ \text{Gamma}\left[\frac{1}{2}-\text{ep}-z\right]\text{Gamma}[-z]\ \text{Gamma}\left[\frac{3}{2}+\text{ep}+z\right]\text{Gamma}[4\,\text{ep}+z]\right)\Big/\right.$
$\left.\left.\text{Gamma}[1-2\,\text{ep}-z],\ \{\{\text{ep}\to 0\},\ \{z\to -1.48302\}\}\right]\right\}$

In[17]:= **MBmerge[%]**

Out[17]= $\left\{\text{MBint}\left[\frac{\text{Gamma}\left[\frac{3}{2}-3\,\text{ep}\right]\text{Gamma}[4\,\text{ep}]\,\text{Gamma}[-1+2\,\text{ep}]\,\text{Gamma}\left[\frac{1}{2}+3\,\text{ep}\right]}{\text{Gamma}[1+2\,\text{ep}]}-\right.\right.$
$\left.\frac{\text{Gamma}\left[\frac{1}{2}-3\,\text{ep}\right]\text{Gamma}[2\,\text{ep}]\,\text{Gamma}\left[\frac{3}{2}+3\,\text{ep}\right]\text{Gamma}[1+4\,\text{ep}]}{\text{Gamma}[2+2\,\text{ep}]},\ \{\{\text{ep}\to 0\},\ \{\}\}\right],$
$\text{MBint}\left[\left(\text{Gamma}[-1-2\,\text{ep}-z]\ \text{Gamma}\left[\frac{1}{2}-\text{ep}-z\right]\text{Gamma}[-z]\ \text{Gamma}\left[\frac{3}{2}+\text{ep}+z\right]\text{Gamma}[4\,\text{ep}+z]\right)\Big/\right.$
$\left.\left.\text{Gamma}[1-2\,\text{ep}-z],\ \{\{\text{ep}\to 0\},\ \{z\to -1.48302\}\}\right]\right\}$

(*   Example 4b   *)

In[18]:= **F = Gamma[-1 / 2 + ep + z] Gamma[1 + ep + z] Gamma[3 / 2 - ep - z] Gamma[-z]**

Out[18]= $\text{Gamma}\left[\frac{3}{2}-\text{ep}-z\right]\text{Gamma}[-z]\ \text{Gamma}\left[-\frac{1}{2}+\text{ep}+z\right]\text{Gamma}[1+\text{ep}+z]$

In[19]:= **MBresolve[F, ep]**

```
CREATING RESIDUES LIST..........0.2188 seconds
EVALUATING RESIDUES..........0.1562 seconds
```

Out[19]= $\left\{\text{MBint}\left[\frac{1}{2}\sqrt{\pi}\ \text{Gamma}\left[-\frac{1}{2}+\text{ep}\right],\ \{\{\text{ep}\to 0\},\ \{\}\}\right],\ \text{MBint}\left[\right.\right.$
$\left.\left.\text{Gamma}\left[\frac{3}{2}-\text{ep}-z\right]\text{Gamma}[-z]\ \text{Gamma}\left[-\frac{1}{2}+\text{ep}+z\right]\text{Gamma}[1+\text{ep}+z],\ \{\{\text{ep}\to 0\},\ \{z\to -0.294497\}\}\right]\right\}$

The massless box diagram with two legs on shell, $p_3^2 = p_4^2 = 0$, and two legs off shell, $p_1^2, p_2^2 \neq 0$

$$B_{1100} = \mathrm{i}\pi^{d/2} \frac{\Gamma(a+\epsilon-2)}{\prod \Gamma(a_l)}$$

$$\times \int_0^\infty \ldots \int_0^\infty \left(\prod_{l=1}^{4} \alpha_l^{a_l - 1} \mathrm{d}\alpha_l \right) \delta \left(\sum_{l=1}^{4} \alpha_l - 1 \right)$$

$$\times (-s\alpha_1\alpha_3 - t\alpha_2\alpha_4 - p_1^2\alpha_1\alpha_2 - p_2^2\alpha_2\alpha_3 - \mathrm{i}0)^{2-a-\epsilon}$$

Apply

$$\frac{1}{(X_1+\ldots+X_n)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi \mathrm{i})^{n-1}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\ldots\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z_2\ldots\mathrm{d}z_n\prod_{i=2}^{n} X$$

$$\times X_1^{-\lambda-z_2-\ldots-z_n}\Gamma(\lambda+z_2+\ldots+z_n)\prod_{i=2}^{n}\Gamma(-z_i)$$

Separate terms with p_1^2 and p_2^2, turn to new variables by

$$\alpha_1=\eta_1\xi_1,\ \alpha_2=\eta_1(1-\xi_1),\ \alpha_3=\eta_2\xi_2,\ \alpha_4=\eta_2(1-\xi_2)$$

and evaluate integrals over parameters to obtain a three fold MB representation

$$B_{1100} = \frac{\mathrm{i}\pi^{d/2}}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)(-s)^{a+\epsilon-2}}$$

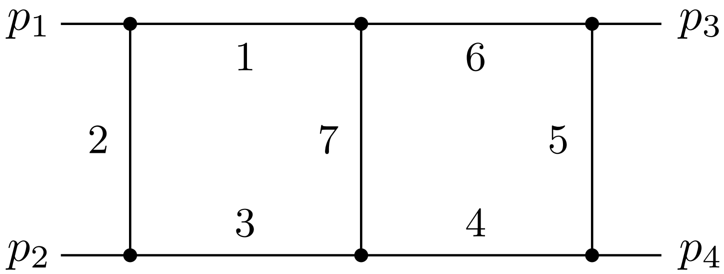
$$\times \frac{1}{(2\pi\mathrm{i})^3}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4 \frac{(-p_1^2)^{z_2}(-p_2^2)^{z_3}(-t)^{z_4}}{(-s)^{z_2+z_3+z_4}}$$

$$\times\Gamma(a+\epsilon-2+z_2+z_3+z_4)\Gamma(a_2+z_2+z_3+z_4)\Gamma(a_4+z_4)$$

$$\times\Gamma(2-\epsilon-a_{234}-z_3-z_4)\Gamma(2-\epsilon-a_{124}-z_2-z_4)$$

$$\times\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\;.$$

Double box with irreducible numerator $(k+p_1+p_2+p_4)^2$



$$B_2(s,t;a_1,\ldots,a_8,\epsilon) \;=\; \int\int \frac{\mathrm{d}^d k\,\mathrm{d}^d l}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}}$$

$$\times \frac{[(k+p_1+p_2+p_4)^2]^{-a_8}}{[(l+p_1+p_2)^2]^{a_4}[(l+p_1+p_2+p_4)^2]^{a_5}(l^2)^{a_6}[(k-l)^2]^{a_7}}$$

$$B_2(s,t;a_1,\ldots,a_8,\epsilon) \;=\; \int \frac{\mathrm{d}^d k\,[(k+p_1+p_2+p_4)^2]^{-a_8}}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}}$$

$$\times B_{1100}(s,(k+p_1+p_2+p_4)^2,k^2,(k+p_1+p_2)^2;a_6,a_7,a_4,a_5,d)$$

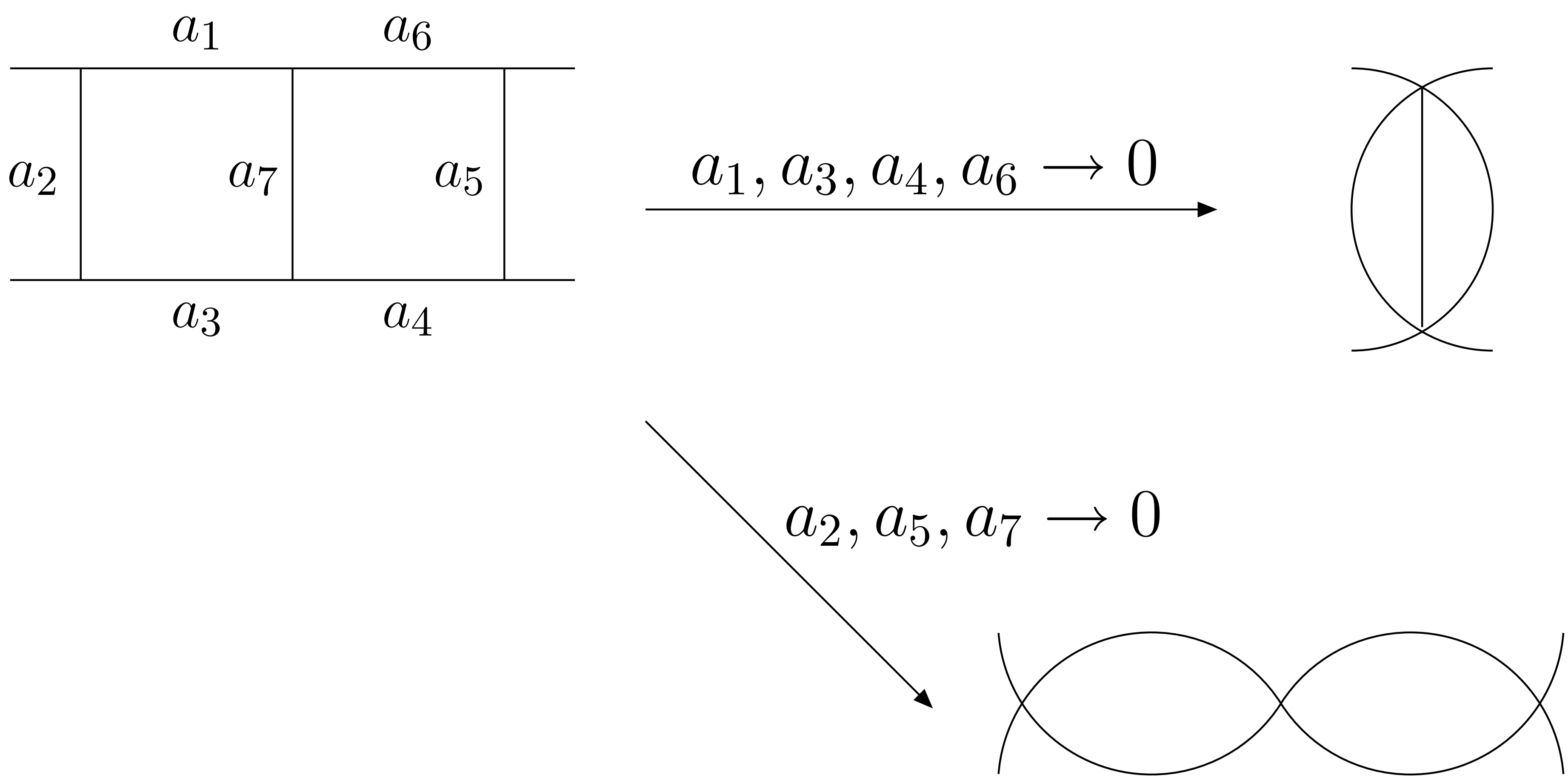
After using the threefold MB representation for B_{1100} and changing the order of integration we obtain an on-shell box integral with indices shifted by z-variables. Apply then the onefold representation for the this box.

Derivation of MB representation loop-by-loop

[A.I. Davydychev & N.I. Ussyukina'93]

AMBRE

[J. Gluza, K. Kajda & T. Riemann'07; T. Riemann: talk today]


a_1
a_6
a_2
a_7
a_5
a_3
a_4
$a_1, a_3, a_4, a_6 \to 0$
$a_2, a_5, a_7 \to 0$

In[1]:=
```
SetDirectory["c:/diskE/job2008/Zurich"];
```

In[2]:=
```
<< MB/MB.m;
<< MB/MBresolve.m
```

```
MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08

MBresolve 1.0

by Alexander Smirnov

last modified 22 Oct 08
```

```
(*   The end of derivation of MB
  representation for the planar massless double box diagram at
  p1^2=p2^2=p3^2=p4^2=0.

     Notation:   S=-s=-(p1+p2)^2, T=-t=-(p1+p3)^2 *)

(*  This is MB representation for the box   *)
```

In[4]:=
```
Box1[a1_, a2_, a3_, a4_] :=
  (S^(2-a1-a2-a3-a4-ep-z1) T^z1 Gamma[a1 + a2 + a3 + a4 - 2 + ep + z1] Gamma[a2 + z1] Gamma[a4 + z1]
     Gamma[2 - a1 - a2 - a4 - ep - z1] Gamma[2 - a2 - a3 - a4 - ep - z1] Gamma[-z1] ) /
   (Gamma[a1] Gamma[a2] Gamma[a3] Gamma[a4] Gamma[4 - a1 - a2 - a3 - a4 - 2 ep]);
```

In[5]:=
```
( 1 / S^(a4 + a5 + a6 + a7 + ep - 2 + z2 + z3 + z4) /
     (Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7] Gamma[4 - a4 - a5 - a6 - a7 - 2 ep])
    Gamma[a4 + a5 + a6 + a7 - 2 + ep + z2 + z3 + z4] Gamma[a7 + z2 + z3 + z4] Gamma[a5 + z4]
    Gamma[2 - a4 - a5 - a7 - ep - z3 - z4] Gamma[2 - a5 - a6 - a7 - ep - z2 - z4]
    Gamma[-z2] Gamma[-z3] Gamma[-z4] ) Box1[a1 - z2, a2, a3 - z3, a8 - z4];
```

In[6]:=
```
Simplify[%]
```

Out[6]=
```
(S^(4-a1-a2-a3-a4-a5-a6-a7-a8-2 ep-z1) T^z1 Gamma[-z1] Gamma[a2 + z1] Gamma[-z2] Gamma[-z3]
   Gamma[a8 + z1 - z4] Gamma[2 - a5 - a6 - a7 - ep - z2 - z4] Gamma[2 - a4 - a5 - a7 - ep - z3 - z4]
   Gamma[-2 + a1 + a2 + a3 + a8 + ep + z1 - z2 - z3 - z4] Gamma[-z4] Gamma[a5 + z4]
   Gamma[2 - a1 - a2 - a8 - ep - z1 + z2 + z4] Gamma[2 - a2 - a3 - a8 - ep - z1 + z3 + z4]
   Gamma[a7 + z2 + z3 + z4] Gamma[-2 + a4 + a5 + a6 + a7 + ep + z2 + z3 + z4]) /
 (Gamma[a2] Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7] Gamma[4 - a4 - a5 - a6 - a7 - 2 ep]
   Gamma[a1 - z2] Gamma[a3 - z3] Gamma[a8 - z4] Gamma[4 - a1 - a2 - a3 - a8 - 2 ep + z2 + z3 + z4])
```

```
(*   Changing variables   *)
```

In[7]:=
```
% /. {z2 → z2 - z4, z3 → z3 - z4};
```

In[8]:=
```
% /. {z3 → z3 + z1};
```

In[9]:=
```
% /. z4 → z4 + z1;
```

In[10]:=
```
% /. z2 → z2 + z1;
```

In[11]:=
```
Simplify[% /. z4 → z4 + z2 + z3]
```

Out[11]=
```
(S^(4-a1-a2-a3-a4-a5-a6-a7-a8-2 ep-z1) T^z1 Gamma[-z1] Gamma[a2 + z1] Gamma[2 - a5 - a6 - a7 - ep - z1 - z2]
   Gamma[2 - a1 - a2 - a8 - ep + z2] Gamma[2 - a4 - a5 - a7 - ep - z1 - z3]
   Gamma[2 - a2 - a3 - a8 - ep + z3] Gamma[a7 + z1 - z4] Gamma[-2 + a4 + a5 + a6 + a7 + ep + z1 - z4]
   Gamma[a8 - z2 - z3 - z4] Gamma[-z1 - z2 - z3 - z4] Gamma[-2 + a1 + a2 + a3 + a8 + ep + z4]
   Gamma[z2 + z4] Gamma[z3 + z4] Gamma[a5 + z1 + z2 + z3 + z4]) /
 (Gamma[a2] Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7]
   Gamma[4 - a4 - a5 - a6 - a7 - 2 ep] Gamma[4 - a1 - a2 - a3 - a8 - 2 ep + z1 - z4]
   Gamma[a8 - z1 - z2 - z3 - z4] Gamma[a3 + z2 + z4] Gamma[a1 + z3 + z4])
```

In[12]:=
```
B2[a1_, a2_, a3_, a4_, a5_, a6_, a7_, a8_] :=
 (S^(4-a1-a2-a3-a4-a5-a6-a7-a8-2 ep-z1) T^z1 Gamma[-z1] Gamma[a2 + z1] Gamma[2 - a5 - a6 - a7 - ep - z1 - z2]
    Gamma[2 - a1 - a2 - a8 - ep + z2] Gamma[2 - a4 - a5 - a7 - ep - z1 - z3]
    Gamma[2 - a2 - a3 - a8 - ep + z3] Gamma[a7 + z1 - z4] Gamma[-2 + a4 + a5 + a6 + a7 + ep + z1 - z4]
    Gamma[a8 - z2 - z3 - z4] Gamma[-z1 - z2 - z3 - z4] Gamma[-2 + a1 + a2 + a3 + a8 + ep + z4]
    Gamma[z2 + z4] Gamma[z3 + z4] Gamma[a5 + z1 + z2 + z3 + z4]) /
  (Gamma[a2] Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7] Gamma[4 - a4 - a5 - a6 - a7 - 2 ep]
    Gamma[4 - a1 - a2 - a3 - a8 - 2 ep + z1 - z4]
    Gamma[a8 - z1 - z2 - z3 - z4] Gamma[a3 + z2 + z4] Gamma[a1 + z3 + z4])

(*   the function in the one-loop integration formula  *)
```

In[13]:=
```
G[a1_, a2_] := Gamma[a1 + a2 + ep - 2] Gamma[2 - ep - a1]
   Gamma[2 - ep - a2] / Gamma[a1] / Gamma[a2] / Gamma[4 - 2 ep - a1 - a2];

(*    a vertical check: shrink vertical lines,
a2,a5,a7→0
                                                        *)
```

In[14]:=
```
B2[a1, a2, a3, a4, a5, a6, a7, 0]
```

Out[14]=
```
(S^(4-a1-a2-a3-a4-a5-a6-a7-2 ep-z1) T^z1 Gamma[-z1] Gamma[a2 + z1] Gamma[2 - a5 - a6 - a7 - ep - z1 - z2]
   Gamma[2 - a1 - a2 - ep + z2] Gamma[2 - a4 - a5 - a7 - ep - z1 - z3] Gamma[2 - a2 - a3 - ep + z3]
   Gamma[a7 + z1 - z4] Gamma[-2 + a4 + a5 + a6 + a7 + ep + z1 - z4] Gamma[-z2 - z3 - z4]
   Gamma[-2 + a1 + a2 + a3 + ep + z4] Gamma[z2 + z4] Gamma[z3 + z4] Gamma[a5 + z1 + z2 + z3 + z4]) /
 (Gamma[a2] Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7] Gamma[4 - a4 - a5 - a6 - a7 - 2 ep]
   Gamma[4 - a1 - a2 - a3 - 2 ep + z1 - z4] Gamma[a3 + z2 + z4] Gamma[a1 + z3 + z4])

(*   a2→0; Gamma[-z1] Gamma[a2+z1]   *)
```

In[15]:=
```
-Residue[B2[a1, a2, a3, a4, a5, a6, a7, 0], {z1, 0}]
```

Out[15]=
```
(S^(4-a1-a2-a3-a4-a5-a6-a7-2 ep) Gamma[2 - a5 - a6 - a7 - ep - z2]
   Gamma[2 - a1 - a2 - ep + z2] Gamma[2 - a4 - a5 - a7 - ep - z3] Gamma[2 - a2 - a3 - ep + z3]
   Gamma[a7 - z4] Gamma[-2 + a4 + a5 + a6 + a7 + ep - z4] Gamma[-z2 - z3 - z4]
   Gamma[-2 + a1 + a2 + a3 + ep + z4] Gamma[z2 + z4] Gamma[z3 + z4] Gamma[a5 + z2 + z3 + z4]) /
 (Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7] Gamma[4 - a4 - a5 - a6 - a7 - 2 ep]
   Gamma[4 - a1 - a2 - a3 - 2 ep - z4] Gamma[a3 + z2 + z4] Gamma[a1 + z3 + z4])
```

In[16]:=
```
% /. a2 → 0
```

Out[16]=
```
(S^(4-a1-a3-a4-a5-a6-a7-2 ep) Gamma[2 - a5 - a6 - a7 - ep - z2]
   Gamma[2 - a1 - ep + z2] Gamma[2 - a4 - a5 - a7 - ep - z3] Gamma[2 - a3 - ep + z3]
   Gamma[a7 - z4] Gamma[-2 + a4 + a5 + a6 + a7 + ep - z4] Gamma[-z2 - z3 - z4]
   Gamma[-2 + a1 + a3 + ep + z4] Gamma[z2 + z4] Gamma[z3 + z4] Gamma[a5 + z2 + z3 + z4]) /
 (Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7] Gamma[4 - a4 - a5 - a6 - a7 - 2 ep]
   Gamma[4 - a1 - a3 - 2 ep - z4] Gamma[a3 + z2 + z4] Gamma[a1 + z3 + z4])

(*   a5→0; Gamma[a5+z2+z3+z4] Gamma[-z2-z3-z4]   *)
```

In[17]:= **-Residue[%, {z4, -z2 - z3}]**

Out[17]= (S^(4-a1-a3-a4-a5-a6-a7-2 ep) Gamma[2 - a5 - a6 - a7 - ep - z2] Gamma[-z2] Gamma[2 - a1 - ep + z2]
Gamma[2 - a4 - a5 - a7 - ep - z3] Gamma[-2 + a1 + a3 + ep - z2 - z3] Gamma[-z3]
Gamma[2 - a3 - ep + z3] Gamma[a7 + z2 + z3] Gamma[-2 + a4 + a5 + a6 + a7 + ep + z2 + z3]) /
(Gamma[a4] Gamma[a6] Gamma[a7] Gamma[4 - a4 - a5 - a6 - a7 - 2 ep]
Gamma[a1 - z2] Gamma[a3 - z3] Gamma[4 - a1 - a3 - 2 ep + z2 + z3])

In[18]:= **% /. a5 → 0**

Out[18]= (S^(4-a1-a3-a4-a6-a7-2 ep) Gamma[2 - a6 - a7 - ep - z2] Gamma[-z2] Gamma[2 - a1 - ep + z2]
Gamma[2 - a4 - a7 - ep - z3] Gamma[-2 + a1 + a3 + ep - z2 - z3] Gamma[-z3]
Gamma[2 - a3 - ep + z3] Gamma[a7 + z2 + z3] Gamma[-2 + a4 + a6 + a7 + ep + z2 + z3]) /
(Gamma[a4] Gamma[a6] Gamma[a7] Gamma[4 - a4 - a6 - a7 - 2 ep]
Gamma[a1 - z2] Gamma[a3 - z3] Gamma[4 - a1 - a3 - 2 ep + z2 + z3])

In[19]:= **-Residue[%, {z3, 0}]**

Out[19]= (S^(4-a1-a3-a4-a6-a7-2 ep) Gamma[2 - a3 - ep] Gamma[2 - a4 - a7 - ep]
Gamma[2 - a6 - a7 - ep - z2] Gamma[-2 + a1 + a3 + ep - z2] Gamma[-z2]
Gamma[a7 + z2] Gamma[2 - a1 - ep + z2] Gamma[-2 + a4 + a6 + a7 + ep + z2]) /
(Gamma[a3] Gamma[a4] Gamma[a6] Gamma[a7] Gamma[4 - a4 - a6 - a7 - 2 ep]
Gamma[a1 - z2] Gamma[4 - a1 - a3 - 2 ep + z2])

In[20]:= **-Residue[%, {z2, 0}]**

Out[20]= (S^(4-a1-a3-a4-a6-a7-2 ep) Gamma[2 - a1 - ep] Gamma[2 - a3 - ep] Gamma[2 - a4 - a7 - ep]
Gamma[2 - a6 - a7 - ep] Gamma[-2 + a1 + a3 + ep] Gamma[-2 + a4 + a6 + a7 + ep]) /
(Gamma[a1] Gamma[a3] Gamma[a4] Gamma[a6] Gamma[4 - a1 - a3 - 2 ep] Gamma[4 - a4 - a6 - a7 - 2 ep])

In[21]:= **% /. a7 → 0**

Out[21]= (S^(4-a1-a3-a4-a6-2 ep) Gamma[2 - a1 - ep] Gamma[2 - a3 - ep] Gamma[2 - a4 - ep]
Gamma[2 - a6 - ep] Gamma[-2 + a1 + a3 + ep] Gamma[-2 + a4 + a6 + ep]) /
(Gamma[a1] Gamma[a3] Gamma[a4] Gamma[a6] Gamma[4 - a1 - a3 - 2 ep] Gamma[4 - a4 - a6 - 2 ep])

In[22]:= **G[a1, a3] G[a4, a6] / %**

Out[22]= S^(-4+a1+a3+a4+a6+2 ep)

(*   a7→0; Gamma[a7+z2+z3] Gamma[-z2]Gamma[-z3]   *)

(*   a horizontal check: shrink horizontal lines,
a1,a3,a4,a6→0
***)**

In[23]:= **B2[a1, a2, a3, a4, a5, a6, a7, 0]**

Out[23]= (S^(4-a1-a2-a3-a4-a5-a6-a7-2 ep-z1) T^z1 Gamma[-z1] Gamma[a2 + z1] Gamma[2 - a5 - a6 - a7 - ep - z1 - z2]
Gamma[2 - a1 - a2 - ep + z2] Gamma[2 - a4 - a5 - a7 - ep - z1 - z3] Gamma[2 - a2 - a3 - ep + z3]
Gamma[a7 + z1 - z4] Gamma[-2 + a4 + a5 + a6 + a7 + ep + z1 - z4] Gamma[-z2 - z3 - z4]
Gamma[-2 + a1 + a2 + a3 + ep + z4] Gamma[z2 + z4] Gamma[z3 + z4] Gamma[a5 + z1 + z2 + z3 + z4]) /
(Gamma[a2] Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7] Gamma[4 - a4 - a5 - a6 - a7 - 2 ep]
Gamma[4 - a1 - a2 - a3 - 2 ep + z1 - z4] Gamma[a3 + z2 + z4] Gamma[a1 + z3 + z4])

Gamma[-2 + a4 + a5 + a6 + a7 + ep + z1 - z4] Gamma[2 - a4 - a5 - a7 - ep - z1 - z3] Gamma[z3 + z4]

-2 + a4 + a5 + a6 + a7 + ep + z1 - z4 + 2 - a4 - a5 - a7 - ep - z1 - z3 + z3 + z4

a6

```
In[24]:= Residue[B2[a1, a2, a3, a4, a5, a6, a7, 0], {z3, -z4}]

Out[24]= (S^(4-a1-a2-a3-a4-a5-a6-a7-2 ep-z1) T^z1 Gamma[-z1] Gamma[a2 + z1]
    Gamma[2 - a5 - a6 - a7 - ep - z1 - z2] Gamma[-z2] Gamma[2 - a1 - a2 - ep + z2] Gamma[a5 + z1 + z2]
    Gamma[2 - a2 - a3 - ep - z4] Gamma[a7 + z1 - z4] Gamma[-2 + a4 + a5 + a6 + a7 + ep + z1 - z4]
    Gamma[-2 + a1 + a2 + a3 + ep + z4] Gamma[2 - a4 - a5 - a7 - ep - z1 + z4] Gamma[z2 + z4]) /
  (Gamma[a1] Gamma[a2] Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7]
    Gamma[4 - a4 - a5 - a6 - a7 - 2 ep] Gamma[4 - a1 - a2 - a3 - 2 ep + z1 - z4] Gamma[a3 + z2 + z4])

In[25]:= -Residue[%, {z4, -2 + a4 + a5 + a6 + a7 + ep + z1}]

Out[25]= (S^(4-a1-a2-a3-a4-a5-a6-a7-2 ep-z1) T^z1 Gamma[2 - a4 - a5 - a6 - ep]
    Gamma[4 - a2 - a3 - a4 - a5 - a6 - a7 - 2 ep - z1] Gamma[-z1]
    Gamma[a2 + z1] Gamma[-4 + a1 + a2 + a3 + a4 + a5 + a6 + a7 + 2 ep + z1]
    Gamma[2 - a5 - a6 - a7 - ep - z1 - z2] Gamma[-z2] Gamma[2 - a1 - a2 - ep + z2]
    Gamma[a5 + z1 + z2] Gamma[-2 + a4 + a5 + a6 + a7 + ep + z1 + z2]) /
  (Gamma[a1] Gamma[a2] Gamma[a4] Gamma[a5] Gamma[a7]
    Gamma[6 - a1 - a2 - a3 - a4 - a5 - a6 - a7 - 3 ep] Gamma[4 - a4 - a5 - a6 - a7 - 2 ep]
    Gamma[-2 + a3 + a4 + a5 + a6 + a7 + ep + z1 + z2])

In[26]:= % /. a6 → 0

Out[26]= (S^(4-a1-a2-a3-a4-a5-a7-2 ep-z1) T^z1 Gamma[2 - a4 - a5 - ep]
    Gamma[4 - a2 - a3 - a4 - a5 - a7 - 2 ep - z1] Gamma[-z1] Gamma[a2 + z1]
    Gamma[-4 + a1 + a2 + a3 + a4 + a5 + a7 + 2 ep + z1] Gamma[2 - a5 - a7 - ep - z1 - z2] Gamma[-z2]
    Gamma[2 - a1 - a2 - ep + z2] Gamma[a5 + z1 + z2] Gamma[-2 + a4 + a5 + a7 + ep + z1 + z2]) /
  (Gamma[a1] Gamma[a2] Gamma[a4] Gamma[a5] Gamma[a7] Gamma[6 - a1 - a2 - a3 - a4 - a5 - a7 - 3 ep]
    Gamma[4 - a4 - a5 - a7 - 2 ep] Gamma[-2 + a3 + a4 + a5 + a7 + ep + z1 + z2])

(* a1,a3,a4→0  *)

(*    let a4→0    *)

Gamma[-2 + a4 + a5 + a7 + ep + z1 + z2] Gamma[2 - a5 - a7 - ep - z1 - z2]

-2 + a4 + a5 + a7 + ep + z1 + z2 + 2 - a5 - a7 - ep - z1 - z2

a4

In[27]:= -Residue[%, {z2, 2 - a5 - a7 - ep - z1}]

Out[27]= (S^(4-a1-a2-a3-a4-a5-a7-2 ep-z1) T^z1 Gamma[2 - a4 - a5 - ep] Gamma[2 - a7 - ep]
    Gamma[4 - a1 - a2 - a5 - a7 - 2 ep - z1] Gamma[4 - a2 - a3 - a4 - a5 - a7 - 2 ep - z1] Gamma[-z1]
    Gamma[a2 + z1] Gamma[-2 + a5 + a7 + ep + z1] Gamma[-4 + a1 + a2 + a3 + a4 + a5 + a7 + 2 ep + z1]) /
  (Gamma[a1] Gamma[a2] Gamma[a3 + a4] Gamma[a5] Gamma[a7]
    Gamma[6 - a1 - a2 - a3 - a4 - a5 - a7 - 3 ep] Gamma[4 - a4 - a5 - a7 - 2 ep])

In[28]:= % /. a4 → 0

Out[28]= (S^(4-a1-a2-a3-a5-a7-2 ep-z1) T^z1 Gamma[2 - a5 - ep] Gamma[2 - a7 - ep]
    Gamma[4 - a1 - a2 - a5 - a7 - 2 ep - z1] Gamma[4 - a2 - a3 - a5 - a7 - 2 ep - z1] Gamma[-z1]
    Gamma[a2 + z1] Gamma[-2 + a5 + a7 + ep + z1] Gamma[-4 + a1 + a2 + a3 + a5 + a7 + 2 ep + z1]) /
  (Gamma[a1] Gamma[a2] Gamma[a3] Gamma[a5] Gamma[a7]
    Gamma[6 - a1 - a2 - a3 - a5 - a7 - 3 ep] Gamma[4 - a5 - a7 - 2 ep])

(* a1,a3→0  *)

(*    let a1→0    *)

Gamma[-4 + a1 + a2 + a3 + a5 + a7 + 2 ep + z1] Gamma[4 - a2 - a3 - a5 - a7 - 2 ep - z1]
```

```
-4 + a1 + a2 + a3 + a5 + a7 + 2 ep + z1 + 4 - a2 - a3 - a5 - a7 - 2 ep - z1

a1
```

```
In[29]:= -Residue[%, {z1, 4 - a2 - a3 - a5 - a7 - 2 ep}]

Out[29]= (S^(-a1) T^(4-a2-a3-a5-a7-2 ep) Gamma[-a1 + a3] Gamma[4 - a3 - a5 - a7 - 2 ep] Gamma[2 - a2 - a3 - ep]
    Gamma[2 - a5 - ep] Gamma[2 - a7 - ep] Gamma[-4 + a2 + a3 + a5 + a7 + 2 ep]) / (Gamma[a2]
    Gamma[a3] Gamma[a5] Gamma[a7] Gamma[6 - a1 - a2 - a3 - a5 - a7 - 3 ep] Gamma[4 - a5 - a7 - 2 ep])

In[30]:= % /. a1 → 0

Out[30]= (T^(4-a2-a3-a5-a7-2 ep) Gamma[4 - a3 - a5 - a7 - 2 ep] Gamma[2 - a2 - a3 - ep]
    Gamma[2 - a5 - ep] Gamma[2 - a7 - ep] Gamma[-4 + a2 + a3 + a5 + a7 + 2 ep]) /
   (Gamma[a2] Gamma[a5] Gamma[a7] Gamma[6 - a2 - a3 - a5 - a7 - 3 ep] Gamma[4 - a5 - a7 - 2 ep])

In[31]:= % /. a3 → 0

Out[31]= (T^(4-a2-a5-a7-2 ep) Gamma[2 - a2 - ep]
    Gamma[2 - a5 - ep] Gamma[2 - a7 - ep] Gamma[-4 + a2 + a5 + a7 + 2 ep]) /
   (Gamma[a2] Gamma[a5] Gamma[a7] Gamma[6 - a2 - a5 - a7 - 3 ep])

In[32]:= G[a2, a7] G[a2 + a7 + ep - 2, a5] / %

Out[32]= T^(-4+a2+a5+a7+2 ep)
```

```
(*  In addition to the usual factor  (I Pi^(d/2))^2,
let us pull out the factor

  S^(4-a1-a2-a3-a4-a5-a6-a7-a8-2 ep).

   Let us turn to the variable x=T/S = t/s      *)
```

```
In[33]:= K2[a1_, a2_, a3_, a4_, a5_, a6_, a7_, a8_] :=
  ( x^z1 Gamma[-z1] Gamma[a2 + z1] Gamma[2 - a5 - a6 - a7 - ep - z1 - z2]
    Gamma[2 - a1 - a2 - a8 - ep + z2] Gamma[2 - a4 - a5 - a7 - ep - z1 - z3]
    Gamma[2 - a2 - a3 - a8 - ep + z3] Gamma[a7 + z1 - z4] Gamma[-2 + a4 + a5 + a6 + a7 + ep + z1 - z4]
    Gamma[a8 - z2 - z3 - z4] Gamma[-z1 - z2 - z3 - z4] Gamma[-2 + a1 + a2 + a3 + a8 + ep + z4]
    Gamma[z2 + z4] Gamma[z3 + z4] Gamma[a5 + z1 + z2 + z3 + z4]) /
   (Gamma[a2] Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7] Gamma[4 - a4 - a5 - a6 - a7 - 2 ep]
    Gamma[4 - a1 - a2 - a3 - a8 - 2 ep + z1 - z4]
    Gamma[a8 - z1 - z2 - z3 - z4] Gamma[a3 + z2 + z4] Gamma[a1 + z3 + z4])
```

```
(*  The 2-box with the powers of the propagators equal to one *)
```

```
In[34]:= B2 = x K2[1, 1, 1, 1, 1, 1, 1, 0]

Out[34]= (x^(1+z1) Gamma[-z1] Gamma[1 + z1] Gamma[-1 - ep - z1 - z2] Gamma[-ep + z2] Gamma[-1 - ep - z1 - z3]
    Gamma[-ep + z3] Gamma[1 + z1 - z4] Gamma[2 + ep + z1 - z4] Gamma[-z2 - z3 - z4]
    Gamma[1 + ep + z4] Gamma[z2 + z4] Gamma[z3 + z4] Gamma[1 + z1 + z2 + z3 + z4]) /
   (Gamma[-2 ep] Gamma[1 - 2 ep + z1 - z4] Gamma[1 + z2 + z4] Gamma[1 + z3 + z4])
```

```
(*      auxiliary functions        *)
```

```
In[35]:= SortByDimension[l_List] := Sort[l, Length[#1[[2, 2]]] > Length[#2[[2, 2]]] &];
CoeffEps[X_, n_] := (X /. X[[1]] → Simplify[Coefficient[X[[1]], ep, n]]);
MBDimension[int_MBint] := Length[int[[2, 2]]];
```

In[38]:= **B2rules = MBoptimizedRules[B2, ep → 0, {}, {ep}]**

Out[38]= $\left\{\left\{ep \to -\frac{9}{16}\right\}, \left\{z1 \to -\frac{1}{2}, z2 \to -\frac{5}{16}, z3 \to -\frac{3}{8}, z4 \to \frac{7}{16}\right\}\right\}$

In[39]:= **B2cont = MBcontinue[B2, ep → 0, B2rules];**

```
Level 1
Taking -residue in z2 = -1 - ep - z1
Taking +residue in z2 = ep
Taking -residue in z3 = -1 - ep - z1
Taking +residue in z3 = ep
Level 2
Integral {1}
Taking -residue in z3 = -1 - ep - z1
Taking +residue in z4 = 1 + ep + z1
Integral {2}
Taking -residue in z1 = -1 - 2 ep
Taking -residue in z3 = -1 - ep - z1
Taking -residue in z4 = -ep - z3
Integral {3}
Taking +residue in z4 = 1 + ep + z1
Integral {4}
Taking -residue in z1 = -1 - 2 ep
Taking -residue in z2 = -1 - ep - z1
Taking +residue in z2 = ep
Taking -residue in z4 = -ep - z2
Level 3
Integral {1, 1}
Taking +residue in z4 = 1 + 2 ep + z1
Taking +residue in z4 = 1 + ep + z1
Integral {1, 2}
Integral {2, 1}
Taking -residue in z4 = -2 ep
Taking -residue in z4 = -ep - z3
Integral {2, 2}
Taking +residue in z4 = 1 + ep + z1
Integral {2, 3}
Integral {3, 1}
Integral {4, 1}
Taking -residue in z4 = -2 ep
```

```
Taking -residue in z4 = -ep - z2
Integral {4, 2}
Taking +residue in z4 = 1 + ep + z1
Integral {4, 3}
Taking -residue in z1 = -1 - 2 ep
Taking -residue in z4 = -2 ep
Integral {4, 4}
Level 4
Integral {1, 1, 1}
Integral {1, 1, 2}
Integral {2, 1, 1}
Taking +residue in z3 = 3 ep
Taking +residue in z3 = 2 ep
Integral {2, 1, 2}
Integral {2, 2, 1}
Integral {4, 1, 1}
Taking +residue in z2 = 3 ep
Taking +residue in z2 = 2 ep
Integral {4, 1, 2}
Integral {4, 2, 1}
Integral {4, 3, 1}
Taking -residue in z4 = -4 ep
...no contribution
Taking -residue in z4 = -2 ep
Integral {4, 3, 2}
Level 5
Integral {2, 1, 1, 1}
Integral {2, 1, 1, 2}
Integral {4, 1, 1, 1}
Integral {4, 1, 1, 2}
Integral {4, 3, 1, 1}
30 integral(s) found
```

```
In[40]:= B2select = MBpreselect[B2cont, {ep, 0, 0}]

In[41]:= B2exp = Simplify[MBexpand[B2select, E^(2 EulerGamma ep), {ep, 0, 0}]]
```

In[42]:= **B2expS = SortByDimension[MBmerge[B2exp]]**

Out[42]=
$$\Big\{\text{MBint}\Big[\frac{\text{Gamma}[-z3]\,\text{Gamma}[z3]\,\text{Gamma}[1-z4]\,\text{Gamma}[-z3-z4]\,\text{Gamma}[z4]\,\text{Gamma}[z3+z4]^2}{\text{Gamma}[1+z3+z4]},\ \Big\{\{ep\to 0\},\ \Big\{z3\to-\frac{3}{8},\ z4\to\frac{7}{16}\Big\}\Big\}\Big],$$

$$\text{MBint}\Big[\frac{\text{Gamma}[-z2]\,\text{Gamma}[z2]\,\text{Gamma}[1-z4]\,\text{Gamma}[-z2-z4]\,\text{Gamma}[z4]\,\text{Gamma}[z2+z4]^2}{\text{Gamma}[1+z2+z4]},\ \Big\{\{ep\to 0\},\ \Big\{z2\to-\frac{5}{16},\ z4\to\frac{7}{16}\Big\}\Big\}\Big],$$

$$\text{MBint}\Big[2\,x^{1+z1}\,\text{Gamma}[-1-z1]\,\text{Gamma}[-z1]\,\text{Gamma}[1+z1]^2\,\text{Gamma}[-1-z1-z3]\,\text{Gamma}[-z3]\,\text{Gamma}[z3]\,\text{Gamma}[2+z1+z3],\ \Big\{\{ep\to 0\},\ \Big\{z1\to-\frac{1}{2},\ z3\to-\frac{3}{8}\Big\}\Big\}\Big],$$

$$\text{MBint}\Big[2\,x^{1+z1}\,\text{Gamma}[-1-z1]\,\text{Gamma}[-z1]\,\text{Gamma}[1+z1]^2\,\text{Gamma}[-1-z1-z2]\,\text{Gamma}[-z2]\,\text{Gamma}[z2]\,\text{Gamma}[2+z1+z2],\ \Big\{\{ep\to 0\},\ \Big\{z1\to-\frac{1}{2},\ z2\to-\frac{5}{16}\Big\}\Big\}\Big],$$

$$\text{MBint}\Big[-\frac{1}{2\,ep\,\text{Gamma}[1+z4]}\text{Gamma}[1-z4]\,\text{Gamma}[-z4]\,\text{Gamma}[z4]^3\,(1+4\,ep\,\text{EulerGamma}+ep\,\text{PolyGamma}[0,1-z4]+4\,ep\,\text{PolyGamma}[0,z4]-ep\,\text{PolyGamma}[0,1+z4]),\ \Big\{\{ep\to 0\},\ \Big\{z4\to\frac{7}{16}\Big\}\Big\}\Big],$$

$$\text{MBint}\Big[\frac{1}{12\,\text{Gamma}[1+z3]}\text{Gamma}[-z3]^2\,\text{Gamma}[z3]\Big(24\,\text{Gamma}[z3]^2+\frac{1}{ep^2}\text{Gamma}[1+z3]^2\big(6+12\,ep\,\text{EulerGamma}+12\,ep^2\,\text{EulerGamma}^2+4\,ep^2\,\pi^2-12\,ep\,\text{Log}[x]-24\,ep^2\,\text{EulerGamma}\,\text{Log}[x]+12\,ep^2\,\text{Log}[x]^2+3\,ep^2\,\text{PolyGamma}[0,-z3]^2+6\,ep\,(1+2\,ep\,\text{EulerGamma}-2\,ep\,\text{Log}[x])\,\text{PolyGamma}[0,z3]+3\,ep^2\,\text{PolyGamma}[0,z3]^2+6\,ep\,\text{PolyGamma}[0,-z3]\,(1+2\,ep\,\text{EulerGamma}-2\,ep\,\text{Log}[x]+ep\,\text{PolyGamma}[0,z3])+3\,ep^2\,\text{PolyGamma}[1,-z3]-21\,ep^2\,\text{PolyGamma}[1,z3]\big)\Big),\ \Big\{\{ep\to 0\},\ \Big\{z3\to-\frac{3}{8}\Big\}\Big\}\Big],$$

$$\text{MBint}\Big[\frac{1}{12\,\text{Gamma}[1+z2]}\text{Gamma}[-z2]^2\,\text{Gamma}[z2]\Big(24\,\text{Gamma}[z2]^2+\frac{1}{ep^2}\text{Gamma}[1+z2]^2\big(6+12\,ep\,\text{EulerGamma}+12\,ep^2\,\text{EulerGamma}^2+4\,ep^2\,\pi^2-12\,ep\,\text{Log}[x]-24\,ep^2\,\text{EulerGamma}\,\text{Log}[x]+12\,ep^2\,\text{Log}[x]^2+3\,ep^2\,\text{PolyGamma}[0,-z2]^2+6\,ep\,(1+2\,ep\,\text{EulerGamma}-2\,ep\,\text{Log}[x])\,\text{PolyGamma}[0,z2]+3\,ep^2\,\text{PolyGamma}[0,z2]^2+6\,ep\,\text{PolyGamma}[0,-z2]\,(1+2\,ep\,\text{EulerGamma}-2\,ep\,\text{Log}[x]+ep\,\text{PolyGamma}[0,z2])+3\,ep^2\,\text{PolyGamma}[1,-z2]-21\,ep^2\,\text{PolyGamma}[1,z2]\big)\Big),\ \Big\{\{ep\to 0\},\ \Big\{z2\to-\frac{5}{16}\Big\}\Big\}\Big],$$

$$\text{MBint}\Big[\frac{1}{ep}4\,x^{1+z1}\,\text{Gamma}[-1-z1]^2\,\text{Gamma}[-z1]\,\text{Gamma}[1+z1]^2\,(3\,ep\,\text{Gamma}[1+z1]+\text{Gamma}[2+z1]\,(-1+4\,ep\,\text{PolyGamma}[0,-1-z1]-2\,ep\,\text{PolyGamma}[0,1+z1]-2\,ep\,\text{PolyGamma}[0,2+z1])),\ \Big\{\{ep\to 0\},\ \Big\{z1\to-\frac{1}{2}\Big\}\Big\}\Big],$$

$$\text{MBint}\Big[-\frac{1}{18\,ep^4}\big(-72+42\,ep^2\,\pi^2+20\,ep^4\,\pi^4+6\,ep^2\,\big(-6+17\,ep^2\,\pi^2\big)\,\text{Log}[x]^2-12\,ep^3\,\text{Log}[x]^3+24\,ep^4\,\text{Log}[x]^4-213\,ep^3\,\text{PolyGamma}[2,1]+3\,ep\,\text{Log}[x]\,\big(30-31\,ep^2\,\pi^2+94\,ep^3\,\text{PolyGamma}[2,1]\big)\big),\ \{\{ep\to 0\},\ \{\}\}\Big]\Big\}$$

In[43]:= **Length[B2expS]**

Out[43]= 9

In[44]:= **MBDimension /@ B2expS**

Out[44]= {2, 2, 2, 2, 1, 1, 1, 1, 0}

In[45]:= **B2expS[[1]]**

Out[45]= $\text{MBint}\left[\frac{\text{Gamma}[-z3]\,\text{Gamma}[z3]\,\text{Gamma}[1-z4]\,\text{Gamma}[-z3-z4]\,\text{Gamma}[z4]\,\text{Gamma}[z3+z4]^2}{\text{Gamma}[1+z3+z4]},\right.$
$\left.\left\{\{ep\to 0\},\left\{z3\to -\frac{3}{8},\ z4\to \frac{7}{16}\right\}\right\}\right]$

In[46]:= **Barnes2[B2expS[[1]], z4]**

Out[46]= $\text{MBint}\left[\frac{1}{6}\pi^2\,\text{Gamma}[-z3]^2\,\text{Gamma}[z3]\,\text{Gamma}[1+z3]-\right.$
$\left.\text{Gamma}[-z3]^2\,\text{Gamma}[z3]\,\text{Gamma}[1+z3]\,\text{PolyGamma}[1,1+z3],\ \left\{\{ep\to 0\},\left\{z3\to -\frac{3}{8}\right\}\right\}\right]$

In[47]:= **res01 =** $\frac{17}{4}$ **Zeta[4]**

Out[47]= $\frac{17\,\pi^4}{360}$

In[48]:= **% // N**

Out[48]= 4.59987

In[49]:= **NIntegrate[Barnes2[B2expS[[1]], z4] [[1]] / (2 Pi) /.** $\left\{z3\to -\frac{3}{8}+I*y1\right\}$**,**
{y1, -Infinity, Infinity}]

Out[49]= 4.59987 + 0. ⅈ

In[50]:= **B2expS[[2]]**

Out[50]= $\text{MBint}\left[\frac{\text{Gamma}[-z2]\,\text{Gamma}[z2]\,\text{Gamma}[1-z4]\,\text{Gamma}[-z2-z4]\,\text{Gamma}[z4]\,\text{Gamma}[z2+z4]^2}{\text{Gamma}[1+z2+z4]},\right.$
$\left.\left\{\{ep\to 0\},\left\{z2\to -\frac{5}{16},\ z4\to \frac{7}{16}\right\}\right\}\right]$

In[51]:= **Barnes2[B2expS[[2]], z4]**

Out[51]= $\text{MBint}\left[\frac{1}{6}\pi^2\,\text{Gamma}[-z2]^2\,\text{Gamma}[z2]\,\text{Gamma}[1+z2]-\right.$
$\left.\text{Gamma}[-z2]^2\,\text{Gamma}[z2]\,\text{Gamma}[1+z2]\,\text{PolyGamma}[1,1+z2],\ \left\{\{ep\to 0\},\left\{z2\to -\frac{5}{16}\right\}\right\}\right]$

In[52]:= **res02 =** $\frac{17}{4}$ **Zeta[4]**

Out[52]= $\frac{17\,\pi^4}{360}$

In[53]:= **% // N**

Out[53]= 4.59987

In[54]:= NIntegrate[Barnes2[B2expS[[2]], z4] [[1]] / (2 Pi) /. {z2 → -5/16 + I * y1},
{y1, -Infinity, Infinity}]

Out[54]= $4.59987 - 2.88547 \times 10^{-13}\, i$

In[55]:= B2expS[[3]]

Out[55]= MBint[2 x^{1+z1} Gamma[-1 - z1] Gamma[-z1] Gamma[1 + z1]^2 Gamma[-1 - z1 - z3]
Gamma[-z3] Gamma[z3] Gamma[2 + z1 + z3], {{ep → 0}, {z1 → -1/2, z3 → -3/8}}]

In[56]:= Barnes1[B2expS[[3]], z3]

Out[56]= MBint[-2 x^{1+z1} Gamma[-1 - z1]^2 Gamma[-z1] Gamma[1 + z1]^3 +
2 EulerGamma x^{1+z1} Gamma[-1 - z1]^2 Gamma[-z1] Gamma[1 + z1]^2 Gamma[2 + z1] +
2 x^{1+z1} Gamma[-1 - z1]^2 Gamma[-z1] Gamma[1 + z1]^2 Gamma[2 + z1] PolyGamma[0, 2 + z1],
{{ep → 0}, {z1 → -1/2}}]

In[57]:= B2expS[[4]]

Out[57]= MBint[2 x^{1+z1} Gamma[-1 - z1] Gamma[-z1] Gamma[1 + z1]^2 Gamma[-1 - z1 - z2]
Gamma[-z2] Gamma[z2] Gamma[2 + z1 + z2], {{ep → 0}, {z1 → -1/2, z2 → -5/16}}]

In[58]:= Barnes1[B2expS[[4]], z2]

Out[58]= MBint[-2 x^{1+z1} Gamma[-1 - z1]^2 Gamma[-z1] Gamma[1 + z1]^3 +
2 EulerGamma x^{1+z1} Gamma[-1 - z1]^2 Gamma[-z1] Gamma[1 + z1]^2 Gamma[2 + z1] +
2 x^{1+z1} Gamma[-1 - z1]^2 Gamma[-z1] Gamma[1 + z1]^2 Gamma[2 + z1] PolyGamma[0, 2 + z1],
{{ep → 0}, {z1 → -1/2}}]

In[59]:= B2expS[[5]]

Out[59]= MBint[-1/(2 ep Gamma[1 + z4])
Gamma[1 - z4] Gamma[-z4] Gamma[z4]^3 (1 + 4 ep EulerGamma + ep PolyGamma[0, 1 - z4] +
4 ep PolyGamma[0, z4] - ep PolyGamma[0, 1 + z4]), {{ep → 0}, {z4 → 7/16}}]

In[60]:= CoeffEps[B2expS[[5]], -1]

Out[60]= MBint[-(Gamma[1 - z4] Gamma[-z4] Gamma[z4]^3)/(2 Gamma[1 + z4]), {{ep → 0}, {z4 → 7/16}}]

In[61]:= CoeffEps[B2expS[[5]], -1] /. {Gamma[1 - z4] → -z4 Gamma[-z4], Gamma[1 + z4] → z4 Gamma[z4]}

Out[61]= MBint[1/2 Gamma[-z4]^2 Gamma[z4]^2, {{ep → 0}, {z4 → 7/16}}]

In[62]:= res11 = Barnes1[%, z4][[1]]

Out[62]= -1/2 PolyGamma[2, 1]

In[63]:= **CoeffEps[B2expS[[5]], 0]**

Out[63]= MBint$\left[-\frac{1}{2\,\text{Gamma}[1+z4]}\text{Gamma}[1-z4]\,\text{Gamma}[-z4]\,\text{Gamma}[z4]^3\right.$
$(4\,\text{EulerGamma}+\text{PolyGamma}[0,1-z4]+4\,\text{PolyGamma}[0,z4]-\text{PolyGamma}[0,1+z4]),$
$\left.\left\{\{ep\to 0\},\left\{z4\to\frac{7}{16}\right\}\right\}\right]$

In[64]:= **res03 = $-\frac{11}{2}$ Zeta[4]**

Out[64]= $-\frac{11\,\pi^4}{180}$

In[65]:= **% // N**

Out[65]= −5.95278

In[66]:= **NIntegrate[CoeffEps[B2expS[[5]], 0][[1]] / (2 Pi) /. {z4 → $\frac{7}{16}$ + I * y1},**
{y1, -Infinity, Infinity}]

Out[66]= −5.95278 + 0. ⅈ

In[67]:= **B2expS[[6]]**

Out[67]= MBint$\left[\frac{1}{12\,\text{Gamma}[1+z3]}\text{Gamma}[-z3]^2\,\text{Gamma}[z3]\right.$
$\left(24\,\text{Gamma}[z3]^2+\frac{1}{ep^2}\text{Gamma}[1+z3]^2\,\left(6+12\,ep\,\text{EulerGamma}+12\,ep^2\,\text{EulerGamma}^2+4\,ep^2\,\pi^2-\right.\right.$
$12\,ep\,\text{Log}[x]-24\,ep^2\,\text{EulerGamma}\,\text{Log}[x]+12\,ep^2\,\text{Log}[x]^2+3\,ep^2\,\text{PolyGamma}[0,-z3]^2+$
$6\,ep\,(1+2\,ep\,\text{EulerGamma}-2\,ep\,\text{Log}[x])\,\text{PolyGamma}[0,z3]+3\,ep^2\,\text{PolyGamma}[0,z3]^2+$
$6\,ep\,\text{PolyGamma}[0,-z3]\,(1+2\,ep\,\text{EulerGamma}-2\,ep\,\text{Log}[x]+ep\,\text{PolyGamma}[0,z3])+$
$\left.\left.3\,ep^2\,\text{PolyGamma}[1,-z3]-21\,ep^2\,\text{PolyGamma}[1,z3]\right)\right),\left.\left\{\{ep\to 0\},\left\{z3\to-\frac{3}{8}\right\}\right\}\right]$

In[68]:= **B2expS[[7]]**

Out[68]= MBint$\left[\frac{1}{12\,\text{Gamma}[1+z2]}\text{Gamma}[-z2]^2\,\text{Gamma}[z2]\right.$
$\left(24\,\text{Gamma}[z2]^2+\frac{1}{ep^2}\text{Gamma}[1+z2]^2\,\left(6+12\,ep\,\text{EulerGamma}+12\,ep^2\,\text{EulerGamma}^2+4\,ep^2\,\pi^2-\right.\right.$
$12\,ep\,\text{Log}[x]-24\,ep^2\,\text{EulerGamma}\,\text{Log}[x]+12\,ep^2\,\text{Log}[x]^2+3\,ep^2\,\text{PolyGamma}[0,-z2]^2+$
$6\,ep\,(1+2\,ep\,\text{EulerGamma}-2\,ep\,\text{Log}[x])\,\text{PolyGamma}[0,z2]+3\,ep^2\,\text{PolyGamma}[0,z2]^2+$
$6\,ep\,\text{PolyGamma}[0,-z2]\,(1+2\,ep\,\text{EulerGamma}-2\,ep\,\text{Log}[x]+ep\,\text{PolyGamma}[0,z2])+$
$\left.\left.3\,ep^2\,\text{PolyGamma}[1,-z2]-21\,ep^2\,\text{PolyGamma}[1,z2]\right)\right),\left.\left\{\{ep\to 0\},\left\{z2\to-\frac{5}{16}\right\}\right\}\right]$

In[69]:= **F67 =**

MBint[Simplify[(B2expS[[6]][[1]] /. z3 → z2) + B2expS[[7]][[1]]], {{ep → 0}, {z2 → $-\frac{1}{2}$}}]

Out[69]= $\text{MBint}\Big[\frac{1}{6\,\text{Gamma}[1+z2]}\text{Gamma}[-z2]^2\,\text{Gamma}[z2]$
$\Big(24\,\text{Gamma}[z2]^2+\frac{1}{ep^2}\text{Gamma}[1+z2]^2\,\big(6+12\,ep\,\text{EulerGamma}+12\,ep^2\,\text{EulerGamma}^2+4\,ep^2\,\pi^2-$
$12\,ep\,\text{Log}[x]-24\,ep^2\,\text{EulerGamma}\,\text{Log}[x]+12\,ep^2\,\text{Log}[x]^2+3\,ep^2\,\text{PolyGamma}[0,\,-z2]^2+$
$6\,ep\,(1+2\,ep\,\text{EulerGamma}-2\,ep\,\text{Log}[x])\,\text{PolyGamma}[0,\,z2]+3\,ep^2\,\text{PolyGamma}[0,\,z2]^2+$
$6\,ep\,\text{PolyGamma}[0,\,-z2]\,(1+2\,ep\,\text{EulerGamma}-2\,ep\,\text{Log}[x]+ep\,\text{PolyGamma}[0,\,z2])+$
$3\,ep^2\,\text{PolyGamma}[1,\,-z2]-21\,ep^2\,\text{PolyGamma}[1,\,z2]\big)\Big),\,\{\{ep\to 0\},\,\{z2\to -\frac{1}{2}\}\}\Big]$

In[70]:= **res04 = $-\frac{\pi^2}{6\,ep^{\wedge}2}+\frac{1}{ep}\left(\frac{\pi^2}{3}\,\text{Log}[x]+\text{Zeta}[3]\right)+\left(-\frac{\pi^2}{3}\,\text{Log}[x]^{\wedge}2-2\,\text{Zeta}[3]\,\text{Log}[x]+\frac{\pi^4}{9}\right)$;**

In[71]:= **res04 /. {ep → 0.3, x → 0.1}**

Out[71]= -40.6045

In[72]:= **NIntegrate[F67[[1]] / (2 Pi) /. {ep → 0.3, x → 0.1, z2 → $-\frac{1}{2}$ + I * y1},**
{y1, -Infinity, Infinity}]

Out[72]= -40.6045 + 0. i

In[73]:= **B2expS[[8]]**

Out[73]= $\text{MBint}\Big[\frac{1}{ep}4\,x^{1+z1}\,\text{Gamma}[-1-z1]^2\,\text{Gamma}[-z1]\,\text{Gamma}[1+z1]^2$
$(3\,ep\,\text{Gamma}[1+z1]+\text{Gamma}[2+z1]\,(-1+4\,ep\,\text{PolyGamma}[0,\,-1-z1]-$
$2\,ep\,\text{PolyGamma}[0,\,1+z1]-2\,ep\,\text{PolyGamma}[0,\,2+z1])),\,\{\{ep\to 0\},\,\{z1\to -\frac{1}{2}\}\}\Big]$

In[74]:= **B2expS[[9]][[1]]**

Out[74]= $-\frac{1}{18\,ep^4}\big(-72+42\,ep^2\,\pi^2+20\,ep^4\,\pi^4+6\,ep^2\,\left(-6+17\,ep^2\,\pi^2\right)\,\text{Log}[x]^2-12\,ep^3\,\text{Log}[x]^3+24\,ep^4\,\text{Log}[x]^4-$
$213\,ep^3\,\text{PolyGamma}[2,\,1]+3\,ep\,\text{Log}[x]\,\left(30-31\,ep^2\,\pi^2+94\,ep^3\,\text{PolyGamma}[2,\,1]\right)\big)$

(* Collecting result for terms with trivial dependence on x *)

In[75]:= **restr = Apart[FullSimplify[res01 + res02 + res11 / ep + res03 + res04 + B2expS[[9]][[1]]], ep]**

Out[75]= $\frac{4}{ep^4}-\frac{5\,\text{Log}[x]}{ep^3}+\frac{-5\,\pi^2+4\,\text{Log}[x]^2}{2\,ep^2}+\frac{33\,\pi^2\,\text{Log}[x]+4\,\text{Log}[x]^3-130\,\text{Zeta}[3]}{6\,ep}+$
$\frac{1}{30}\left(-29\,\pi^4-180\,\pi^2\,\text{Log}[x]^2-40\,\text{Log}[x]^4+880\,\text{Log}[x]\,\text{Zeta}[3]\right)$

(* Collecting terms with nontrivial dependence on x *)

In[76]:= `MBmerge[{Barnes1[B2expS[[3]], z3] , Barnes1[B2expS[[4]], z2] , B2expS[[8]]}][[1]]`

Out[76]= $\text{MBint}\left[\frac{1}{ep} 4\, x^{1+z1}\, \text{Gamma}[-1-z1]^2\, \text{Gamma}[-z1]\, \text{Gamma}[1+z1]^2 \left(2\, ep\, \text{Gamma}[1+z1] + \text{Gamma}[2+z1]\, (-1 + ep\, \text{EulerGamma} + 4\, ep\, \text{PolyGamma}[0, -1-z1] - 2\, ep\, \text{PolyGamma}[0, 1+z1] - ep\, \text{PolyGamma}[0, 2+z1])\right), \left\{\{ep \to 0\}, \left\{z1 \to -\frac{1}{2}\right\}\right\}\right]$

In[77]:= $\text{NIntegrate}\left[\%[[1]] / (2\,\text{Pi}) \;\; /. \left\{ep \to 0.3,\, x \to 0.1,\, z1 \to -\frac{1}{2} + I * y1\right\}, \{y1, -\text{Infinity}, \text{Infinity}\}\right]$

Out[77]= $-7.68026 + 0.\, i$

In[78]:= $\text{resnontr} = -\frac{2\pi^2\,\text{Log}[1+x]}{ep} + \frac{10}{3}\pi^2\,\text{Log}[x]\,\text{Log}[1+x] - \frac{2\,\text{Log}[x]^2\,\text{Log}[1+x]}{ep} + \frac{8}{3}\text{Log}[x]^3\,\text{Log}[1+x] - \pi^2\,\text{Log}[1+x]^2 - \text{Log}[x]^2\,\text{Log}[1+x]^2 - \frac{20}{3}\pi^2\,\text{PolyLog}[2,-x] - \frac{4\,\text{Log}[x]\,\text{PolyLog}[2,-x]}{ep} - 2\,\text{Log}[x]^2\,\text{PolyLog}[2,-x] + \frac{4\,\text{PolyLog}[3,-x]}{ep} + 24\,\text{Log}[x]\,\text{PolyLog}[3,-x] - 4\left(-\frac{1}{2}\text{PolyLog}[2,-x]^2 - \text{Log}[1+x]\,\text{PolyLog}[3,-x]\right) - 44\,\text{PolyLog}[4,-x] + 4\,\text{Log}[x]\,(-\text{Log}[1+x]\,\text{PolyLog}[2,-x] - 2\,\text{PolyLog}[1,2,-x]) + 4\,\text{Log}[x]\,\text{PolyLog}[1,2,-x] - 4\left(\frac{1}{2}\text{PolyLog}[2,-x]^2 - 2\,\text{PolyLog}[2,2,-x]\right) - 4\,\text{PolyLog}[2,2,-x] - 4\,\text{Log}[1+x]\,\text{Zeta}[3];$

In[79]:= `% /. {ep → 0.3, x → 0.1}`

Out[79]= $-7.68026 - 3.59491 \times 10^{-15}\, i$

In[80]:= `resnontr + restr;`

```
Apart[%, ep]
```

$$\frac{4}{ep^4} - \frac{5\,Log[x]}{ep^3} + \frac{-5\,\pi^2 + 4\,Log[x]^2}{2\,ep^2} + \frac{1}{6\,ep}\left(33\,\pi^2\,Log[x] + 4\,Log[x]^3 - 12\,\pi^2\,Log[1+x] - 12\,Log[x]^2\,Log[1+x] - 24\,Log[x]\,PolyLog[2,-x] + 24\,PolyLog[3,-x] - 130\,Zeta[3]\right) + \frac{1}{30}\left(-29\,\pi^4 - 180\,\pi^2\,Log[x]^2 - 40\,Log[x]^4 + 100\,\pi^2\,Log[x]\,Log[1+x] + 80\,Log[x]^3\,Log[1+x] - 30\,\pi^2\,Log[1+x]^2 - 30\,Log[x]^2\,Log[1+x]^2 - 200\,\pi^2\,PolyLog[2,-x] - 60\,Log[x]^2\,PolyLog[2,-x] - 120\,Log[x]\,Log[1+x]\,PolyLog[2,-x] + 720\,Log[x]\,PolyLog[3,-x] + 120\,Log[1+x]\,PolyLog[3,-x] - 1320\,PolyLog[4,-x] - 120\,Log[x]\,PolyLog[1,2,-x] + 120\,PolyLog[2,2,-x] + 880\,Log[x]\,Zeta[3] - 120\,Log[1+x]\,Zeta[3]\right)$$

```
(*              the result              *)
```

$$1/x\left(\frac{4}{ep^4} - \frac{5\,Log[x]}{ep^3} + \frac{-5\,\pi^2 + 4\,Log[x]^2}{2\,ep^2} + \frac{1}{6\,ep}\left(33\,\pi^2\,Log[x] + 4\,Log[x]^3 - 12\,\pi^2\,Log[1+x] - 12\,Log[x]^2\,Log[1+x] - 24\,Log[x]\,PolyLog[2,-x] + 24\,PolyLog[3,-x] - 130\,Zeta[3]\right) + \frac{1}{30}\left(-29\,\pi^4 - 180\,\pi^2\,Log[x]^2 - 40\,Log[x]^4 + 100\,\pi^2\,Log[x]\,Log[1+x] + 80\,Log[x]^3\,Log[1+x] - 30\,\pi^2\,Log[1+x]^2 - 30\,Log[x]^2\,Log[1+x]^2 - 200\,\pi^2\,PolyLog[2,-x] - 60\,Log[x]^2\,PolyLog[2,-x] - 120\,Log[x]\,Log[1+x]\,PolyLog[2,-x] + 720\,Log[x]\,PolyLog[3,-x] + 120\,Log[1+x]\,PolyLog[3,-x] - 1320\,PolyLog[4,-x] - 120\,Log[x]\,PolyLog[1,2,-x] + 120\,PolyLog[2,2,-x] + 880\,Log[x]\,Zeta[3] - 120\,Log[1+x]\,Zeta[3]\right)\right)$$

```
In[1]:= Get["e:/MB/MB.m"];
        Get["e:/MB/MBresolve.m"];
        Get["e:/MB/barnesroutines.m"];

MB 1.2

by Michal Czakon

improvements by Alexander Smirnov

more info in hep-ph/0511200

last modified 2 Jan 09

MBresolve 1.0

by Alexander Smirnov

more info in arXiv:0901.0386

last modified 4 Jan 09

Barnes Routines, v 1.1.0 of June 5, 2009

In[4]:= MBDimension[int_MBint] := Length[int[[2, 2]]];
        SortByDimension[l_List] := Sort[l, Length[#1[[2, 2]]] > Length[#2[[2, 2]]] &];

In[5]:= B2 =
          (x^(1+z1) Gamma[-z1] Gamma[1 + z1] Gamma[-1 - ep - z1 - z2] Gamma[-ep + z2] Gamma[-1 - ep - z1 - z3]
             Gamma[-ep + z3] Gamma[1 + z1 - z4] Gamma[2 + ep + z1 - z4] Gamma[-z2 - z3 - z4]
             Gamma[1 + ep + z4] Gamma[z2 + z4] Gamma[z3 + z4] Gamma[1 + z1 + z2 + z3 + z4]) /
           (Gamma[-2 ep] Gamma[1 - 2 ep + z1 - z4] Gamma[1 + z2 + z4] Gamma[1 + z3 + z4]);

In[6]:= B2rules = MBoptimizedRules[B2, ep → 0, {}, {ep}]
```

Out[6]= $\left\{\left\{ep \to -\frac{9}{16}\right\}, \left\{z1 \to -\frac{1}{2}, z2 \to -\frac{5}{16}, z3 \to -\frac{3}{8}, z4 \to \frac{7}{16}\right\}\right\}$

```
In[7]:= B2cont = MBcontinue[B2, ep → 0, B2rules];

In[8]:= B2select = MBpreselect[B2cont, {ep, 0, 0}]

In[9]:= B2exp = Simplify[MBexpand[B2select, E^(2 EulerGamma ep), {ep, 0, 0}]]

In[10]:= B2expI = DoAllBarnes[B2exp]
```

```
{4 x 2, 59 x 1, 55 x 0}
Looking at z3;z4;z3Doing z3 [BarnesRoutines`Private`xtag$29503]Looking  at z1;z3;z1;z3;Looking a
. [No change of variable needed]
; {2} → {1}
 solution not found
. [No change of variable needed]
; {2} → {1}
 solution not found
. [No change of variable needed]
; {2} → {1}
. [No change of variable needed]
; {2} → {1}
{23 x 1, 13 x 0}
{22 x 1, 13 x 0}
```

Out[10]= $\{\mathrm{MBint}[\frac{4}{ep^4}, \{\{ep \to 0\}, \{\}\}], \mathrm{MBint}[-\frac{5\pi^2}{2\,ep^2}, \{\{ep \to 0\}, \{\}\}],$

$\mathrm{MBint}[-\frac{1}{12}\,\mathrm{EulerGamma}^2\,\pi^2, \{\{ep \to 0\}, \{\}\}], \mathrm{MBint}[-\frac{1061\,\pi^4}{720}, \{\{ep \to 0\}, \{\}\}],$

$\mathrm{MBint}[-\frac{5\,\mathrm{Log}[x]}{ep^3}, \{\{ep \to 0\}, \{\}\}], \mathrm{MBint}[\frac{11\,\pi^2\,\mathrm{Log}[x]}{2\,ep}, \{\{ep \to 0\}, \{\}\}],$

$\mathrm{MBint}[\frac{2\,\mathrm{Log}[x]^2}{ep^2}, \{\{ep \to 0\}, \{\}\}], \mathrm{MBint}[-6\,\pi^2\,\mathrm{Log}[x]^2, \{\{ep \to 0\}, \{\}\}],$

$\mathrm{MBint}[\frac{2\,\mathrm{Log}[x]^3}{3\,ep}, \{\{ep \to 0\}, \{\}\}], \mathrm{MBint}[-\frac{4}{3}\,\mathrm{Log}[x]^4, \{\{ep \to 0\}, \{\}\}],$

$\mathrm{MBint}[\frac{65\,\mathrm{PolyGamma}[2, 1]}{6\,ep}, \{\{ep \to 0\}, \{\}\}],$

$\mathrm{MBint}[5\,\mathrm{EulerGamma}\,\mathrm{PolyGamma}[2, 1], \{\{ep \to 0\}, \{\}\}],$

$\mathrm{MBint}[-\frac{44}{3}\,\mathrm{Log}[x]\,\mathrm{PolyGamma}[2, 1], \{\{ep \to 0\}, \{\}\}],$

$\mathrm{MBint}[8\,x^{1+z1}\,\mathrm{Gamma}[-1-z1]^2\,\mathrm{Gamma}[-z1]\,\mathrm{Gamma}[1+z1]^3, \{\{ep \to 0\}, \{z1 \to -\frac{1}{2}\}\}], \mathrm{MBint}[$

$-\frac{4\,x^{1+z1}\,\mathrm{Gamma}[-1-z1]^2\,\mathrm{Gamma}[-z1]\,\mathrm{Gamma}[1+z1]^2\,\mathrm{Gamma}[2+z1]}{ep}, \{\{ep \to 0\}, \{z1 \to -\frac{1}{2}\}\}],$

```
MBint[4 EulerGamma x^(1+z1) Gamma[-1 - z1]^2 Gamma[-z1] Gamma[1 + z1]^2 Gamma[2 + z1],
  {{ep -> 0}, {z1 -> -1/2}}], MBint[16 x^(1+z1) Gamma[-1 - z1]^2 Gamma[-z1]
   Gamma[1 + z1]^2 Gamma[2 + z1] PolyGamma[0, -1 - z1], {{ep -> 0}, {z1 -> -1/2}}],
MBint[-8 x^(1+z1) Gamma[-1 - z1]^2 Gamma[-z1] Gamma[1 + z1]^2 Gamma[2 + z1] PolyGamma[0, 1 + z1],
  {{ep -> 0}, {z1 -> -1/2}}], MBint[-4 x^(1+z1) Gamma[-1 - z1]^2 Gamma[-z1]
   Gamma[1 + z1]^2 Gamma[2 + z1] PolyGamma[0, 2 + z1], {{ep -> 0}, {z1 -> -1/2}}],
MBint[1/4 Gamma[-z2]^2 Gamma[z2] Gamma[1 + z2] PolyGamma[0, z2]^2, {{ep -> 0}, {z2 -> -5/16}}],
MBint[1/4 Gamma[-z2]^2 Gamma[z2] Gamma[1 + z2] PolyGamma[1, -z2], {{ep -> 0}, {z2 -> -5/16}}],
MBint[-7/4 Gamma[-z2]^2 Gamma[z2] Gamma[1 + z2] PolyGamma[1, z2], {{ep -> 0}, {z2 -> -5/16}}],
MBint[1/4 Gamma[-z3]^2 Gamma[z3] Gamma[1 + z3] PolyGamma[0, z3]^2, {{ep -> 0}, {z3 -> -3/8}}],
MBint[1/4 Gamma[-z3]^2 Gamma[z3] Gamma[1 + z3] PolyGamma[1, -z3], {{ep -> 0}, {z3 -> -3/8}}],
MBint[-7/4 Gamma[-z3]^2 Gamma[z3] Gamma[1 + z3] PolyGamma[1, z3], {{ep -> 0}, {z3 -> -3/8}}],
MBint[-(Gamma[1 - z4] Gamma[-z4] Gamma[z4]^3 PolyGamma[0, 1 - z4])/(2 Gamma[1 + z4]), {{ep -> 0}, {z4 -> 7/16}}],
MBint[(2 Gamma[1 - z4] Gamma[-z4] Gamma[z4]^3 PolyGamma[0, -z4])/Gamma[1 + z4], {{ep -> 0}, {z4 -> 7/16}}],
MBint[(6 Gamma[1 - z4] Gamma[-z4] Gamma[z4]^3 PolyGamma[0, z4])/Gamma[1 + z4], {{ep -> 0}, {z4 -> 7/16}}],
MBint[-2 EulerGamma Gamma[1 - z4] Gamma[-z4] Gamma[z4]^2 PolyGamma[0, 1 + z4],
  {{ep -> 0}, {z4 -> 7/16}}],
MBint[-(7 Gamma[1 - z4] Gamma[-z4] Gamma[z4]^3 PolyGamma[0, 1 + z4])/(2 Gamma[1 + z4]), {{ep -> 0}, {z4 -> 7/16}}],
MBint[-2 Gamma[1 - z4] Gamma[-z4] Gamma[z4]^2 PolyGamma[0, -z4] PolyGamma[0, 1 + z4],
  {{ep -> 0}, {z4 -> 7/16}}], MBint[-2 Gamma[1 - z4] Gamma[-z4] Gamma[z4]^2
```

```
PolyGamma[0, z4] PolyGamma[0, 1 + z4], {{ep -> 0}, {z4 -> 7/16}}],
MBint[Gamma[1 - z4] Gamma[-z4] Gamma[z4]^2 PolyGamma[0, 1 + z4]^2, {{ep -> 0}, {z4 -> 7/16}}],
MBint[Gamma[1 - z4] Gamma[-z4] Gamma[z4]^2 PolyGamma[1, z4], {{ep -> 0}, {z4 -> 7/16}}],
MBint[-3 Gamma[1 - z4] Gamma[-z4] Gamma[z4]^2 PolyGamma[1, 1 + z4], {{ep -> 0}, {z4 -> 7/16}}]}
```

In[11]:= **B2expS = SortByDimension[MBmerge[B2expI]]**

```
Out[11]= {MBint[
   -1/(2 Gamma[1 + z4]) Gamma[1 - z4] Gamma[-z4] Gamma[z4]^2 (Gamma[z4] (PolyGamma[0, 1 - z4] - 4
        PolyGamma[0, -z4] - 12 PolyGamma[0, z4] + 7 PolyGamma[0, 1 + z4]) + 2 Gamma[1 + z4]
      (2 (EulerGamma + PolyGamma[0, -z4] + PolyGamma[0, z4]) PolyGamma[0, 1 + z4] -
       PolyGamma[0, 1 + z4]^2 - PolyGamma[1, z4] + 3 PolyGamma[1, 1 + z4])),
   {{ep -> 0}, {z4 -> 7/16}}], MBint[1/4 Gamma[-z3]^2 Gamma[z3] Gamma[1 + z3]
   (PolyGamma[0, z3]^2 + PolyGamma[1, -z3] - 7 PolyGamma[1, z3]),
   {{ep -> 0}, {z3 -> -3/8}}],
 MBint[1/4 Gamma[-z2]^2 Gamma[z2] Gamma[1 + z2]
   (PolyGamma[0, z2]^2 + PolyGamma[1, -z2] - 7 PolyGamma[1, z2]),
   {{ep -> 0}, {z2 -> -5/16}}],
 MBint[1/ep 4 x^(1+z1) Gamma[-1 - z1]^2 Gamma[-z1] Gamma[1 + z1]^2
   (2 ep Gamma[1 + z1] + Gamma[2 + z1] (-1 + ep EulerGamma + 4 ep PolyGamma[0, -1 - z1] -
       2 ep PolyGamma[0, 1 + z1] - ep PolyGamma[0, 2 + z1])), {{ep -> 0}, {z1 -> -1/2}}],
 MBint[4/ep^4 - (5 π^2)/(2 ep^2) - (EulerGamma^2 π^2)/12 - (1061 π^4)/720 + (2/ep^2 - 6 π^2) Log[x]^2 + (2 Log[x]^3)/(3 ep) -
   (4 Log[x]^4)/3 + Log[x] (-5/ep^3 + (11 π^2)/(2 ep) - 44/3 PolyGamma[2, 1]) +
   (65 PolyGamma[2, 1])/(6 ep) + 5 EulerGamma PolyGamma[2, 1], {{ep -> 0}, {}}]}
```

In[12]:= **MBDimension /@ B2expS**

```
Out[12]= {1, 1, 1, 1, 0}
```

Summing up series with nested sums

$$S_i(n) = \sum_{j=1}^{n} \frac{1}{j^i} \,, \quad S_{ik}(n) = \sum_{j=1}^{n} \frac{S_k(j)}{j^i} \,,$$

$$S_{ikl}(n) = \sum_{j=1}^{n} \frac{S_{kl}(j)}{j^i} \,, \quad S_{iklm}(n) = \sum_{j=1}^{n} \frac{S_{klm}(j)}{j^i}$$

E.g., with one index:

$$\psi(n) = S_1(n-1) - \gamma_{\rm E} \,,$$
$$\psi^{(k)}(n) = (-1)^k k! \left(S_{k+1}(n-1) - \zeta(k+1)\right) \,, \quad k = 1, 2 \,, \ldots \,,$$

`SUMMER` [J.A.M. Vermaseren'00]

`XSummer` [S. Moch and P. Uwer'00]

Harmonic polylogarithms (HPL)

$H_{a_1,a_2,\ldots,a_n}(x) \equiv H(a_1,a_2,\ldots,a_n;x)$, with $a_i = 1, 0, -1$

[E. Remiddi & J.A.M. Vermaseren'00]

are generalizations of the usual polylogarithms $\text{Li}_a(z)$ and Nielsen polylogarithms $S_{a,b}(z)$

$$H(a_1,a_2,\ldots,a_n;x) = \int_0^x f(a_1;t) H(a_2,\ldots,a_n;t)\text{d}t\,,$$

where $f(\pm 1;t) = 1/(1 \mp t)$, $\;f(0;t) = 1/t$,

$$H(\pm 1;x) = \mp \ln(1 \mp x), \quad H(0;x) = \ln x\,,$$

with $a_i = 1, 0, -1$.

HPL are implemented in `Mathematica` [D. Maitre'06]

In[1]:=
```
SetDirectory["c:/diskE/job2008/Zurich"];
```

In[2]:=
```
<< MB/MB.m
```

```
MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08
```

In[3]:=
```
SortByDimension[l_List] := Sort[l, Length[#1[[2, 2]]] > Length[#2[[2, 2]]] &];
CoeffEps[X_, n_] := (X /. X[[1]] → Simplify[Coefficient[X[[1]], ep, n]]);
MBDimension[int_MBint] := Length[int[[2, 2]]];

(*   a 4fold MB representation for the on-shell 2loop
  non-planar vertex diagram derived loop by loop;
  sg=-1    *)
```

In[7]:=
```
V2 = (sg^z4 Gamma[-1 - ep - z1 - z2] Gamma[-z2] Gamma[1 + z1 + z2] Gamma[-1 - ep - z1 - z3]
      Gamma[-z3] Gamma[1 + z1 + z3] Gamma[-1 - 2 ep - z2 - z4] Gamma[-1 - 2 ep - z3 - z4]
      Gamma[-z4] Gamma[2 + 2 ep + z4] Gamma[-z1 + z4] Gamma[2 + ep + z1 + z2 + z3 + z4]) /
   (Gamma[-3 ep] Gamma[-2 ep] Gamma[1 - z2] Gamma[1 - z3]) ;
```

In[8]:=
```
V2rules = MBoptimizedRules[V2, ep → 0, {}, {ep}]
```

MBrules::norules : no rules could be found to regulate this integral

MBrules::norules : no rules could be found to regulate this integral

MBrules::norules : no rules could be found to regulate this integral

General::stop : Further output of MBrules::norules will be suppressed during this calculation. ≫

Out[8]= $\left\{\left\{ep \to -\frac{5}{8}\right\}, \left\{z1 \to -\frac{1}{4}, z2 \to -\frac{1}{2}, z3 \to -\frac{5}{16}, z4 \to -\frac{1}{8}\right\}\right\}$

In[9]:=
```
V2cont = MBcontinue[V2, ep → 0, V2rules];
```

Level 1

Taking −residue in z2 = −1 − ep − z1

Taking −residue in z3 = −1 − ep − z1

Taking −residue in z4 = −1 − 2 ep − z2

Taking −residue in z4 = −1 − 2 ep − z3

Level 2

Integral {1}

Taking −residue in z4 = −ep + z1

Integral {2}

Taking −residue in z2 = −1 − ep − z1

Taking −residue in z4 = −ep + z1

Taking −residue in z4 = −1 − 2 ep − z2

Integral {3}

Taking −residue in z2 = −1 − 2 ep − z1

Integral {4}

Taking −residue in z2 = −1 − ep − z1

Taking −residue in z3 = −1 − 2 ep − z1

Level 3

Integral {1, 1}

Integral {2, 1}

Taking −residue in z4 = −ep + z1

Integral {2, 2}

Integral {2, 3}

Taking −residue in z2 = −1 − 2 ep − z1

Integral {3, 1}

Integral {4, 1}

Taking −residue in z3 = −1 − 2 ep − z1

Integral {4, 2}

Taking −residue in z2 = −1 − 2 ep − z1

Level 4

Integral {2, 1, 1}

Integral {2, 3, 1}

Integral {4, 1, 1}

Integral {4, 2, 1}

16 integral(s) found

```
(*   no   1/ep^4 poles???       *)
```

In[10]:= **V2select4 = MBpreselect[MBmerge[V2cont], {ep, 0, -4}]**

Out[10]= { }

In[11]:= **V2select0 = MBpreselect[MBmerge[V2cont], {ep, 0, 0}]**

In[12]:= **V2select0S = Simplify[SortByDimension[V2select0]]**

Out[12]=
```
{MBint[(sg^z4 Gamma[-ep]^2 Gamma[1 + ep + z1]^2 Gamma[-ep + z1 - z4]^2
      Gamma[-z4] Gamma[2 + 2 ep + z4] Gamma[-z1 + z4] Gamma[-ep - z1 + z4]) /
   (Gamma[-3 ep] Gamma[-2 ep] Gamma[2 + ep + z1]^2), {{ep -> 0}, {z1 -> -1/4, z4 -> -1/8}}],
 MBint[(sg^(-1-2 ep-z3) Gamma[-1 - ep - z1 - z3] Gamma[-z3] Gamma[1 + z1 + z3]
     (sg^(1+ep+z1+z3) Gamma[-ep]^2 Gamma[ep - z1] Gamma[1 + ep + z1]
        Gamma[2 + ep + z1] Gamma[-1 - ep - z1 - z3] Gamma[1 - ep + z1 + z3] +
       Gamma[-2 ep] Gamma[-1 - 2 ep - z1 - z3] (sg^(1+2 ep+z1+z3) Gamma[ep] Gamma[-z1]
          Gamma[2 + ep + z1] Gamma[1 + 2 ep + z1] Gamma[1 - ep + z1 + z3] + Gamma[-ep]
          Gamma[1 + ep + z1] Gamma[1 - z3] Gamma[1 + 2 ep + z3] Gamma[1 + ep + z1 + z3]))) /
   (Gamma[-3 ep] Gamma[-2 ep] Gamma[2 + ep + z1] Gamma[1 - z3]), {{ep -> 0},
   {z1 -> -1/4, z3 -> -5/16}}],
 MBint[(sg^(-1-2 ep-z2) Gamma[-1 - ep - z1 - z2] Gamma[-z2] Gamma[1 + z1 + z2]
     (sg^(1+ep+z1+z2) Gamma[-ep]^2 Gamma[ep - z1] Gamma[1 + ep + z1]
        Gamma[2 + ep + z1] Gamma[-1 - ep - z1 - z2] Gamma[1 - ep + z1 + z2] +
       Gamma[-2 ep] Gamma[-1 - 2 ep - z1 - z2] (sg^(1+2 ep+z1+z2) Gamma[ep] Gamma[-z1]
          Gamma[2 + ep + z1] Gamma[1 + 2 ep + z1] Gamma[1 - ep + z1 + z2] + Gamma[-ep]
          Gamma[1 + ep + z1] Gamma[1 - z2] Gamma[1 + 2 ep + z2] Gamma[1 + ep + z1 + z2]))) /
   (Gamma[-3 ep] Gamma[-2 ep] Gamma[2 + ep + z1] Gamma[1 - z2]), {{ep -> 0},
   {z1 -> -1/4, z2 -> -1/2}}],
 MBint[(sg^(-ep+z1) (sg^ep Gamma[-3 ep] Gamma[-2 ep] Gamma[ep]^2 Gamma[-z1] Gamma[2 + ep + z1]
        Gamma[1 + 2 ep + z1]^2 - Gamma[-ep]^2 Gamma[1 + ep + z1] Gamma[2 + 2 ep + z1]
        (-2 sg^ep Gamma[-2 ep] Gamma[ep] Gamma[-z1] Gamma[1 + 2 ep + z1] + Gamma[-ep]
          Gamma[ep - z1] Gamma[1 + ep + z1] (2 EulerGamma + Log[sg] + PolyGamma[0, -2 ep] +
            PolyGamma[0, -ep] - PolyGamma[0, ep - z1] + PolyGamma[0, 2 + ep + z1])))) /
   (Gamma[-3 ep] Gamma[2 + ep + z1] Gamma[2 + 2 ep + z1]), {{ep -> 0},
   {z1 -> -1/4}}]}
```

In[13]:= **MBDimension /@ V2select0S**

Out[13]= {2, 2, 2, 1}

(*   One-dimensional contribution V2select0S[[4]]   *)

```
In[14]:= V2select0S[[4]]

Out[14]= MBint[((sg^(-ep+z1) (sg^ep Gamma[-3 ep] Gamma[-2 ep] Gamma[ep]^2 Gamma[-z1] Gamma[2 + ep + z1]
            Gamma[1 + 2 ep + z1]^2 - Gamma[-ep]^2 Gamma[1 + ep + z1] Gamma[2 + 2 ep + z1]
            (-2 sg^ep Gamma[-2 ep] Gamma[ep] Gamma[-z1] Gamma[1 + 2 ep + z1] + Gamma[-ep]
               Gamma[ep - z1] Gamma[1 + ep + z1] (2 EulerGamma + Log[sg] + PolyGamma[0, -2 ep] +
                 PolyGamma[0, -ep] - PolyGamma[0, ep - z1] + PolyGamma[0, 2 + ep + z1])))) /
     (Gamma[-3 ep] Gamma[2 + ep + z1] Gamma[2 + 2 ep + z1]),
   {{ep -> 0},
    {z1 -> -1/4}}]

(*            a piece of this:              *)

In[15]:= V20 =
   (2 sg^z1 Gamma[-2 ep] Gamma[-ep]^2 Gamma[ep] Gamma[-z1] Gamma[1 + ep + z1] Gamma[1 + 2 ep + z1]) /
    (Gamma[-3 ep] Gamma[2 + ep + z1]);

(*    no 1/ep^4 poles here???  *)

In[16]:= Series[V20 E^(2 EulerGamma ep), {ep, 0, -4}]

Out[16]= 1/O[ep]^3

In[17]:= V20 /. sg^z1 -> E^(I Pi z1)

Out[17]= (2 e^(i π z1) Gamma[-2 ep] Gamma[-ep]^2 Gamma[ep] Gamma[-z1] Gamma[1 + ep + z1] Gamma[1 + 2 ep + z1]) /
   (Gamma[-3 ep] Gamma[2 + ep + z1])

In[18]:= % /. Gamma[2 + ep + z1] -> Gamma[1 + ep + z1] (1 + ep + z1)

Out[18]= (2 e^(i π z1) Gamma[-2 ep] Gamma[-ep]^2 Gamma[ep] Gamma[-z1] Gamma[1 + 2 ep + z1]) /
         ((1 + ep + z1) Gamma[-3 ep])

In[19]:= (% /. Gamma[-z1] -> (-1)^n / n!) /. z1 -> n

Out[19]= (2 (-1)^n e^(i n π) Gamma[-2 ep] Gamma[-ep]^2 Gamma[ep] Gamma[1 + 2 ep + n]) /
         ((1 + ep + n) n! Gamma[-3 ep])

In[20]:= % /. e^(i n π) -> (-1)^n

Out[20]= (2 (-1)^(2 n) Gamma[-2 ep] Gamma[-ep]^2 Gamma[ep] Gamma[1 + 2 ep + n]) /
         ((1 + ep + n) n! Gamma[-3 ep])

In[21]:= % /. (-1)^(2 n) -> 1

Out[21]= (2 Gamma[-2 ep] Gamma[-ep]^2 Gamma[ep] Gamma[1 + 2 ep + n]) /
         ((1 + ep + n) n! Gamma[-3 ep])

In[22]:= Sum[%, {n, 0, Infinity}]

Out[22]= -(2 π Csc[2 ep π] Gamma[-2 ep] Gamma[-ep]^2 Gamma[ep] Gamma[1 + ep]) /
          (Gamma[1 - ep] Gamma[-3 ep])
```

In[23]:=
```
Simplify[Normal[Series[% E^(2 EulerGamma ep), {ep, 0, -4}]]]
```

Out[23]= $-\frac{3}{2\,ep^4}$

```
(*    a singularity in epsilon arises when integrating over large values of z *)

(*    do not use the loop-by-loop strategy of
  deriving MB representations for nonplanar diagrams  *)
```

$$\frac{1}{2\pi i}\int_C \frac{\Gamma(1+2\epsilon+z)\Gamma(-z)}{1+\epsilon+z}(-1)^z \mathsf{d}z$$

$$\frac{1}{2\pi i}\int_C \frac{\Gamma(1+2\epsilon+z)\Gamma(-z)}{1+\epsilon+z}(-1)^z \mathrm{d}z$$

$z = x + iy$; ϵ real

$$\frac{1}{2\pi i}\int_C \frac{\Gamma(1+2\epsilon+z)\Gamma(-z)}{1+\epsilon+z}(-1)^z \mathrm{d}z$$

$z = x + iy$; ϵ real

$$= \frac{1}{2\pi}\int_{-i\infty}^{+i\infty} \frac{\Gamma(1+2\epsilon+x+iy)\Gamma(-x-iy)}{1+\epsilon+x+iy}\mathrm{e}^{-i\pi x+\pi y}\mathrm{d}y$$

$$\frac{1}{2\pi i}\int_C \frac{\Gamma(1+2\epsilon+z)\Gamma(-z)}{1+\epsilon+z}(-1)^z \mathrm{d}z$$

$z = x + iy$; ϵ real

$$= \frac{1}{2\pi}\int_{-i\infty}^{+i\infty} \frac{\Gamma(1+2\epsilon+x+iy)\Gamma(-x-iy)}{1+\epsilon+x+iy}\mathrm{e}^{-i\pi x+\pi y}\mathrm{d}y$$

$$\Gamma(x \pm iy) \sim \sqrt{2\pi}\mathrm{e}^{\pm i\frac{\pi}{4}(2x-1)}\mathrm{e}^{\pm iy(\ln y-1)}\mathrm{e}^{-\frac{\pi}{2}y}$$

when $y \to +\infty$

$$\frac{1}{2\pi i}\int_C \frac{\Gamma(1+2\epsilon+z)\Gamma(-z)}{1+\epsilon+z}(-1)^z \mathrm{d}z$$

$z = x + iy$; ϵ real

$$= \frac{1}{2\pi}\int_{-i\infty}^{+i\infty} \frac{\Gamma(1+2\epsilon+x+iy)\Gamma(-x-iy)}{1+\epsilon+x+iy}\mathrm{e}^{-i\pi x+\pi y}\mathrm{d}y$$

$$\Gamma(x \pm iy) \sim \sqrt{2\pi}\mathrm{e}^{\pm i\frac{\pi}{4}(2x-1)}\mathrm{e}^{\pm iy(\ln y-1)}\mathrm{e}^{-\frac{\pi}{2}y}$$

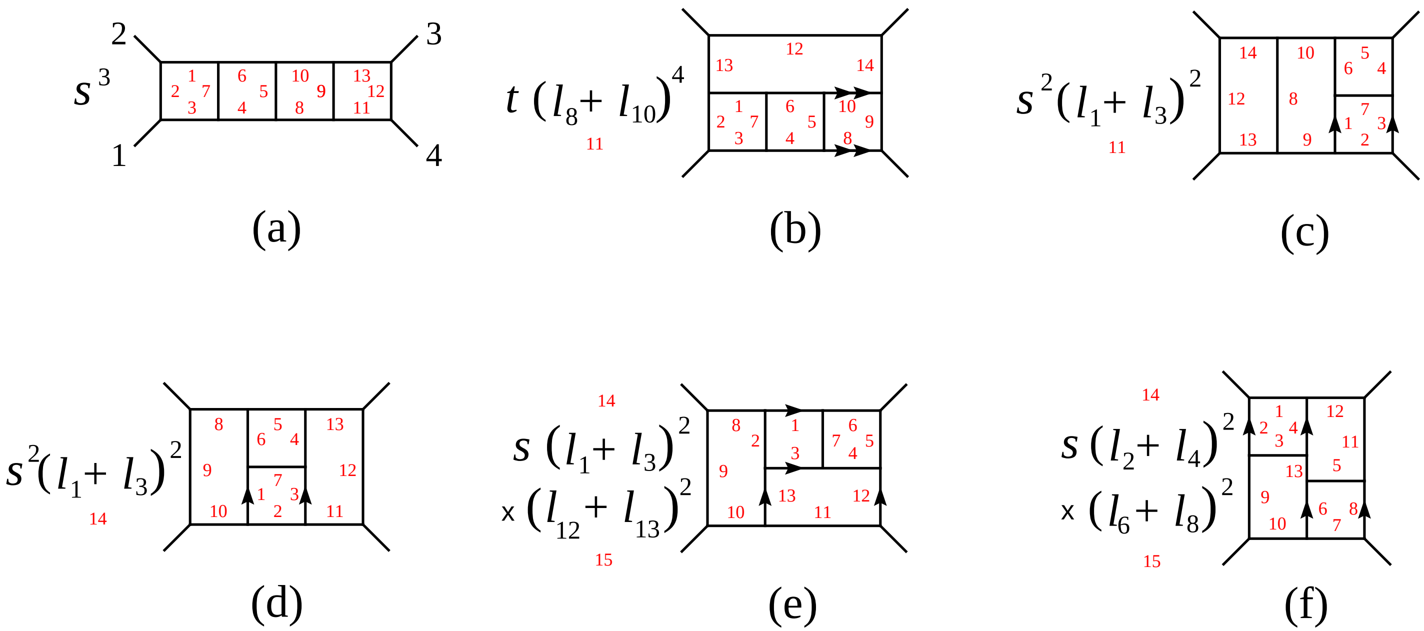
when $y \to +\infty$

The integrand behaves like

$$2\pi\frac{1}{y^{1-2\epsilon}}$$

The four-loop cusp (soft) anomalous dimension

[Z. Bern, M. Czakon, L. Dixon, D.A. Kosower, & V.A. Smirnov'06]



```
<< MB/MB.m ";
<<MB/MBresolve.m";
```

```
MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08

MBresolve 1.0

by Alexander Smirnov

last modified 22 Oct 08
```

```
F1 = -(S^(-5-4 ep-z7) T^z7 Gamma[1 + z1] Gamma[-1 - ep - z1 - z2] Gamma[-z2] Gamma[-1 - ep - z1 - z3]
     Gamma[-z3] Gamma[1 + z1 + z2 + z3] Gamma[2 + ep + z1 + z2 + z3] Gamma[z10 - z4]
     Gamma[-z1 + z4] Gamma[-ep + z1 + z2 - z4 - z5] Gamma[-z5] Gamma[-ep + z1 + z3 - z4 - z6]
     Gamma[-z6] Gamma[1 + z4 + z5 + z6] Gamma[1 + ep - z1 - z2 - z3 + z4 + z5 + z6]
     Gamma[-z7] Gamma[1 + z7] Gamma[-z10 + z7] Gamma[-ep - z10 + z4 + z5 - z8]
     Gamma[-z8] Gamma[-ep + z10 - z7 + z8] Gamma[-ep - z10 + z4 + z6 - z9]
     Gamma[1 + ep - z10 + z7 - z8 - z9] Gamma[-z9] Gamma[-ep + z10 - z7 + z9]
     Gamma[1 + z10 + z8 + z9] Gamma[1 + ep + z10 - z4 - z5 - z6 + z8 + z9]) /
   (Gamma[-2 ep] Gamma[1 - z2] Gamma[1 - z3] Gamma[1 - 2 ep + z1 + z2 + z3]
    Gamma[1 - z5] Gamma[1 - z6] Gamma[1 - 2 ep + z4 + z5 + z6]
    Gamma[1 - z8] Gamma[1 - z9] Gamma[1 - 2 ep + z10 + z8 + z9]);
```

```
F1cont = MBresolve[F1, ep]
```

```
CREATING RESIDUES LIST..........654.3281 seconds
EVALUATING RESIDUES..........16.9375 seconds
```

A very large output was generated. Here is a sample of it:

```
{MBint[(EulerGamma^3 T^(-1-4 ep) Gamma[-2 ep]^2 Gamma[-ep]^4 Gamma[1+ep]^2 Gamma[1+4 ep]) /
       (S^4 Gamma[1-ep]^2 Gamma[-4 ep]) + <<180>> +
   (T^(-1-4 ep) Gamma[-2 ep]^2 Gamma[-ep]^4 <<5>> <<1>> <<1>>]^2 Gamma[1+4 ep] PolyGamma[2,-2 ep]) /
   (2 S^4 Gamma[1-ep]^2 Gamma[-4 ep]), {{ep -> 0}, {}}],
 <<654>>, MBint[-(S^(-5-<<1>> <<1>> z7) <<27>> <<1>>) / (Gamma[-2 ep] <<8>> <<1>>), {{<<1>>}, <<1>>}]}
```

Show Less | Show More | Show Full Output | Set Size Limit...

```
Length[F1cont]
```

```
656
```

```
F1exp = MBexpand[F1select, E^(4 EulerGamma ep), {ep, 0, 0}] // Timing
```

A very large output was generated. Here is a sample of it:

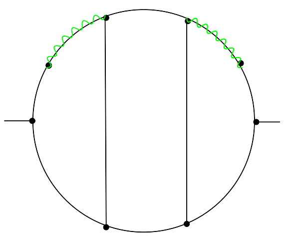
$$\left\{2337.28, \left\{\text{MBint}\left[-\frac{19}{16\,ep^8\,S^4\,T}-\frac{11\,\pi^2}{24\,ep^6\,S^4\,T}+\frac{167\,\pi^4}{720\,ep^4\,S^4\,T}+\frac{14213\,\pi^6}{7560\,ep^2\,S^4\,T}+\frac{975257\,\pi^8}{226800\,S^4\,T}+\ll 117\gg+\frac{974\,\text{Log}[S]\,\text{PolyGamma}[6,1]}{315\,S^4\,T}+\frac{54983\,\text{Log}[T]\,\text{PolyGamma}[6,1]}{2520\,S^4\,T},\{\{ep\to 0\},\{\}\}\right],\ll 553\gg,\text{MBint}\left[-\frac{2\ll 21\gg\,\text{Gamma}[1-z4-z5-z6+z7+z8]}{\text{Gamma}[1-z2]\,\text{Gamma}[1-z3]\ll 1\gg\text{Gamma}[1-z6]},\{\{ep\to 0\},\{\ll 1\gg\}\}\right]\right\}\right\}$$

Show Less | Show More | Show Full Output | Set Size Limit...

```
F1rules = MBoptimizedRules[F1, ep → 0, {}, {ep}]

F1rules = MBcorrectContours[ MBoptimizedRules[F1, ep → 0, {}, {ep}] , 10 000]

M1cont = MBcontinue[F1, ep → 0, F1rules]

2200" integral(s) found"
```

Evaluating Feynman integrals contributing to the three-loop static quark potential [A.V. Smirnov, V.A. Smirnov, and M. Steinhauser'08]

For example,



$$\frac{(i\pi^{d/2})^3}{(q^2)^3 v^2}\left[\frac{56\pi^4}{135\epsilon}+\frac{112\pi^4}{135}+\frac{16\pi^2\zeta(3)}{9}+\frac{8\zeta(5)}{3}+O(\epsilon)\right]$$

Linear propagators $\frac{1}{v\cdot k+i0}$ in addition to usual massless propagators $\frac{1}{k^2+i0}$

$$v\cdot q=0$$

An open problem: how to derive systematically MB representations for non-planar diagrams.

MB tools at http://projects.hepforge.org/mbtools/:

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MB.m (updated)

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MBresolve.m

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MB.m (updated)

MBresolve.m

MBasymptotics.m [M. Czakon]

An open problem: how to derive systematically MB representations for non-planar diagrams.

MB tools at http://projects.hepforge.org/mbtools/:

MB.m (updated)

MBresolve.m

MBasymptotics.m [M. Czakon]

barnesroutines.m [D. Kosower]
(applying Barnes lemmas automatically)

An open problem: how to derive systematically MB representations for non-planar diagrams.

MB tools at http://projects.hepforge.org/mbtools/:

MB.m (updated)

MBresolve.m

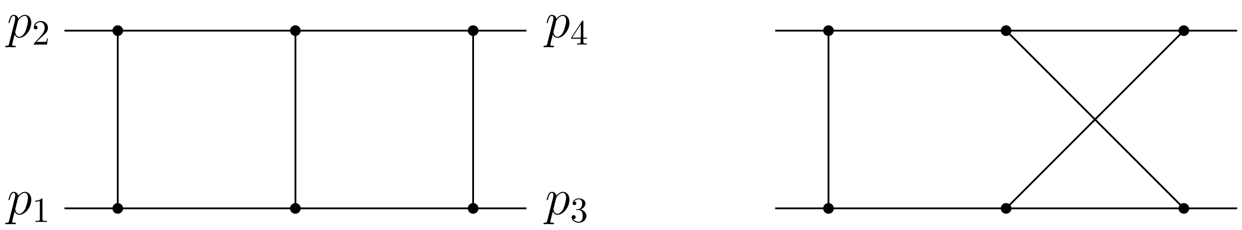
MBasymptotics.m [M. Czakon]

barnesroutines.m [D. Kosower]
(applying Barnes lemmas automatically)

to be continued

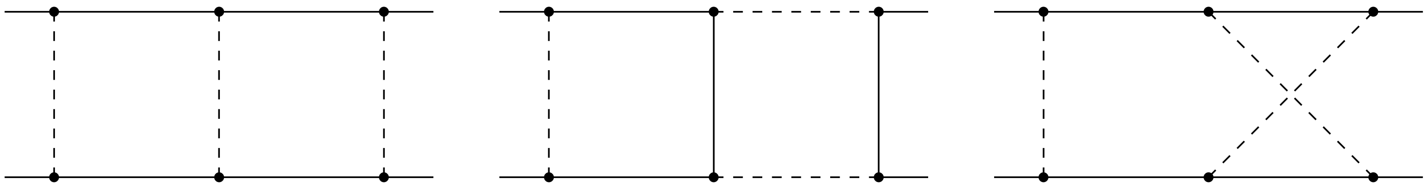
additional slides

Examples and results



Massless on-shell ($p_i^2 = 0, \; i = 1,2,3,4$) double boxes: done in 1999-2000, with multiple subsequent applications. Master integrals calculated with the help of MB representation [V.A. Smirnov'99, J.B Tausk'99, V.A. Smirnov & O.L. Veretin'99]

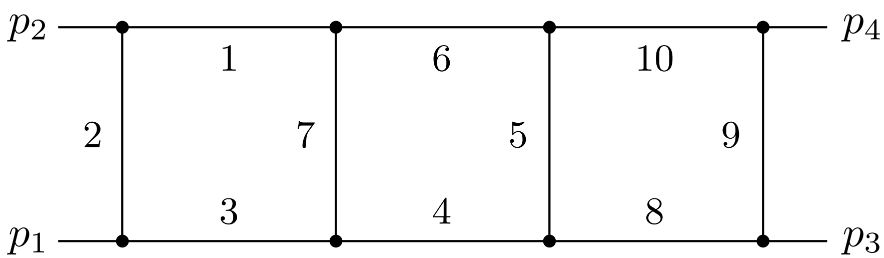
Massive on-shell 2-boxes, $p_i^2 = m^2, \; i = 1,2,3,4$



- first results obtained by MB

  [V.A. Smirnov'02,04; G. Heinrich & V.A. Smirnov'04]
- Reduction to master integrals by Laporta's algorithm

  [M. Czakon, J. Gluza & T. Riemann'04]
- Evaluating the master integrals by differential equations and MB [M. Czakon, J. Gluza & T. Riemann'05-08]



The general planar triple box Feynman integral

$$T(a_1,\ldots,a_{10};s,t;\epsilon) = \int\int\int \frac{\mathrm{d}^d k\,\mathrm{d}^d l\,\mathrm{d}^d r}{[k^2]^{a_1}[(k+p_2)^2]^{a_2}}$$

$$\times \frac{1}{[(k+p_1+p_2)^2]^{a_3}[(l+p_1+p_2)^2]^{a_4}[(r-l)^2]^{a_5}[l^2]^{a_6}[(k-l)^2]^{a_7}}$$

$$\times \frac{1}{[(r+p_1+p_2)^2]^{a_8}[(r+p_1+p_2+p_3)^2]^{a_9}[r^2]^{a_{10}}}$$

General 7fold MB representation:

$$T(a_1,\ldots,a_{10};s,t,m^2;\epsilon) \quad = \quad \frac{\left(i\pi^{d/2}\right)^3(-1)^a}{\prod_{j=2,5,7,8,9,10}\Gamma(a_j)\Gamma(4-a_{589(10)}-2\epsilon)(-s)^{a-6+3\epsilon}}$$

$$\times\frac{1}{(2\pi i)^7}\int_{-i\infty}^{+i\infty}\mathrm{d}w\prod_{j=2}^{7}\mathrm{d}z_j\left(\frac{t}{s}\right)^w\frac{\Gamma(a_2+w)\Gamma(-w)\Gamma(z_2+z_4)\Gamma(z_3+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)}$$

$$\times\frac{\Gamma(2-a_1-a_2-\epsilon+z_2)\Gamma(2-a_2-a_3-\epsilon+z_3)\Gamma(a_7+w-z_4)}{\Gamma(4-a_1-a_2-a_3-2\epsilon+w-z_4)\Gamma(a_6-z_5)\Gamma(a_4-z_6)}$$

$$\times\Gamma(+a_1+a_2+a_3-2+\epsilon+z_4)\Gamma(w+z_2+z_3+z_4-z_7)\Gamma(-z_5)\Gamma(-z_6)$$

$$\times\Gamma(2-a_5-a_9-a_{10}-\epsilon-z_5-z_7)\Gamma(2-a_5-a_8-a_9-\epsilon-z_6-z_7)$$

$$\times\Gamma(a_4+a_6+a_7-2+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(a_9+z_7)$$

$$\times\Gamma(4-a_4-a_6-a_7-2\epsilon+z_5+z_6+z_7)$$

$$\times\Gamma(2-a_6-a_7-\epsilon-w-z_2+z_5+z_7)\Gamma(2-a_4-a_7-\epsilon-w-z_3+z_6+z_7)$$

$$\times\Gamma(a_5+z_5+z_6+z_7)\Gamma(a_5+a_8+a_9+a_{10}-2+\epsilon+z_5+z_6+z_7)\,,$$

General 7fold MB representation:

$$T(a_1,\ldots,a_{10};s,t,m^2;\epsilon) \quad = \quad \frac{\left(i\pi^{d/2}\right)^3(-1)^a}{\prod_{j=2,5,7,8,9,10}\Gamma(a_j)\Gamma(4-a_{589(10)}-2\epsilon)(-s)^{a-6+3\epsilon}}$$

$$\times\frac{1}{(2\pi i)^7}\int_{-i\infty}^{+i\infty}\mathrm{d}w\prod_{j=2}^{7}\mathrm{d}z_j\left(\frac{t}{s}\right)^w\frac{\Gamma(a_2+w)\Gamma(-w)\Gamma(z_2+z_4)\Gamma(z_3+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)}$$

$$\times\frac{\Gamma(2-a_1-a_2-\epsilon+z_2)\Gamma(2-a_2-a_3-\epsilon+z_3)\Gamma(a_7+w-z_4)}{\Gamma(4-a_1-a_2-a_3-2\epsilon+w-z_4)\Gamma(a_6-z_5)\Gamma(a_4-z_6)}$$

$$\times\Gamma(+a_1+a_2+a_3-2+\epsilon+z_4)\Gamma(w+z_2+z_3+z_4-z_7)\Gamma(-z_5)\Gamma(-z_6)$$

$$\times\Gamma(2-a_5-a_9-a_{10}-\epsilon-z_5-z_7)\Gamma(2-a_5-a_8-a_9-\epsilon-z_6-z_7)$$

$$\times\Gamma(a_4+a_6+a_7-2+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(a_9+z_7)$$

$$\times\Gamma(4-a_4-a_6-a_7-2\epsilon+z_5+z_6+z_7)$$

$$\times\Gamma(2-a_6-a_7-\epsilon-w-z_2+z_5+z_7)\Gamma(2-a_4-a_7-\epsilon-w-z_3+z_6+z_7)$$

$$\times\Gamma(a_5+z_5+z_6+z_7)\Gamma(a_5+a_8+a_9+a_{10}-2+\epsilon+z_5+z_6+z_7)\,,$$

The master triple box:

$$
\begin{aligned}
& T(1,1,\ldots,1;s,t;\epsilon) \\
&= \frac{\left(i\pi^{d/2}\right)^3}{\Gamma(-2\epsilon)(-s)^{4+3\epsilon}} \frac{1}{(2\pi i)^7} \int_{-i\infty}^{+i\infty} \mathrm{d}w \prod_{j=2}^{7} \mathrm{d}z_j \left(\frac{t}{s}\right)^w \frac{\Gamma(1+w)\Gamma(-w)}{\Gamma(1-2\epsilon+w-z_4)} \\
&\times \frac{\Gamma(-\epsilon+z_2)\Gamma(-\epsilon+z_3)\Gamma(1+w-z_4)\Gamma(-z_2-z_3-z_4)\Gamma(1+\epsilon+z_4)}{\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)} \\
&\times \frac{\Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(-z_5)\Gamma(-z_6)\Gamma(w+z_2+z_3+z_4-z_7)}{\Gamma(1-z_5)\Gamma(1-z_6)\Gamma(1-2\epsilon+z_5+z_6+z_7)} \\
&\times \Gamma(-1-\epsilon-z_5-z_7)\Gamma(-1-\epsilon-z_6-z_7)\Gamma(1+z_7) \\
&\times \Gamma(1+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(-\epsilon-w-z_2+z_5+z_7) \\
&\times \Gamma(-\epsilon-w-z_3+z_6+z_7)\Gamma(1+z_5+z_6+z_7)\Gamma(2+\epsilon+z_5+z_6+z_7)
\end{aligned}
$$

Result

[V.A. Smirnov'03]

$$T(1,1,\ldots,1;s,t;\epsilon) = -\frac{\left(i\pi^{d/2}e^{-\gamma_{\rm E}\epsilon}\right)^3}{s^3(-t)^{1+3\epsilon}} \sum_{i=0}^{6} \frac{c_j(x,L)}{\epsilon^j} \,,$$

where $x = -t/s$, $L = \ln(s/t)$, and

$$c_6 = \frac{16}{9}\,, \quad c_5 = -\frac{5}{3}L\,, \quad c_4 = -\frac{3}{2}\pi^2\,,$$

$$c_3 = 3(H_{0,0,1}(x) + LH_{0,1}(x)) + \frac{3}{2}(L^2+\pi^2)H_1(x) - \frac{11}{12}\pi^2 L - \frac{131}{9}\zeta_3\,,$$

$$c_2 = -3\left(17H_{0,0,0,1}(x) + H_{0,0,1,1}(x) + H_{0,1,0,1}(x) + H_{1,0,0,1}(x)\right)$$

$$-L\left(37H_{0,0,1}(x) + 3H_{0,1,1}(x) + 3H_{1,0,1}(x)\right) - \frac{3}{2}(L^2+\pi^2)H_{1,1}(x)$$

$$-\left(\frac{23}{2}L^2 + 8\pi^2\right)H_{0,1}(x) - \left(\frac{3}{2}L^3 + \pi^2 L - 3\zeta_3\right)H_1(x) + \frac{49}{3}\zeta_3 L - \frac{1411}{1080}\pi^4\,,$$

$$c_1 = 3\left(81H_{0,0,0,0,1}(x) + 41H_{0,0,0,1,1}(x) + 37H_{0,0,1,0,1}(x) + H_{0,0,1,1,1}(x)\right.$$
$$+33H_{0,1,0,0,1}(x) + H_{0,1,0,1,1}(x) + H_{0,1,1,0,1}(x) + 29H_{1,0,0,0,1}(x)$$
$$\left.+H_{1,0,0,1,1}(x) + H_{1,0,1,0,1}(x) + H_{1,1,0,0,1}(x)\right) + L\left(177H_{0,0,0,1}(x) + 85H_{0,0,1,1}(x\right.$$
$$\left.+73H_{0,1,0,1}(x) + 3H_{0,1,1,1}(x) + 61H_{1,0,0,1}(x) + 3H_{1,0,1,1}(x) + 3H_{1,1,0,1}(x)\right)$$
$$+\left(\frac{119}{2}L^2 + \frac{139}{12}\pi^2\right)H_{0,0,1}(x) + \left(\frac{47}{2}L^2 + 20\pi^2\right)H_{0,1,1}(x)$$
$$+\left(\frac{35}{2}L^2 + 14\pi^2\right)H_{1,0,1}(x) + \frac{3}{2}\left(L^2 + \pi^2\right)H_{1,1,1}(x)$$
$$+\left(\frac{23}{2}L^3 + \frac{83}{12}\pi^2 L - 96\zeta_3\right)H_{0,1}(x) + \left(\frac{3}{2}L^3 + \pi^2 L - 3\zeta_3\right)H_{1,1}(x)$$
$$+\left(\frac{9}{8}L^4 + \frac{25}{8}\pi^2 L^2 - 58\zeta_3 L + \frac{13}{8}\pi^4\right)H_1(x) - \frac{503}{1440}\pi^4 L + \frac{73}{4}\pi^2\zeta_3 - \frac{301}{15}\zeta_5\,,$$

$$
\begin{aligned}
c_0 = &-(951H_{0,0,0,0,0,1}(x)+819H_{0,0,0,0,1,1}(x)+699H_{0,0,0,1,0,1}(x)+195H_{0,0,0,1,1,1}(x)\\
&+547H_{0,0,1,0,0,1}(x)+231H_{0,0,1,0,1,1}(x)+159H_{0,0,1,1,0,1}(x)+3H_{0,0,1,1,1,1}(x)\\
&+363H_{0,1,0,0,0,1}(x)+267H_{0,1,0,0,1,1}(x)+195H_{0,1,0,1,0,1}(x)+3H_{0,1,0,1,1,1}(x)\\
&+123H_{0,1,1,0,0,1}(x)+3H_{0,1,1,0,1,1}(x)+3H_{0,1,1,1,0,1}(x)+147H_{1,0,0,0,0,1}(x)\\
&+303H_{1,0,0,0,1,1}(x)+231H_{1,0,0,1,0,1}(x)+3H_{1,0,0,1,1,1}(x)+159H_{1,0,1,0,0,1}(x)\\
&+3H_{1,0,1,0,1,1}(x)+3H_{1,0,1,1,0,1}(x)+87H_{1,1,0,0,0,1}(x)+3H_{1,1,0,0,1,1}(x)\\
&+3H_{1,1,0,1,0,1}(x)+3H_{1,1,1,0,0,1}(x))\\
&-L\,(729H_{0,0,0,0,1}(x)+537H_{0,0,0,1,1}(x)+445H_{0,0,1,0,1}(x)+133H_{0,0,1,1,1}(x)\\
&+321H_{0,1,0,0,1}(x)+169H_{0,1,0,1,1}(x)+97H_{0,1,1,0,1}(x)+3H_{0,1,1,1,1}(x)\\
&+165H_{1,0,0,0,1}(x)+205H_{1,0,0,1,1}(x)+133H_{1,0,1,0,1}(x)+3H_{1,0,1,1,1}(x)\\
&+61H_{1,1,0,0,1}(x)+3H_{1,1,0,1,1}(x)+3H_{1,1,1,0,1}(x))\\
&-\left(\frac{531}{2}L^2+\frac{89}{4}\pi^2\right)H_{0,0,0,1}(x)-\left(\frac{311}{2}L^2+\frac{619}{12}\pi^2\right)H_{0,0,1,1}(x)\\
&-\left(\frac{247}{2}L^2+\frac{307}{12}\pi^2\right)H_{0,1,0,1}(x)-\left(\frac{71}{2}L^2+32\pi^2\right)H_{0,1,1,1}(x)
\end{aligned}
$$

$$
-\left(\frac{151}{2}L^2-\frac{197}{12}\pi^2\right)H_{1,0,0,1}(x)-\left(\frac{107}{2}L^2+50\pi^2\right)H_{1,0,1,1}(x)
$$

$$
-\left(\frac{35}{2}L^2+14\pi^2\right)H_{1,1,0,1}(x)-\frac{3}{2}\left(L^2+\pi^2\right)H_{1,1,1,1}(x)
$$

$$
-\left(\frac{119}{2}L^3+\frac{317}{12}\pi^2L-455\zeta_3\right)H_{0,0,1}(x)-\left(\frac{47}{2}L^3+\frac{179}{12}\pi^2L\right.
$$

$$
\left.-120\zeta_3\right)H_{0,1,1}(x)-\left(\frac{35}{2}L^3+\frac{35}{12}\pi^2L-156\zeta_3\right)H_{1,0,1}(x)-\left(\frac{3}{2}L^3+\pi^2L\right.
$$

$$
\left.-3\zeta_3\right)H_{1,1,1}(x)-\left(\frac{69}{8}L^4+\frac{101}{8}\pi^2L^2-291\zeta_3L+\frac{559}{90}\pi^4\right)H_{0,1}(x)
$$

$$
-\left(\frac{9}{8}L^4+\frac{25}{8}\pi^2L^2-58\zeta_3L+\frac{13}{8}\pi^4\right)H_{1,1}(x)
$$

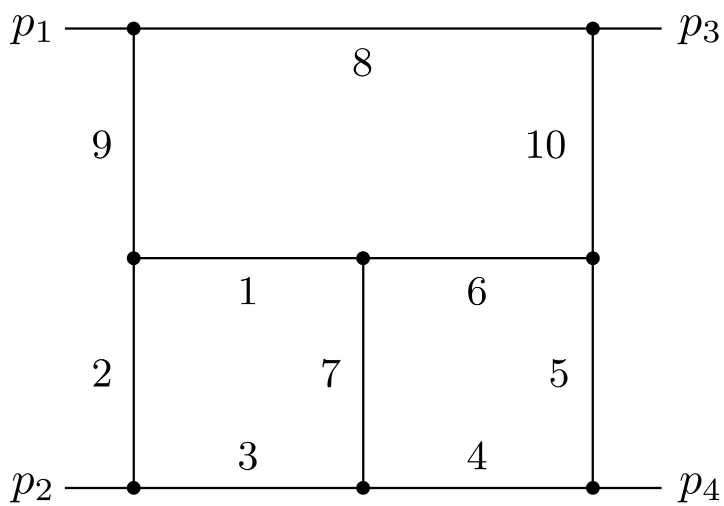
$$
-\left(\frac{27}{40}L^5+\frac{25}{8}\pi^2L^3-\frac{183}{2}\zeta_3L^2+\frac{131}{60}\pi^4L-\frac{37}{12}\pi^2\zeta_3+57\zeta_5\right)H_1(x)
$$

$$
+\left(\frac{223}{12}\pi^2\zeta_3+149\zeta_5\right)L+\frac{167}{9}\zeta_3^2-\frac{624607}{544320}\pi^6\,.
$$

Iteration relations in $N = 4$ SUSY YM.
Iteration relations in two loops

[C. Anastasiou, L.J. Dixon, Z. Bern & D.A. Kosower'03,04]

To check such relations in three loops one more diagram was needed: the 'tennis court' graph with numerator $(l_1 + l_3)^2$



[Z. Bern, L.J. Dixon & V.A. Smirnov'05]

$$W(s,t;1,\ldots,1,-1,\epsilon) = -\frac{\left(\mathrm{i}\pi^{d/2}\right)^3}{\Gamma(-2\epsilon)(-s)^{1+3\epsilon}t^2}$$

$$\times \frac{1}{(2\pi\mathrm{i})^8}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\ldots\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}w\,\mathrm{d}z_1 \prod_{j=2}^{7}\mathrm{d}z_j\Gamma(-z_j)\left(\frac{t}{s}\right)^w \Gamma(1+3\epsilon+w)$$

$$\times \frac{\Gamma(-3\epsilon-w)\Gamma(1+z_1+z_2+z_3)\Gamma(-1-\epsilon-z_1-z_3)\Gamma(1+z_1+z_4)}{\Gamma(1-z_2)\Gamma(1-z_3)\Gamma(1-z_6)\Gamma(1-2\epsilon+z_1+z_2+z_3)}$$

$$\times \frac{\Gamma(-1-\epsilon-z_1-z_2-z_4)\Gamma(2+\epsilon+z_1+z_2+z_3+z_4)}{\Gamma(-1-4\epsilon-z_5)\Gamma(1-z_4-z_7)\Gamma(2+2\epsilon+z_4+z_5+z_6+z_7)}$$

$$\times\Gamma(-\epsilon+z_1+z_3-z_5)\Gamma(2-w+z_5)\Gamma(-1+w-z_5-z_6)$$

$$\times\Gamma(z_5+z_7-z_1)\Gamma(1+z_5+z_6)\Gamma(-1+w-z_4-z_5-z_7)$$

$$\times\Gamma(-\epsilon+z_1+z_2-z_5-z_6-z_7)\Gamma(1-\epsilon-w+z_4+z_5+z_6+z_7)$$

$$\times\Gamma(1+\epsilon-z_1-z_2-z_3+z_5+z_6+z_7)$$

Result:

$$W(s,t;1,\ldots,1,-1,\epsilon) = -\frac{\left(\mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^3}{(-s)^{1+3\epsilon}\,t^2}\sum_{i=0}^{6}\frac{c_j}{\epsilon^j}\,,$$

where

$$\begin{aligned}
c_6 &= \frac{16}{9}\,,\quad c_5 = -\frac{13}{6}\ln x\,,\quad c_4 = -\frac{19}{12}\pi^2 + \frac{1}{2}\ln^2 x \\
c_3 &= \frac{5}{2}\left[\mathsf{Li}_3(-x) - \ln x\,\mathsf{Li}_2(-x)\right] + \frac{7}{12}\ln^3 x - \frac{5}{4}\ln^2 x\ln(1+x) \\
&\quad + \frac{157}{72}\pi^2\ln x - \frac{5}{4}\pi^2\ln(1+x) - \frac{241}{18}\zeta(3)\ \ldots
\end{aligned}$$