Functional equations for Feynman integrals

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Outline of the talk

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- General method for obtaining functional equations
- Functional equations for one-loop propagator integral
- Functional equations for one-loop vertex type integral
- Functional equations for one-loop box integrals
- Summary

Most of theoretical predictions in the elementary particle physics are based on perturbation theory:

$$\sigma(\{p_j p_k\}, \{m_k^2\}) = \sum_k g^k M_k$$

where g is a small coupling constant, $\{m_k^2\}$ - masses of particles, $\{p_j p_k\}$ - scalar products of external momenta p_r.

To evaluate the coefficients M_k, the so-called, Feynman diagram technique was proposed by Feynman 60 years ago.

To find contribution to M_k one should draw Feynman diagrams appropriate to a given process. Each diagram may be a sum of millions of integrals with different powers of propagators, masses, scalar invariants.

By using recurrence relations these integrals are reducible to tens or hundreds of "basic integrals". "Basic integrals" are functions depending on several variables.

Many radiative corrections needed for comparison of theoretical predictions with experimental data expected from LHC correspond to diagrams with five and more external legs. In this case integrals are functions of 5-10 variables depending on physical kinematics.

Analytic as well as numeric evaluation of these integrals is very problematic.

- analytic or fast precise calculation of integrals with several masses and kinematic variables is needed
- no regular methods for analytic continuation of integrals with several variables (scalar products, masses) to different kinematical regions

A possible tool to solve these problems:

Functional equations

For arbitrary value of the space-time dimension d Feynman integrals are combinations of different hypergeometric functions: generalized hypergeometric function ${}_pF_{p-1}$, Appell functions F_1, F_2, F_3, F_4, Horn functions, Lauricella functions F_C, F_D, Lauricella - Saran functions, e.t.c.

Functional equations for Feynman integrals are in fact functional equations for combinations of hypergeometric functions.

General method

Feynman integrals satisfy recurrence relations (Petersson (1966), t'Hooft and Veltman (1972), Chetyrkin and Tkachov (1981), Tarasov (1996)). F.V. Tkachov, Phys.Lett. **100B** (1981) 65; K.G. Chetyrkin and F.V. Tkachov, Nucl.Phys.**192** (1981) 159.

Basic idea of the methods: use the relation (t'Hooft , Veltman)

$$\int d^d k_1 \ldots \int d^d k_L \frac{\partial}{\partial k_{j\mu}} \frac{k_{j\mu}}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}} = 0$$

differentiate w.r.t. k, make substitutions:

$$k_1 q_1 = \frac{1}{2}\{[(k_1 + q_1)^2 - m_1^2] - [k_1^2 - m_1^2] - q_1^2\}$$

or transform scalar products to integrals with shifted space-time dimension $d \rightarrow d + 2$.

General method

As a result we obtain many relations connecting integrals with different powers of propagators. The general form of these relations:

$$\sum_j Q_j I_{j,n} = \sum_{k,r<n} R_{k,r} I_{k,r}$$

where Q_j, R_k are polynomials in masses, scalar products of external momenta, dimension of space-time d, powers of propagators ν_l.

$I_{k,r}$ are some integrals with r external legs.

In recurrence relation one can always arrange integrals according to the number of external legs, internal lines i.e. in fact according to the complexity of their calculation.

General method

General method for obtainig functional equations:

Remove integrals with n external legs (lines) by choosing scalar invariants, masses, d and ν_j i.e. satisfy conditions:

$$Q_j = 0$$

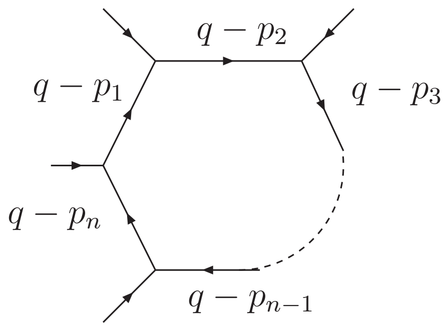
and keep integrals with less number of external legs (lines), i.e:

$$R_k \neq 0$$

In some cases not all of Q_j can be made zero. However also in these cases functional equations do exist.

One-loop n-point integrals

Let's consider one-loop n-point integrals



$$I_n^{(d)} = \int \frac{d^dq}{i\pi^{d/2}} \prod_{j=1}^{n} \frac{1}{c_j^{\nu_j}}, \qquad c_j = (q-p_j)^2 - m_j^2 + i\epsilon.$$

p_j external momentum flowing through j-th propagator with mass m_j.

One-loop n-point integrals

Integrals $I_n^{(d)}$ satisfy the following generalized recurrence relation O.T. in Phys.Rev.D54 (1996) p.6479

$$G_{n-1}\nu_j \mathbf{j}^+ I_n^{(d+2)} - (\partial_j \Delta_n) I_n^{(d)} = \sum_{k=1}^{n} (\partial_j \partial_k \Delta_n) \mathbf{k}^- I_n^{(d)},$$

where $\mathbf{j}^{\pm}$ shift indices $\nu_j \to \nu_j \pm 1$,

$$\partial_j \equiv \frac{\partial}{\partial m_j^2},$$

One-loop n-point integrals

$$G_{n-1} = -2^n \begin{vmatrix} p_1 p_1 & p_1 p_2 & \ldots & p_1 p_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_1 p_{n-1} & p_2 p_{n-1} & \ldots & p_{n-1} p_{n-1} \end{vmatrix},$$

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix},$$

$$Y_{ij} = m_i^2 + m_j^2 - s_{ij}, \qquad \boxed{\boxed{s_{ij} = (p_i - p_j)^2}},$$

One-loop n-point integrals

Gram determinant G_{n-1} and modified Cayley determinant Δ_n are polynomials in s_{ij} and masses m_j^2.

To obtain functional equation we must choose s_{ij} and m_j^2 in such a way that

$$G_{n-1} = 0, \qquad \partial_j \Delta_n = 0.$$

Functional equation for 1-loop propagator

Equation at $n=3,\ j=1$:

$$\begin{aligned}
&G_2 \mathbf{1}^+ I_3^{(d+2)}(m_1^2,m_2^2,m_3^2,s_{23},s_{13},s_{12}) \\
&\quad -(\partial_1\Delta_3) I_3^{(d)}(m_1^2,m_2^2,m_3^2,s_{23},s_{13},s_{12}) \\
&= 2(s_{12}+s_{23}-s_{13}) I_2^{(d)}(m_1^2,m_2^2,s_{12}) \\
&\quad + 2(s_{13}+s_{23}-s_{12}) I_2^{(d)}(m_1^2,m_3^2,s_{13}) - 4s_{23} I_2^{(d)}(m_2^2,m_3^2,s_{23}).
\end{aligned}$$

where

$$\begin{aligned}
G_2 &= 2s_{12}^2+2s_{13}^2+2s_{23}^2-4s_{13}s_{23}-4s_{12}s_{13}-4s_{23}s_{12}, \\
\Delta_3 &= 2(m_2^2-m_3^2)[(m_1^2-m_2^2)s_{13}-(m_1^2-m_3^2)s_{12}]-2m_1^2s_{23}^2-2m_3^2s_{12}^2 \\
&\quad -2m_2^2s_{13}^2-2(m_1^2-m_3^2)(m_1^2-m_2^2)s_{23}+2(m_2^2+m_3^2)s_{12}s_{13} \\
&\quad +2(m_3^2+m_1^2)s_{23}s_{12}+2(m_2^2+m_1^2)s_{13}s_{23}-2s_{12}s_{13}s_{23}
\end{aligned}$$

Functional equation for 1-loop propagator

Coefficients in front of I_3 depend on 6 variables s_{12}, s_{13}, s_{23}, m_1^2, m_2^2, m_3^2. To exclude I_3 we must solve system of equations

$$
\begin{aligned}
G_2 &= 2s_{12}^2 + 2s_{13}^2 + 2s_{23}^2 - 4s_{13}s_{23} - 4s_{12}s_{13} - 4s_{23}s_{12} = 0, \\
\partial_1 \Delta_3 &= -2s_{23}^2 - 4m_1^2 s_{23} + 2m_2^2 s_{23} + 2m_3^2 s_{23} + 2s_{12}m_3^2 \\
&\quad +2m_2^2 s_{13} - 2m_3^2 s_{13} + 2s_{13}s_{23} - 2m_2^2 s_{12} + 2s_{23}s_{12} = 0
\end{aligned}
$$

This system can be resolved with respect to s_{13}, s_{23}. There are 3 solutions. One solution

$$s_{13} = s_{12}, \qquad s_{23} = 0,$$

gives no functional equations.

Functional equation for 1-loop propagator

There are two other solutions

$$s_{13} = s_{13}(m_1^2, m_2^2, m_3^2, s_{12}) = \frac{\Delta_{12} + 2s_{12}(m_1^2 + m_3^2) - (s_{12} + m_1^2 - m_2^2)\lambda}{2s_{12}},$$

$$s_{23} = s_{23}(m_1^2, m_2^2, m_3^2, s_{12}) = \frac{\Delta_{12} + 2s_{12}(m_2^2 + m_3^2) + (s_{12} - m_1^2 + m_2^2)\lambda}{2s_{12}}.$$

where

$$\lambda = \pm\sqrt{\Delta_{12} + 4s_{12}m_3^2}.$$

$$\Delta_{ij} = s_{ij}^2 + m_i^4 + m_j^4 - 2s_{ij}m_i^2 - 2s_{ij}m_j^2 - 2m_i^2m_j^2.$$

Functional equation for 1-loop propagator

These solutions give the following functional equation

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = \frac{s_{12}+m_1^2-m_2^2-\lambda}{2s_{12}}\, I_2^{(d)}(m_1^2, m_3^2, s_{13}(m_1^2, m_2^2, m_3^2, s_{12})) + \frac{s_{12}-m_1^2+m_2^2+\lambda}{2s_{12}}\, I_2^{(d)}(m_2^2, m_3^2, s_{23}(m_1^2, m_2^2, m_3^2, s_{12})).$$

- How useful are functional equations for Feynman integrals?
- How one can use these functional equations?

By rescaling $I_2^{(d)}$ this equation can be reduced to Sincov's type functional equation.

Functional equations for one-loop integrals with more external legs can be transformed to generalized Sincov equations.

A method of solution of these kind of equations was proposed by russian mathematician Sincov in 1903.

Functional equation for 1-loop propagator

Example of Sincov equation (derived in 1903) for functions with 2 variables.

Theorem *The general system of solutions of the functional equation*

$$F(x,z) = G(x,y) + H(y,z)$$

is

$$F(x,z) = h(z) - f(x), \quad G(x,y) = q(y) - f(x), \quad H(y,z) = h(z) - g(y), \tag{1}$$

where f, g *and* h *are arbitrary functions*

Proof: Making $y = a$ in the equation and calling $h(z) = H(a,z)$ and $f(x) = -G(x,a)$ we get

$$F(x,z) = G(x,a) + H(a,z) = h(z) - f(x),$$

and with $r(y) = H(y,b)$ we have

$$G(x,y) = F(x,b) - H(y,b) = h(b) - f(x) - r(y),$$
$$H(y,z) = F(c,z) - G(c,y) = h(z) - h(b) + r(y),$$

and calling $q(y) = h(b) - r(y)$ expression (1) is obtained.

Functional equation for 1-loop propagator

Important: Solution of functional equation can be expressed in terms of functions with lesser number of arguments!

One can identify $I_2^{(d)}$ integrals with different arguments as:

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) \sim F(m_1^2, m_2^2),$$
$$I_2^{(d)}(m_1^2, m_3^2, s_{13}(m_1^2, m_2^2, m_3^2, s_{12}) \sim G(m_1^2, m_3^2),$$
$$I_2^{(d)}(m_2^2, m_3^2, s_{23}(m_1^2, m_2^2, m_3^2, s_{12}) \sim H(m_2^2, m_3^2),$$

The choice of points a, b, c is important in calculation of Feynman integrals. In our example we consider $m_3^2 = 0$.

$m_3^2 = 0$

Substituting $m_3^2 = 0$ into functional equation gives:

$$\begin{aligned} I_2^{(d)}(m_1^2, m_2^2, s_{12}) &= \frac{s_{12} + m_1^2 - m_2^2 - \alpha_{12}}{2s_{12}} \; I_2^{(d)}(m_1^2, 0, s_{13}) \\ &+ \frac{s_{12} - m_1^2 + m_2^2 + \alpha_{12}}{2s_{12}} \; I_2^{(d)}(0, m_2^2, s_{23}) \end{aligned}$$

where

$$\begin{aligned} s_{13} &= \frac{\Delta_{12} + 2s_{12}m_1^2 - (s_{12} + m_1^2 - m_2^2)\alpha_{12}}{2s_{12}}, \\ s_{23} &= \frac{\Delta_{12} + 2s_{12}m_2^2 + (s_{12} - m_1^2 + m_2^2)\alpha_{12}}{2s_{12}}, \\ \alpha_{12} &= \pm\sqrt{\Delta_{12}}. \end{aligned}$$

Integral with arbitrary momentum and masses can be expressed in terms of integrals with one massless propagator!!!

Functional equation for 1-loop propagator

Analytical result for the integral $I_2^{(d)}(0, m^2, p^2)$ is known Bollini and Giambiagi (1972b), Boos and Davydychev (1990rg):

$$I_2^{(d)}(0, m^2, p^2) = I_2^{(d)}(0, m^2, 0)\, {}_2F_1\left[\begin{array}{cc} 1, 2 - \frac{d}{2}\,; & \dfrac{p^2}{m^2} \\ \frac{d}{2}\,; & \end{array}\right].$$

where

$$I_2^{(d)}(0, m^2, 0) = -\Gamma\left(1 - \frac{d}{2}\right) m^{d-4}.$$

Substituting this expression for $I_2^{(d)}(0, m^2, p^2)$ into functional equation we descover complete agreement with the known result for $I_2^{(d)}(m_1^2, m_2^2, s_{12})$

Functional equation for 1-loop propagator

Setting $m_2 = 0$ in the previous functional relation gives:

$$I_2^{(d)}(m_1^2, 0, s_{12}) = \frac{m_1^2}{s_{12}} I_2^{(d)}\left(m_1^2, 0, \frac{m_1^4}{s_{12}}\right) + \frac{(s_{12} - m_1^2)}{s_{12}} I_2^{(d)}\left(0, 0, \frac{(s_{12} - m_1^2)^2}{s_{12}}\right).$$

where

$$I_2^{(d)}(0, 0, p^2) = \frac{\Gamma\left(2 - \frac{d}{2}\right)\Gamma^2\left(\frac{d}{2} - 1\right)}{\Gamma(d-2)} (-p^2)^{\frac{d}{2}-2}.$$

Integral $I_2^{(d)}$ on the r.h.s. has inverse argument. In fact this equation corresponds to the well known formula for analytic continuation:

$$ {}_2F_1\left[\begin{matrix} 1, 2 - \frac{d}{2}\,; \\ \frac{d}{2}\,; \end{matrix} \; z\right] = \frac{1}{z}\; {}_2F_1\left[\begin{matrix} 1, 2 - \frac{d}{2}\,; \\ \frac{d}{2}\,; \end{matrix} \; \frac{1}{z}\right] + \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2} - 1\right)}{\Gamma(d-2)} (-z)^{\frac{d}{2}-2} \left(1 - \frac{1}{z}\right)^{d-3}.$$

One-loop vertex integral

Consider the recurrence relation at $n = 4$ and $j = 1$

$$
\begin{aligned}
& G_3\, \mathbf{1}^+ I_4^{(d+2)}(m_1^2, m_2^2, m_3^2, m_4^2, \{s_{ij}\}) \\
& -(\partial_1 \Delta_4) I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, \{s_{ij}\}) = \\
& \qquad (\partial_1^2 \Delta_3)\; I_3^{(d)}(m_2^2, m_3^2, m_4^2, s_{34}, s_{24}, s_{23}) \\
& \qquad + (\partial_1 \partial_2 \Delta_3)\; I_3^{(d)}(m_1^2, m_3^2, m_4^2, s_{34}, s_{14}, s_{13}) \\
& \qquad + (\partial_1 \partial_3 \Delta_3)\; I_3^{(d)}(m_1^2, m_2^2, m_4^2, s_{24}, s_{14}, s_{12}) \\
& \qquad + (\partial_1 \partial_4 \Delta_3)\; I_3^{(d)}(m_1^2, m_2^2, m_3^2, s_{23}, s_{13}, s_{12}),
\end{aligned}
$$

where

$$
\{s_{ij}\} = \{s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34}\}
$$

Integrals $I_4^{(d)}$ depend on 10 variables, and integrals $I_3^{(d)}$ only on 6 variables.

One-loop vertex integral

Again derivatives are polynomials in s_{ij} and masses:

$$\begin{aligned}
\partial_1^2\Delta_3 &= 2[(s_{23}-s_{24})^2 - 2(s_{24}+s_{23})s_{34} + s_{34}^2], \\
\partial_1\partial_2\Delta_3 &= 2[(s_{23}-s_{24})(s_{14}-s_{13}) + (s_{24}+s_{23}+s_{13}-2s_{12})s_{34} \\
&\qquad - s_{34}^2 + s_{14}s_{34}], \\
\partial_1\partial_3\Delta_3 &= 2[(s_{12}-s_{24}-2s_{13})s_{24} + s_{23}(s_{24}-s_{12}) \\
&\qquad + (s_{23}+s_{24}-s_{34})s_{14} + (s_{24}+s_{12})s_{34}], \\
\partial_1\partial_4\Delta_3 &= 2[s_{13}(s_{24}+s_{23}) + s_{23}(s_{12}+s_{24}-s_{23}-2s_{14}) - s_{12}s_{24} \\
&\qquad + (s_{23}+s_{12}-s_{13})s_{34}].
\end{aligned}$$

One-loop vertex integral

To obtain functional equations for $I_3(m_1^2, m_2^2, m_3^{,} s_{23}, s_{13}, s_{12})$ we must solve system of equations

$$G_3 = 0, \quad \partial_1 \Delta_4 = 0.$$

This system can be solved with respect to s_{14}, s_{34}. In general case solutions of this system and also functional equations are rather lengthy. For this reason we consider simplified situation. We set from the very beginning $m_4^2 = 0$ and exclude term with $\partial_1 \partial_2 \Delta_4$.

$$G_3 = 0, \qquad \partial_1 \Delta_4 = 0, \qquad \partial_1 \partial_2 \Delta_4 = 0,$$

This system can be resolved with respect to s_{14}, s_{34}, s_{24}

One-loop vertex integral

$$
\begin{aligned}
s_{14} &= s_{14}^{(13)}, \qquad\qquad s_{34} = s_{34}^{(13)}, \\
s_{24} &= s_{24}(m_1^2, m_3^2, s_{23}, s_{13}, s_{12}) \\
&= \frac{1}{2}(s_{12} + s_{23} - m_1^2 - m_3^2) \\
&\quad + \frac{(s_{12} - s_{23} - m_1^2 + m_3^2)(m_3^2 - m_1^2 + \alpha_{13})}{2s_{13}},
\end{aligned}
$$

where

$$
\begin{aligned}
s_{14}^{(ij)} &= \frac{\Delta_{ij} + 2m_i^2 s_{ij} - (s_{ij} + m_i^2 - m_j^2)\alpha_{ij}}{2s_{ij}}, \\
s_{34}^{(ij)} &= \frac{\Delta_{ij} + 2m_j^2 s_{ij} + (s_{ij} + m_j^2 - m_i^2)\alpha_{ij}}{2s_{ij}}, \\
\alpha_{ij} &= \pm\sqrt{\Delta_{ij}},
\end{aligned}
$$

One-loop vertex integral

Substituting this solution into initial equation gives:

$$I_3^{(d)}(m_1^2, m_2^2, m_3^2, s_{23}, s_{13}, s_{12}) =$$
$$\frac{s_{13} + m_3^2 - m_1^2 + \alpha_{13}}{2s_{13}}$$
$$\times I_3^{(d)}(m_2^2, m_3^2, 0, s_{34}^{(13)}, s_{24}(m_1^2, m_3^2, s_{23}, s_{13}, s_{12}), s_{23})$$
$$+ \frac{s_{13} - m_3^2 + m_1^2 - \alpha_{13}}{2s_{13}}$$
$$\times I_3^{(d)}(m_1^2, m_2^2, 0, s_{24}(m_1^2, m_3^2, s_{23}, s_{13}, s_{12}), s_{14}^{(13)}, s_{12})$$

Again as it was for integral $I_2^{(d)}$ integral $I_3^{(d)}$ with arbitrary arguments can be expressed in terms of integrals with at least one propagator massless!!!

One-loop vertex integral

Setting in this functional equation $m_2 = 0$ gives:

$$
\begin{aligned}
I_3^{(d)}(m_1^2, 0, m_3^2, s_{23}, s_{13}, s_{12}) = \\
\frac{s_{13} - m_1^2 + m_3^2 + \alpha_{13}}{2s_{13}} \\
\times I_3^{(d)}(0, m_3^2, 0, s_{34}^{(13)}, s_{24}(m_1^2, m_3^2, s_{23}, s_{13}, s_{12}), s_{23}) \\
+ \frac{s_{13} + m_1^2 - m_3^2 - \alpha_{13}}{2s_{13}} \\
\times I_3^{(d)}(m_1^2, 0, 0, s_{24}(m_1^2, m_3^2, s_{23}, s_{13}, s_{12}), s_{14}^{(13)}, s_{12})
\end{aligned}
$$

Therefore if $s_{ij} \neq 0$ then $I_3^{(d)}$ can be expressed in terms of integrals with two massless propagators!!! In fact this restriction can be avoided by making appropriate analytic continuation.

One-loop vertex integral

Moreover, in the very initial equation one can set form the very beginning $m_1 = m_3 = m_4 = 0$ and $s_{24} = 0$ and obtain functional equation:

$$I_3^{(d)}(0, m_2^2, 0, s_{23}, s_{13}, s_{12}) =$$
$$+\frac{m_2^2 s_{13} - s_{12} m_2^2 + s_{12} s_{23} - (s_{12} + m_2^2)\beta}{\Lambda_2} I_3^{(d)}(0, m_2^2, 0, 0, \kappa_{14}, s_{12})$$
$$+\frac{m_2^2 s_{23} - s_{23}^2 + (s_{23} + m_2^2)\beta}{\Lambda_2} I_3^{(d)}(0, m_2^2, 0, 0, \alpha, s_{23})$$
$$+\frac{s_{13} s_{23} + (s_{12} - s_{23})\beta}{\Lambda_2} I_3^{(d)}(0, 0, 0, \alpha, \kappa_{14}, s_{13}),$$

where $\Lambda_2, \kappa_{14}, \kappa_{34}, \beta$ made of polynomials in s_{ij} and radicals from polynomials. Integrals on the r.h.s. have not only zero masses but also some scalar invariants equal to zero!!!

One-loop vertex integral

As we see, if $s_{ij} \neq 0$ then integrals $I_3^{(d)}(m_1^2, m_2^2, m_3^2 s_{23}, s_{13}, s_{12})$ can be expressed in terms of integrals with two massless propagators and one external momentum squared equal to zero. Analytic expression for such an integral is:

$$I_3^{(d)}(0, m^2, 0, 0, s_{13}, s_{12}) =$$

$$= I_2^{(d)}(0, m^2, 0) F_1\left(1, 1, 2 - \frac{d}{2}, \frac{d}{2}; \frac{s_{12} - s_{13}}{m^2}, \frac{s_{12}}{m^2}\right)$$

$$+ \frac{I_2^{(d)}(0, 0, s_{13})}{m^2} \, {}_2F_1\left[\begin{array}{cc} 1, \frac{d-2}{2}; & \frac{p_{12} - p_{13}}{m^2} \\ d - 2; & \end{array}\right]$$

where F_1 is well known Appell's function.

One-loop vertex integral

To analytically continue integral $I_3^{(d)}$ with arbitrary arguments we must know how to analytically continue much simpler integral. Analytic continuation of F_1 and ${}_2F_1$ are known. But we can use for analytic continuation functional equations.

If $s_{12} \geq m^2$, then use functional equation:

$$I_3^{(d)}(0, m^2, 0;\ 0, s_{13}, s_{12}) =$$
$$\frac{m^2}{s_{12}}\ I_3^{(d)}\left(0, m^2, 0;\ 0, \frac{m^2(s_{13} - s_{12} + m^2)}{s_{12}}, \frac{m^4}{s_{12}}\right)$$
$$+\frac{(s_{12} - m^2)}{s_{12}}\ I_3^{(d)}\left(0, 0, 0;\ \frac{m^2(s_{13} - s_{12} + m^2)}{s_{12}}, \frac{(s_{12} - m^2)^2}{s_{12}}, s_{13}\right).$$

One-loop vertex integral

If $|s_{12}-s_{13}| \geq m^2$ and $s_{12} \leq m^2$, the following relation can be applied

$$
\begin{aligned}
& I_3^{(d)}(0,m^2,0;\ 0,s_{13},s_{12}) = \\
& \frac{s_{12}m^2}{m^2 s_{13}+s_{12}s_{13}-s_{12}^2} I_3^{(d)}\left(0,m^2,0;\ 0,\frac{s_{12}^2(s_{13}-s_{12}+m^2)}{m^2 s_{13}+s_{12}s_{13}-s_{12}^2},s_{12}\right) \\
& +\frac{s_{12}(s_{13}-s_{12})}{m^2 s_{13}+s_{12}s_{13}-s_{12}^2} \\
& \times I_3^{(d)}\left(0,0,0;\ \frac{m^2(s_{13}-s_{12})^2}{m^2 s_{13}+s_{12}s_{13}-s_{12}^2},\frac{s_{12}^2(s_{13}-s_{12}+m^2)}{m^2 s_{13}+s_{12}s_{13}-s_{12}^2},s_{13}\right) \\
& +\frac{m^2(s_{13}-s_{12})}{m^2 s_{13}+s_{12}s_{13}-s_{12}^2} I_3^{(d)}\left(0,m^2,0;\ 0,\frac{m^2(s_{13}-s_{12})^2}{m^2 s_{13}+s_{12}s_{13}-s_{12}^2},0\right).
\end{aligned}
$$

The last but one argument of the first integral on the r.h.s is finite for large $|s_{13}| \geq m^2$. The 2-d and the 3-d integrals on the r.h.s can be expressed in terms of ${}_2F_1$.

In other kinematical domains a combination of both relations is needed.

Functional equation for box integrals

Functional equations for one-loop integrals corresponding to diagrams with four external legs can be derived from the equation for five point functions:

$$
\begin{aligned}
& G_4\, \mathbf{j}^+ I_5^{(d+2)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2; \{s_{kr}\}) \\
& -(\partial_j \Delta_5) I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2; \{s_{kr}\}) \\
= \quad & (\partial_j \partial_1 \Delta_5)\, I_4^{(d)}(m_2^2, m_3^2, m_4^2, m_5^2; s_{23}, s_{34}, s_{45}, s_{25};\ s_{35}, s_{24}) \\
& + (\partial_j \partial_2 \Delta_5)\, I_4^{(d)}(m_1^2, m_3^2, m_4^2, m_5^2; s_{13}, s_{34}, s_{45}, s_{15};\ s_{35}, s_{14}) \\
& + (\partial_j \partial_3 \Delta_5)\, I_4^{(d)}(m_1^2, m_2^2, m_4^2, m_5^2; s_{12}, s_{24}, s_{45}, s_{15};\ s_{25}, s_{14}) \\
& + (\partial_j \partial_4 \Delta_5)\, I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_5^2; s_{12}, s_{23}, s_{35}, s_{15};\ s_{25}, s_{13}) \\
& + (\partial_j \partial_5 \Delta_5)\, I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2; s_{12}, s_{23}, s_{34}, s_{14};\ s_{24}, s_{13}).
\end{aligned}
$$

The integrals $I_4^{(d)}(\{m_j^2\},\{s_{kr}\})$ are

$$I_4^{(d)}(m_n^2, m_j^2, m_k^2, m_l^2;\ s_{nj}, s_{jk}, s_{kl}, s_{nl};\ s_{jl}, s_{nk})$$
$$= \int \frac{d^dq}{i\pi^{d/2}} \frac{1}{[(q-p_n)^2-m_n^2][(q-p_j)^2-m_j^2][(q-p_k)^2-m_k^2][(q-p_l)^2-m_l^2]}$$

The integral $I_5^{(d)}(\{m_j^2\},\{s_{kr}\})$ depends on **15** kinematical variables, while the integral $I_4^{(d)}(\{m_j^2\},\{s_{kr}\})$ depends on **10** variables.

To obtain a functional equation for the integral $I_4^{(d)}(\{m_j^2\},\{s_{kr}\})$ with all 10 kinematical variables arbitrary, we can impose conditions on some 5 variables. To eliminate terms with $I_5^{(d)}(\{m_j^2\},\{s_{kr}\})$ from the equation two conditions must be fulfilled:

$$G_4 = 0, \qquad \partial_j \Delta_5 = 0,$$

thus fixing two kinematical variables.

Functional equations for $B(s,t)$ and $D_2(t,s)$

By choosing remaining three variables one can either obtain a functional equation connecting the integral of interest with integrals which are easy to evaluate or to reduce the number of terms in the functional equation by requiring some derivatives $\partial_i \partial_j \Delta_5$ to be zero. The investigation of possible choices can be done on a computer.

The solutions of these equations and functional equations are rather lengthy. For this reason we consider simplified situation. We consider “basic integrals” needed for calculation of rather important radiative corrections required for LHC. Namely we consider integrals needed for the one-loop radiative corrections to the process $e^+e^- \to e^+e^-$, the so-called Bhabha scattering, and heavy-quark production.

Functional equations for $B(s,t)$ and $D_2(t,s)$

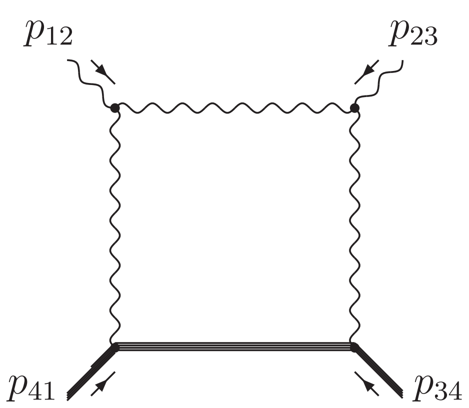
In what follows, we use the following short-hand notation for integrals needed in the calculation of Bhabha scattering, and heavy-quark production:

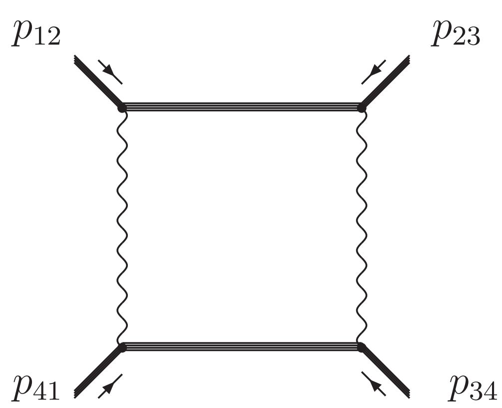
$$
\begin{aligned}
B(s,t) &= I_4^{(d)}(0,m^2,0,m^2;\; m^2,m^2,m^2,m^2;\; s,t), \\
D_2(s,t) &= I_4^{(d)}(0,0,0,m^2;\; 0,0,m^2,m^2;\; t,s).
\end{aligned}
$$

The diagrams corresponding to these integrals are:

$s = (p_{12} + p_{41})^2$

$t = (p_{12} + p_{23})^2$





Functional equations for $B(s,t)$ and $D_2(t,s)$

Substituting

$$j = 2, \quad m_2^2 = m_3^2 = m_5^2 = 0, s_{12} = s_{23} = s_{34} = s_{14} = m^2,$$

$$s_{24} = s, s_{13} = t,$$

into the starting equation and choosing $s_{15}, s_{25}, s_{35}, s_{45}$ from the conditions

$$G_4 = 0, \qquad \partial_2 \Delta_5 = 0, \qquad \partial_1 \partial_2 \Delta_5 = 0, \qquad \partial_2 \partial_3 \Delta_5 = 0,$$

we arrive at the following equation:

$$B(s,t) = \frac{m^2}{s}(1+\alpha_+)D_2(t, m^2\alpha_+) + \frac{m^2}{s}(1+\alpha_-)D_2(t, m^2\alpha_-),$$

where

$$\alpha_\pm = \frac{1 \pm \beta_s}{1 \mp \beta_s}, \qquad \beta_s = \sqrt{1 - \frac{4m^2}{s}}.$$

Thus, we have a relation connecting the integral $B(s,t)$ with the integral $D_2(s,t)$.

Functional equations for $B(s,t)$ and $D_2(t,s)$

It turns out that the integral $D_2(s,t)$ satisfies the following functional equation:

$$D_2(t,s) = \frac{m^2}{s}\, D_2\left(t, \frac{m^4}{s}\right) + \frac{s-m^2}{s}\, I_4^{(d)}\left(0,0,0,0;\ 0,0,0,0;\ \frac{(s-m^2)^2}{s}, t\right),$$

which can be obtained from the initial equation by setting

$$j=5,\ \ m_1^2 = m_2^2 = m_3^2 = m_5^2 = s_{12} = s_{23} = 0,$$

$$s_{34} = s_{14} = m_4^2 = m^2, \quad s_{24} = s,\ s_{13} = t$$

and imposing the conditions

$$G_4 = \partial_5 \Delta_5 = \partial_1 \partial_5 \Delta_5 = \partial_3 \partial_5 \Delta_5 = 0.$$

The last integral corresponds to the box integral with all propagators massless and the squares of all external momenta equal to zero.

Functional equations for $B(s,t)$ and $D_2(t,s)$

By using this equation and taking into account the relation $\alpha_+\alpha_- = 1$, one can write the integral with argument α_+ as

$$D_2(t,m^2\alpha_+) = \alpha_-\, D_2(t,m^2\alpha_-)+(1-\alpha_-)\, I_4^{(d)}\left(0,0,0,0;\ 0,0,0,0;\ s-4m^2,t\right).$$

Exploiting this relation gives

$$B(s,t) = (1-\beta_s)D_2(t,m^2\alpha_-) + \beta_s\, I_4^{(d)}(0,0,0,0;\ 0,0,0,0;\ s-4m^2,t).$$

This equation canbe inverted

$$D_2(s,t) = \frac{t+m^2}{2t}\, B\left(\frac{(t+m^2)^2}{t},s\right) + \frac{t-m^2}{2t}\, I_4^{(d)}\left(0,0,0,0;\ 0,0,0,0;\ \frac{(t-m^2)^2}{t},s\right).$$

We remark that, for $\varepsilon = (4-d)/2 \to 0$, the integral D_2 has a pole proportional to $1/\varepsilon^2$, while the leading singularity of the integral $B(s,t)$ is $1/\varepsilon$. The leading $1/\varepsilon^2$ singularity on the right-hand side comes from the massless integral I_4.

Summary

Summary:

- New kind of relations for Feynman integrals are proposed
- Functional equations allows one to reduce complicated integrals with many variables to simpler integrals
- Functional equations can be used for analytic continuation of complicated integrals
- Functional equations are valid for arbitrary space-time dimension d and therefore $\varepsilon = (4-d)/2$ expansion for integrals with many mass scales can be significantly simplified.

Summary

Summary:

- We expect that the proposed general method can be mapped directly to hypergeometric functions. We have seen that Feynman integrals are expressible in terms of hypergeometri functions - therefore the proposed equations are in fact functional equations for combinations of hypergeometric functions.
- It will be also interesting to formulate such a method for some classes of holonomic functions.