Factorization method in the model of unstable particles

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Contents

- The model of unstable particles (UP) with a smeared mass
- Effect of factorization in the processes with UP in intermediate states
- Factorization method in the model of UP with a smeared mass
- Conclusion and discussions

The model of unstable particles

The conventional descriptions of unstable particles (UP's) are usually performed with the help of the S- matrix with complex pole or by dressed propagator.

There are also another approaches:

1. Time-asymmetric QFT of UP's (A.R.Bohm et al., Nucl. Phys. B 581, 91 (2000)).
2. Effective theory (M.Beneke et al., Phys. Rev. Lett. 93, 011602 (2004)).
3. Phenomenological QF model of UP's (V.I.Kuksa, Int. J. Mod. Phys. A 24, 1185 (2009)).

In this talk, we consider some features of the model of UP's with a smeared mass. This features lead to the factorization effects in the description of processes with UP in an intermediate state. The model under consideration is based on the time-energy uncertainty relation(UR).

Because of specific nature of time-energy UR, here we touch this item in close analogy with the consideration in (S.M. Bilenky et al., arXiv:0803.0527). Formally, all UR's are based on the Cauchy-Schwarz inequality:

$$\Delta f \cdot \Delta g \geqslant \frac{1}{2}|\langle\Psi|[\hat{f},\hat{g}]|\Psi\rangle|, \tag{1.1}$$

However, various UR's have different physical nature. Heisenberg UR for the momentum and coordinate follows from (1.1) and commutation relation:

$$[\hat{p},\hat{q}] = -i\hbar \quad \longrightarrow \quad \Delta p \cdot \Delta q \geqslant \frac{1}{2}\hbar. \tag{1.2}$$

The time-energy UR has a completely different character, due to the time is not an operator but parameter in Quantum Mechanics. This relation follows from (1.1) and equation for time-depended operator $Q(t)$ in Heisenberg representation:

$$i\hbar\frac{d\hat{Q}(t)}{dt} = [\hat{Q}(t),\hat{H}] \tag{1.3}$$

From (1.1) and (1.3) it follows formal relation:

$$\Delta E \cdot \Delta t \geqslant \frac{1}{2}\hbar, \;\; \Delta t = \frac{\Delta Q(t)}{|d\bar{Q}(t)/dt}|. \tag{1.4}$$

For an excited state, Δt is the life-time of the state (P. Bush, arXiv:quant-ph/0105049).

The first model of UP, based on the energy-time UR, was suggested in (P.T.Matthews and A.Salam, Phys. Rev. 112, 283 (1958)). Time-dependent wave function of UP in their rest system was written in term of its Fourier transform:

$$\Phi(t) \sim \exp\{iMt - \Gamma|t|/2\} \longrightarrow \frac{\Gamma}{2\pi}\int \frac{\exp\{-imt\}}{(m-M)^2 + \Gamma^2/4} dm, \tag{1.5}$$

where $\Gamma = 1/\tau$ is decay width of UP. The right-hand side of Eq.(1.5) may be interpreted as a distribution of mass values, with a spread, δm, related to the mean life $\delta\tau = 1/\Gamma$, by uncertainty relation:

$$\delta m \cdot \delta\tau \sim 1, \;\; \text{or} \;\; \delta m \sim \Gamma \;\; (c = \hbar = 1). \tag{1.6}$$

Now, we consider the main elements of the model of UP with a smeared mass. The model field function of the UP is the superposition of the standard ones:

$$\Phi_a(x) = \int \Phi_a(x,\mu)\,\omega(\mu)\,d\mu, \tag{1.7}$$

where $\omega(\mu)$ is some weight function and spectral component is:

$$\Phi_a(x,\mu) = \frac{1}{(2\pi)^{3/2}} \int \Phi_a(k)\delta(k^2-\mu)e^{ikx}\,dk. \tag{1.8}$$

Using the representation (1.8), we suppose that for an arbitrary mass parameter μ the spectral component of field function, $\Phi_a(x,\mu)$, satisfies to the Klein-Gordon equation:

$$(\Box-\mu)\Phi_a(x,\mu)=0,\ \ k^0=\pm\sqrt{k^2+\mu}. \tag{1.9}$$

In another words, within the framework of the model, UP is on a smeared mass shell with an arbitrary $\mu = k^2$. The third element of the model is the commutation relations:

$$[\dot{\Phi}^-_a(\bar{k},\mu),\ \Phi^+_b(\bar{q},\mu^{'})]_\pm = \delta(\mu-\mu^{'})\delta(\bar{k}-\bar{q})\delta_{ab}, \tag{1.10}$$

The model Green function has a spectral form, in particular, for the case of scalar UP it is:

$$D(x) = \int D(x,\mu)\,\rho(\mu)\,d\mu, \;\; \rho(\mu) = |\omega(\mu)|^2, \tag{1.11}$$

where $D(x,\mu)$ is defined in a standard way for the fixed $m^2 = \mu$ and $\rho(\mu)$ is probability density of mass parameter μ. The amplitude of the process with UP in a final or initial state has the form:

$$A(k,\mu) = \omega(\mu) A^{st}(k,\mu), \tag{1.12}$$

where $A^{st}(k,\mu)$ is amplitude at fixed μ, which is defined in a standard way.

Now, we consider the definition of $\rho(\mu)$ from the matching of the model scalar propagator to the standard dressed one:

$$\int \frac{\rho(\mu)d\mu}{k^2-\mu+i\epsilon} \longleftrightarrow \frac{1}{k^2-M_0^2-\Pi(k^2)}\,, \tag{1.13}$$

where $\Pi(k^2)$ is the conventional polarization function. It was shown (V.I. Kuksa, Int. J. Mod. Phys. A 24, 1185 (2009)) that the correspondence (1.13) leads to the definition:

$$\rho(\mu) = \frac{1}{\pi}\frac{Im\Pi(\mu)}{[\mu-M^2(\mu)]^2+[Im\Pi(\mu)]^2}\,, \tag{1.14}$$

where $M^2(\mu)=M_0^2+Re\Pi(\mu)$. So, the smearing of mass is generated by stochastic nature of UP's interaction with vacuum. The mean value of mass is defined by the real part of polarization operator and dispersion of mass by the imaginary one.

The relations between scalar, vector and spinor Green function, which follows from the equation of motion, together with the definition (1.14) lead to the correspondences

$$\int \frac{-g_{mn}+k_m k_n/\mu}{k^2-\mu+i\epsilon}\rho(\mu)\,d\mu \longleftrightarrow \frac{-g_{mn}+k_m k_n/k^2}{k^2-M^2(k^2)-iIm\Pi(k^2)}\,. \tag{1.15}$$

and

$$\int \frac{\hat{k}+\sqrt{\mu}}{k^2-\mu+i\epsilon}\,\rho(\mu)d\mu \longleftrightarrow \frac{\hat{k}+k}{k^2-M^2(k^2)-ik\Sigma(k^2)}\,, \tag{1.16}$$

The transition (1.13) - (1.16) define some effective theory of UP's. The correspondence between standard and model expressions for the cases of vector (in unitary gauge) and spinor UP is given by the transition $m \leftrightarrow k$, where $k=\sqrt{k_i k^i}$:

$$\eta_{mn}(m) = -g_{mn}+k_m k_n/m^2,\;\; \hat{\eta}(m)=\hat{k}+m \;\; \text{(Standard)};$$

$$\eta_{mn}(k) = -g_{mn}+k_m k_n/k^2,\;\; \hat{\eta}(k)=\hat{k}+k \;\; \text{(Model)}. \tag{1.17}$$

The structure of the model propagators lead to the effect of exact factorization, while the standard ones lead to approximate factorization.

II. EFFECTS OF FACTORIZATION IN THE PROCESSES WITH UP IN THE INTERMEDIATE STATE

Exact factorization is stipulated by the following properties of the model:

a)the smearing of mass shell;

b)the structure of the model functions $\eta_{mn}(k)$ and $\hat{\eta}(k)$, Eq.(1.17).

The first factor allows us to describe UP in an intermediate state with the momentum k as the particle in a final or initial states with the mass $m^2 = k^2$. These states are described by the following polarization matrixes:

$$\sum_{a=1}^{3} e^a_m(\bar{k})\dot{e}^a_n(\bar{k}) = -g_{mn} + \frac{k_m k_n}{k^2} \quad \text{(vector UP)};$$
$$\sum_{a=1}^{2} u^{a,\mp}_i(\bar{k})\bar{u}^{a,\pm}_k(\bar{k}) = \frac{1}{2k^0}(\hat{k} + k)_{ik} \quad \text{(spinor UP)}. \tag{2.1}$$

The second factor is the coincidence of the expressions for the propagator numerators (1.17) and for the polarization matrixes (2.1). It allows us to represent the amplitude of the process with UP in an intermediate state in a partially factorized form:

$$M(p,p^{'},q) = K\sum_{a}\frac{M_{1}^{(a)}(p,q)\cdot M_{2}^{(a)}(p^{'},q)}{P(q^2,M^2)}., \tag{2.2}$$

Note that full factorization occurs in integrated cross-section or decay rate.

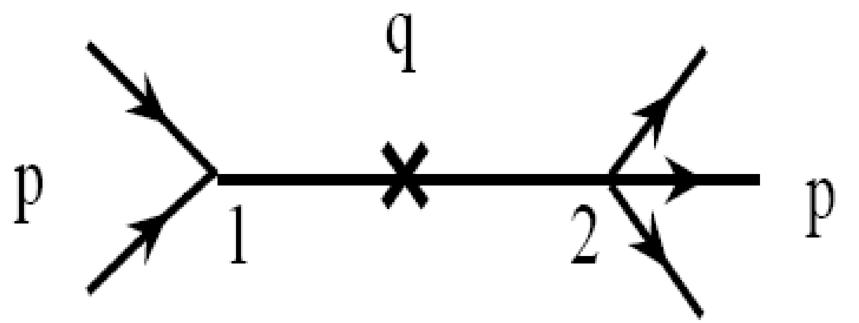

FIG. 1: Factorization in reducible diagram

Now, we demonstrate the factorization effect in the case of the simplest basic elements of the tree processes. The first element is two-particle scattering with UP of any type in an intermediate state: $a+b\rightarrow R\rightarrow c+d$ (V.I. Kuksa, Int. J. Mod. Phys. A 23, 4509 (2008)).

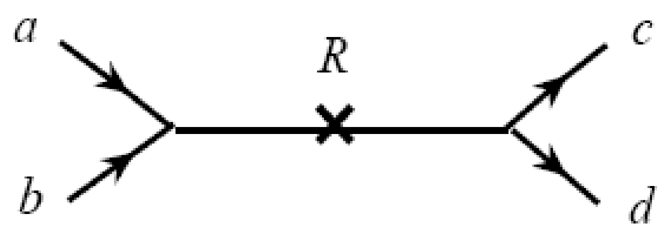

FIG. 2: Factorization in $2\rightarrow 2$ scattering diagram

By straightforward calculation it was shown that the cross-section for all permissible combinations of particles (a,b,R,c,d) can be represented in universal factorized form:

$$\sigma(ab\rightarrow R\rightarrow cd)=\frac{16\pi(2J_a+1)(2J_b+1)}{(2J_R+1)\bar{\lambda}^2(m_a,m_b;\sqrt{s})}\frac{\Gamma^{ab}_R(s)\Gamma^{cd}_R(s)}{|P_R(s)|^2}. \qquad (2.3)$$

Here, $s=(p_a+p_b)^2$, $P_R(s)$ is propagator's denominator of unstable particle R, $\lambda(m_a,m_b;\sqrt{s})$ is normalized Callen function and $\Gamma^{ab}_R(s)=\Gamma(R(s)\rightarrow ab)$ is decay rate of the particle R

The second basic element is three-particle decay with UP in an intermediate state - $\Phi \rightarrow \phi_1 R \rightarrow \phi_1\phi_2\phi_3$, where R is UP of any kind.

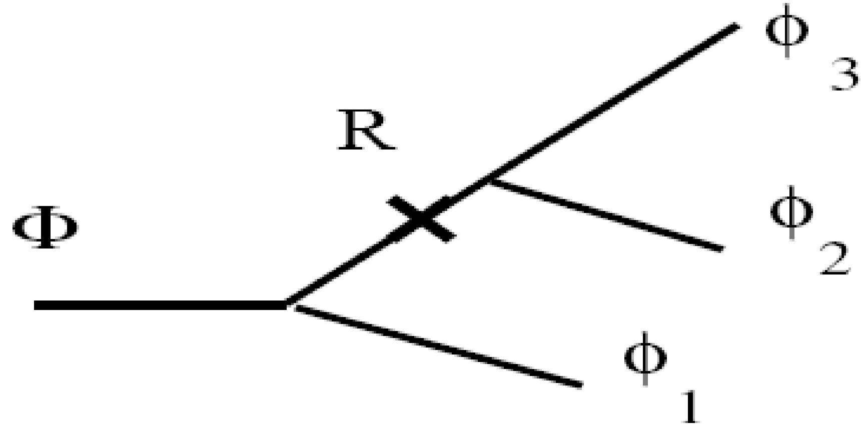

FIG. 3: Factorization in $1 \rightarrow 3$ decay diagram

By straightforward calculations it was shown that the decay rate can be represented in the universal factorized form (V.I. Kuksa, Phys. Lett. B 633, 545 (2006)):

$$\Gamma(\Phi \rightarrow \phi_1\phi_2\phi_3) = \int_{q_1^2}^{q_2^2} \Gamma(\Phi \rightarrow \phi_1 R(q))\, \frac{q\, \Gamma(R(q) \rightarrow \phi_2\phi_3)}{\pi |P_R(q)|^2}\, dq^2\,, \tag{2.4}$$

where $q_1 = m_2 + m_3$ and $q_2 = m_\Phi - m_1$.

By means of the summation over decay channels of R, from Eq.(2.4) we get the well-known convolution formula for the decays with UP in a final state:

$$\Gamma(\Phi \to \phi_1 R) = \int_{q_1^2}^{q_2^2} \Gamma(\Phi \to \phi_1 R(q))\, \rho_R(q)\, dq^2 \,. \tag{2.5}$$

In Eq.(2.5) the smearing of mass of unstable state R is described by the probability density $\rho_R(q)$:

$$\rho_R(q) = \frac{q\, \Gamma_R^{tot}(q)}{\pi\, |P_R(q)|^2}. \tag{2.6}$$

Substituting the relation $q\Gamma^{tot}(q) = Im\Pi(q)$ into Eq.(2.6) leads to the definition (1.14). Note, also, that in the two-particle scattering and three-particle decay rate the factorization is exact within the framework of the model and approximate in the standard treatment (convolution method).

III. FACTORIZATION METHOD IN THE MODEL OF UP'S WITH SMEARED MASS

The method is based on exact factorization of the simplest processes with UP in an intermediate state. It is realized by combination of the universal formulae for cross-section (2.3) and decay rate (2.4). Further, we consider some examples of such processes.

$$1)\, a+b \to R_1 \to c+R_2 \to c+d+f. \qquad (3.1)$$

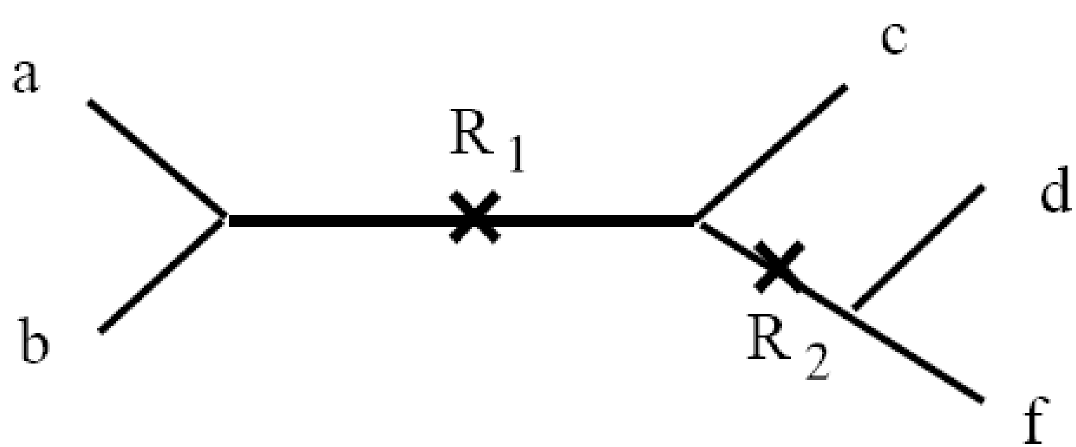

FIG. 4: Factorization in $2 \to 3$ scattering-decay diagram

The cross-section of this process is direct combination of the expressions (2.3) and (2.4) (V.I. Kuksa, Phys. Atom. Nucl., 72, 1063 (2009)):

$$\sigma(ab \to R_1 \to cdf) = \frac{16 k_a k_b}{k_R \bar{\lambda}^2(m_a, m_b; \sqrt{s})} \frac{\Gamma^{ab}_{R_1}(s)}{|P_{R_1}(s)|^2} \int_{q_1^2}^{q_2^2} \Gamma(R_1(s) \to c R_2(q)) \frac{q\, \Gamma^{df}_{R_2}(q)}{|P_{R_2}(q)|^2} dq^2, \qquad (3.2)$$

where $k_a = 2J_a + 1$. It should be noted that the factorization effectively reduces the number of independent kinematical variables. In the standard approach for the process $2 \to 3$ the number of such variables $N = 3n - 4 = 5$ (R.Kumar, Phys. Rev. 185,1865 (1969)), while the approach suggested gives $N = 1$.

$$2)\,\Phi \to a + R_1 \to a + b + R_2 \to a + b + c + d. \qquad (3.3)$$

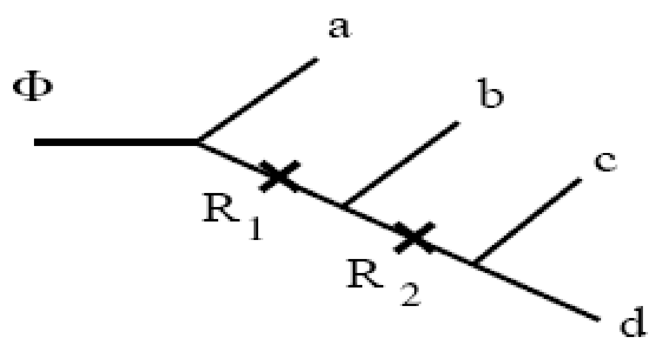

FIG. 5: Factorization in $1 \to 4$ decay diagram

The decay rate of this decay-chain process is given by the formula (2.4) doubling:

$$\Gamma(\Phi \to abcd) = \frac{1}{\pi^2} \int_{q_1^2}^{q_2^2} \frac{q\,\Gamma(\Phi \to aR_1(q))}{|P_{R_1}(q)|^2} \times$$
$$\times \int_{g_1^2}^{g_2^2} \Gamma(R_1(q) \to bR_2(g))\, \frac{g\,\Gamma(R_2(g) \to cd)}{|P_{R_2}(g)|^2}\, dg^2\, dq^2. \qquad (3.4)$$

Note that in the general case of n-particles decay the number of kinematic variables is $N = 3n - 7 = 5$ (R.Kumar, Phys. Rev. 185,1865 (1969)), while the method gives $N = 2$.

$$3)\, e^+e^- \to ZZ \to \sum_{i,k} \bar{f}_i f_i \bar{f}_k f_k. \tag{3.5}$$

The process of Z-pair production (or four-fermion production in double-pole approximation, DPA).

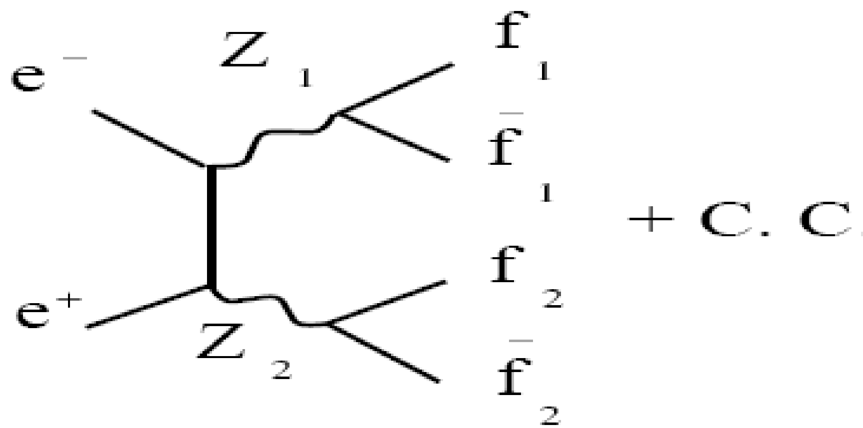

FIG. 6: Z-pair production process

The cross-section of the full process in DPA is:

$$\sigma^{tr}(e^+e^- \to ZZ) = \int\int \sigma^{tr}(e^+e^- \to Z_1(m_1)Z_2(m_2))\, \rho_Z(m_1)\, \rho_Z(m_2)\, dm_1\, dm_2. \tag{3.6}$$

In Eq. (3.6):

$$\rho_Z(m) = \frac{1}{\pi} \frac{m\, \Gamma_Z^{tot}(m)}{(m^2 - M_Z^2)^2 + (m\, \Gamma_Z^{tot}(m))^2}. \tag{3.7}$$

Using the FM, one can describes the complicate decay-chain and scattering processes in a simple way. The same results can occur within the frame of standard treatment as the approximations. Such approximations are known as narrow-width approximation (NWA), convolution method (CM), decay-chain method (DCM) and semi-analytical approach (SAA). All these approximations get a strict analytical formulation within the framework of FM. For instance, the NWA includes five assumptions which was considered in detail in (D. Berdine, N. Kauer and D. Rainwater, hep-ph/0703058). Factorization method contains one assumption - non-factorizable corrections are small (the fifth assumption of the NWA).

Now, we evaluate an error of the method, which is defined as the deviation of results from standard ones. For the scalar UP an error equal zero in accordance with the definition. For the vector UP an error is caused by the difference:

$$\delta\eta_{\mu\nu} = \eta_{\mu\nu}(q^2) - \eta_{\mu\nu}(m^2) = q_\mu q_\nu \frac{m^2 - q^2}{m^2 q^2}. \tag{3.8}$$

In the processes of meson-pairs production $e^+e^- \to \rho^0, \omega, ... \to \pi^+\pi^-, K^+K^-, \rho^+\rho^-, ...$ the deviation equals to zero too, due to zero contribution of the transverse parts of amplitudes in both cases:

$$M^{trans}(q) \sim \bar{e}^-(p_1)\hat{q}e^-(p_2) = \bar{e}^-(p_1)(\hat{p}_1 + \hat{p}_2)e^-(p_2) = 0. \tag{3.9}$$

In the case of the high-energy collisions $e^+e^- \to Z \to f\bar{f}$ (we neglect $\gamma - Z$ interference) the transverse part of amplitude is:

$$M^{trans}(q) \sim \bar{e}^-(p_1)\hat{q}(c_e - \gamma_5)e^-(p_2) \cdot \bar{f}^+(k_1)\hat{q}(c_f - \gamma_5)f^+(k_2) \tag{3.10}$$

and we get from (3.8):

$$\delta M \sim \frac{m_e m_f}{M_Z^2}\,\frac{M_Z^2 - q^2}{q^2}. \tag{3.11}$$

Thus, an error of FM at the vicinity of resonance is always small. However, at $q^2 \sim m_f^2$ it can be noticeable, $\delta M \sim m_e/m_f$.

A relative deviation of the model partial cross-section in the boson-pair production with consequence decay boson to fermion pair is:

$$\epsilon_f \sim 4\frac{m_f}{M}[1 - M\int_{m_f^2}^{s}\frac{\rho(q^2)}{q}\,dq^2], \tag{3.12}$$

where M is boson mass. For the case $f = \tau$ an error is maximal, $\epsilon_\tau \sim 10^{-3}$.

A relative deviations of the model partial decay rates for the case of μ and τ decays are follow:

$$\epsilon(\mu \to e\nu\bar{\nu}) \approx 5\cdot 10^{-4};\;\; \epsilon(\tau \to e\nu\bar{\nu}) \approx 3\cdot 10^{-6};\;\; \epsilon(\tau \to \mu\nu\bar{\nu}) \approx 3\cdot 10^{-2}. \tag{3.13}$$

So, an error in the last case is large. In the case of spinor UP in an intermediate state an error is an order of $(M_f - q)/M_f$. It can be large when q is far from the resonance region.

IV. CONCLUSION

1) The model of UP's leads to effective theory of UP's with specific structure of vector and spinor propagators. This structure causes the effect of exact factorization in the processes with UP in an intermediate state.

2) Factorization method is simple and convenient tool for description of the complicate scattering and decay-chain processes.

3) FM can be used as some analytical analog of NWA, which allows us to evaluate the error of the approach in a simple way.