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The property of maximal transcendentality in calculations. Evaluation of anomalous dimensions of the Wilson operators in $\mathcal{N} = 4$ SUSY.

OUTLINE

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1. Introduction

A. Deep-inelastic lepton-hadron scattering (DIS) cross-section:

$$\sigma \sim L^{\mu\nu} F^{\mu\nu}$$

Hadron part $F^{\mu\nu}$ ($Q^2 = -q^2 > 0$, $x = Q^2/[2(pq)]$):

$$F^{\mu\nu} = (-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}) F_1(x, Q^2) - (p^\mu - \frac{(pq)}{q^2} q^\mu)(p^\nu - \frac{(pq)}{q^2} q^\nu) \frac{2x}{q^2} F_2(x, Q^2) + ...,$$

where $F_k(x, Q^2)$ ($k = 1, 2, 3, L$) - are DIS structure functions (SF) and q and p are photon and hadron (parton) momentums.

B. Wilson operator expansion: Mellin moments $M_k(j,Q^2)$ of DIS SF $F_k(x,Q^2)$ can be represented as sum

$$M_k(j,Q^2) = \sum_{a=ns,s,g} \underbrace{C_k^a(j,Q^2/\mu^2)}_{\text{Coeff. function}} A_a(j,\mu^2),$$

where $A_a(j,\mu^2) =< N|\mathcal{O}^a_{\mu_1,\ldots,\mu_j}|N >$ are matrix elements of the Wilson operators $\mathcal{O}^a_{\mu_1,\ldots,\mu_j}$.

C. The matrix elements $A_a(j,\mu^2)$ are Mellin moments of the unpolarized and polarized parton densities $f_a(x,\mu^2)$ and $\tilde{f}_a(x,\mu^2)$.

DGLAP equations:

$$\frac{d}{d\ln Q^2} f_a(x,Q^2) = \int_x^1 \frac{dy}{y} \sum_b W_{b\to a}(x/y)\, f_b(y,Q^2)\,,$$
$$\frac{d}{d\ln Q^2} \tilde{f}_a(x,Q^2) = \int_x^1 \frac{dy}{y} \sum_b \tilde{W}_{b\to a}(x/y)\, \tilde{f}_b(y,Q^2)\,. \qquad (1)$$

The anomalous dimensions (AD) $\gamma_{ab}(j)$ of the twist-2 Wilson operators $\mathcal{O}^a_{\mu_1,\dots,\mu_j}$ (hereafter $a_s = \alpha_s/(4\pi)$)

$$\gamma_{ab}(j) = \int_0^1 dx\; x^{j-1} W_{b\to a}(x) = \sum_{m=0}^{\infty} \gamma_{ab}^{(m)}(j) a_s^m,$$
$$\tilde{\gamma}_{ab}(j) = \int_0^1 dx\; x^{j-1} \tilde{W}_{b\to a}(x) = \sum_{m=0}^{\infty} \tilde{\gamma}_{ab}^{(m)}(j) a_s^m.$$

2. Results

A. Wilson twist-2 operators in the $\mathcal{N}=4$ SUSY

$$
\begin{aligned}
\mathcal{O}^{g}_{\mu_1,...,\mu_j} &= \hat{S} G^{a}_{\rho\mu_1}\mathcal{D}_{\mu_2}\mathcal{D}_{\mu_3}...\mathcal{D}_{\mu_{j-1}}G^{a}_{\rho\mu_j}\,,\\
\tilde{\mathcal{O}}^{g}_{\mu_1,...,\mu_j} &= \hat{S} G^{a}_{\rho\mu_1}\mathcal{D}_{\mu_2}\mathcal{D}_{\mu_3}...\mathcal{D}_{\mu_{j-1}}\tilde{G}^{a}_{\rho\mu_j}\,,\\
\mathcal{O}^{\lambda}_{\mu_1,...,\mu_j} &= \hat{S}\bar{\lambda}^{a}_{i}\gamma_{\mu_1}\mathcal{D}_{\mu_2}...\mathcal{D}_{\mu_j}\lambda^{a\,i}\,,\\
\tilde{\mathcal{O}}^{\lambda}_{\mu_1,...,\mu_j} &= \hat{S}\bar{\lambda}^{a}_{i}\gamma_5\gamma_{\mu_1}\mathcal{D}_{\mu_2}...\mathcal{D}_{\mu_j}\lambda^{a\,i}\,,\\
\mathcal{O}^{\phi}_{\mu_1,...,\mu_j} &= \hat{S}\bar{\phi}^{a}_{r}\mathcal{D}_{\mu_1}\mathcal{D}_{\mu_2}...\mathcal{D}_{\mu_j}\phi^{a}_{r}\,,
\end{aligned}
$$

where $\mathcal{D}_\mu$ are covariant derivatives.

The LO AD (L.Lipatov, 1999) ($S(j) \equiv S_1(j) = \Psi(j+1) - \Psi(1)$):

for unpolarized case

$$\gamma_{gg}^{(0)}(j) = \frac{4}{j+1} - \frac{4}{j+2} - \frac{8}{j} - 4S(j-2), \qquad \gamma_{q\varphi}^{(0)}(j) = \frac{8}{j},$$

$$\gamma_{\lambda g}^{(0)}(j) = \frac{8}{j} - \frac{16}{j+1} + \frac{16}{j+2} \qquad \gamma_{\varphi g}^{(0)}(j) = \frac{12}{j+1} - \frac{12}{j+2},$$

$$\gamma_{g\lambda}^{(0)}(j) = \frac{4}{j-1} - \frac{4}{j} + \frac{2}{j+1}, \qquad \gamma_{\lambda\lambda}^{(0)}(j) = \frac{4}{j} - \frac{8}{j+1} - 4S(j-1),$$

$$\gamma_{\varphi\lambda}^{(0)}(j) = \frac{6}{j+1}, \qquad \gamma_{\varphi\varphi}^{(0)}(j) = -4S(j) \qquad \gamma_{g\varphi}^{(0)}(j) = \frac{4}{j-1} - \frac{4}{j},$$

for polarized case

$$\widetilde{\gamma}^{(0)}_{gg}(j) = \frac{8}{j} - \frac{8}{j+1} - 4S(j), \qquad \widetilde{\gamma}^{a,(0)}_{\lambda g}(j) = \frac{16}{j+1} - \frac{8}{j},$$

$$\widetilde{\gamma}^{(0)}_{g\lambda}(j) = \frac{4}{j} - \frac{2}{j+1}, \qquad \widetilde{\gamma}^{(0)}_{\lambda\lambda}(j) = \frac{4}{j+1} - \frac{4}{j} - 4S(j)$$

These matrices can be diagonalized (L.Lipatov, 1999)

$$[D\Gamma D^{-1}]^{N=4}_{\mathbf{unpol}} = \begin{vmatrix} -4S_1(j-2) & 0 & 0 \\ 0 & -4S_1(j) & 0 \\ 0 & 0 & -4S_1(j+2) \end{vmatrix}$$

$$[D\Gamma D^{-1}]^{N=4}_{\mathbf{pol}} = \begin{vmatrix} -4S_1(j-1) & 0 \\ 0 & -4S_1(j+1) \end{vmatrix},$$

The LO AD of all multiplicatively renornalized operators can be extracted through one universal function

$$\gamma^{(0)}_{uni}(j) \;=\; -\,4S(j-2) \equiv -4(\Psi(j-1)-\Psi(1)) \equiv -4\sum_{r=1}^{j-2}\frac{1}{r}$$

B. QCD results $\rightarrow \mathcal{N}=1$ SUSY results by the SUSY relations for the QCD color factors $C_F = C_A = N_c$, $T_f = N_c/2$.
The $\mathcal{N}=4$ SUSY needs contributions from scalars.

The universal AD $\gamma_{uni}(j)$ can be extracted directly from the QCD results without finding the scalar particle contribution (A.K., L.Lipatov, 2002), because

- the relation between DGLAP and BFKL dynamics in the $\mathcal{N}=4$ SYM (Lipatov idea),
- the specific properties of the kernel for the BFKL equation in this model.

C. BFKL equation (A.K., L.Lipatov, 2000)

QCD in $\overline{MS}$ scheme

$$\omega^{QCD}_{\overline{MS}} = 4a_s(q^2)[\chi(n,\gamma) + \delta^{QCD}_{\overline{MS}}(n,\gamma)a_s(q^2)] \,.$$

LO part

$$\chi(n,\gamma) \;=\; 2\Psi(1) - \Psi(\gamma + \frac{n}{2}) - \Psi(1 - \gamma + \frac{n}{2})$$

NLO part

$$
\begin{aligned}
\delta^{QCD}_{\overline{MS}}(n,\gamma) &= -(\frac{11}{3}-\frac{2}{3}\frac{n_f}{N_c})\frac{1}{2}(\chi^2(n,\gamma)-\Psi'(\gamma+\frac{n}{2})+\Psi'(1-\gamma+\frac{n}{2})) \\
&+ (\frac{67}{9}-2\zeta(2)-\frac{10}{9}\frac{n_f}{N_c})\chi(n,\gamma)+6\zeta(3) \\
&+ \frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\{(1+\frac{n_f}{N_c^3})\frac{\gamma(1-\gamma)}{2(3-2\gamma)(1+2\gamma)}\cdot\delta_n^2 \\
&- (3+(1+\frac{n_f}{N_c^3})\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)})\cdot\delta_n^0\} \\
&+ \Psi''(\gamma+\frac{n}{2})+\Psi''(1-\gamma+\frac{n}{2})-2\Phi(n,\gamma)-2\Phi(n,1-\gamma),
\end{aligned}
$$

where δ_n^m is the Kroneker symbol, and $\Psi(z)$, $\Psi'(z)$ and $\Psi''(z)$ are the Euler Ψ -function and its derivatives.

$\mathcal{N}=4$ SUSY in $\overline{DR}$ scheme ($a_s \longrightarrow \hat{a}_s = a_s + \frac{1}{3}a_s^2$)

$$\omega_{\overline{DR}}^{N=4} = 4\hat{a}_s[\chi(n,\gamma) + \delta_{\overline{DR}}^{N=4}(n,\gamma)\hat{a}_s]\,.$$

$$\begin{aligned}\delta_{\overline{DR}}^{N=4}(n,\gamma) &= 6\zeta(3) + \Psi''(\gamma + \frac{n}{2}) + \Psi''(1-\gamma+\frac{n}{2}) \\ &- 2\Phi(n,\gamma) - 2\Phi(n,1-\gamma) - 2\zeta(2)\chi(n,\gamma)\end{aligned}$$

In the $\mathcal{N}=4$ SUSY, the eigenvalues of the BFKL equation are analytic functions of the conformal spin $|n|$.

Transcendentality principe. (A.K., L.Lipatov, 2002)

In the $\overline{\mathrm{DR}}$-scheme, there is no mixing among the functions of different transcendentality levels i, i.e. all special functions at the NLO correction contain only sums of the terms $\sim 1/\gamma^i$ $(i = 3)$.

More precisely, if we introduce the transcendentality level for the eigenvalues of integral kernels of the BFKL equations:

$$\Psi \sim 1/\gamma, \quad \Psi' \sim \beta' \sim \zeta(2) \sim 1/\gamma^2, \quad \Psi'' \sim \beta'' \sim \zeta(3) \sim 1/\gamma^3,$$

then for the BFKL kernel in LO and in NLO the corresponding levels are $i = 1$ and $i = 3$, respectively.

Lipatov idea. In $\mathcal{N}=4$ SYM there is a relation between the BFKL and DGLAP equations: BFKL results reproduce DGLAP ones at the nonphysical values $|n| \to -(1+r), r>0$.

Transcendentality principe. (A.K., L.Lipatov, 2002)
So, $\gamma^{(0)}_{uni}(j)$, $\gamma^{(1)}_{uni}(j)$ and $\gamma^{(2)}_{uni}(j)$ are should contain (in the $\overline{\mathrm{DR}}$-scheme) also only the functions assumed to be of the types $\sim 1/j^i$ with the levels $i=1$, $i=3$ and $i=5$, respectively.

Further, the universal AD $\gamma^{(0)}_{uni}(j)$, $\gamma^{(1)}_{uni}(j)$ and $\gamma^{(2)}_{uni}(j)$ should be equal to a combination of the *most complicated contributions* to the QCD AD in LO, NLO and NNLO (i.e. the functions with a maximal value of the transcendentality levels $i=1$, $i=3$ and $i=5$, respectively) with the SUSY relation for the QCD color factors $C_F = C_A = N_c$.

D. The universal anomalous dimension $\gamma_{uni}(j)$ for $\mathcal{N}=4$ SYM is (based on (A.Moch, J.A.M.L.Vermaseren, A.Vogt, 2004))

$$\gamma(j) \equiv \gamma_{uni}(j) \;=\; \hat{a}\gamma_{uni}^{(0)}(j) + \hat{a}^2\gamma_{uni}^{(1)}(j) + \hat{a}^3\gamma_{uni}^{(2)}(j) + ...,$$

where (A.K., L.Lipatov, A.Onischenko, V.Velizhanin, 2004)

$$\frac{1}{4}\gamma^{(0)}_{uni}(j+2) = -S_1,$$
$$\frac{1}{8}\gamma^{(1)}_{uni}(j+2) = (S_3+\overline{S}_{-3}) - 2\,\overline{S}_{-2,1} + 2\,S_1(S_2+\overline{S}_{-2}),$$
$$\begin{aligned}\frac{1}{32}\gamma^{(2)}_{uni}(j+2) = {} & 2\,\overline{S}_{-3}\,S_2 - S_5 - 2\,\overline{S}_{-2}\,S_3 - 3\,\overline{S}_{-5} + 24\,\overline{S}_{-2,1,1,1} \\ & +6(\overline{S}_{-4,1}+\overline{S}_{-3,2}+\overline{S}_{-2,3}) - 12(\overline{S}_{-3,1,1}+\overline{S}_{-2,1,2}+\overline{S}_{-2,2,1}) \\ & -(S_2+2\,S_1^2)(3\,\overline{S}_{-3}+S_3-2\,\overline{S}_{-2,1}) - S_1(8\,\overline{S}_{-4}+\overline{S}^2_{-2} \\ & +4\,S_2\,\overline{S}_{-2}+2\,S_2^2+3\,S_4-12\,\overline{S}_{-3,1}-10\,\overline{S}_{-2,2}+16\,\overline{S}_{-2,1,1}),\end{aligned}$$

$S_{\pm a} \equiv S_{\pm a}(j)$, $S_{\pm a,b} \equiv S_{\pm a,b}(j)$, $S_{\pm a,b,c} \equiv S_{\pm a,b,c}(j)$ are harmonic sums

$$S_a(j) = \sum_{m=1}^{j}\frac{1}{m^a}, \quad S_{a,b,c,\cdots}(j) = \sum_{m=1}^{j}\frac{1}{m^a}\,S_{b,c,\cdots}(m),$$
$$S_{-a}(j) = \sum_{m=1}^{j}\frac{(-1)^m}{m^a}, \quad S_{-a,b,c,\cdots}(j) = \sum_{m=1}^{j}\frac{(-1)^m}{m^a}\,S_{b,c,\cdots}(m),$$
$$\overline{S}_{-a,b,c,\cdots}(j) = (-1)^j\,S_{-a,b,c,\ldots}(j) + S_{-a,b,c,\cdots}(\infty)\,(1-(-1)^j)$$

DD. Calculations of master integrals (MI) without direct cal-cualtions (J.Fleischer, A.K., O.Veretin, 1998)

$$MI \sim \sum_{n=1} C_n \, x^n \, \{F_0(n) + [\ln x \, F_{1,1}(n) + \frac{1}{\varepsilon} F_{1,2}(n)]$$
$$+[\ln^2 x \, F_{2,1}(n) + \frac{1}{\varepsilon} \ln x \, F_{2,2}(n) + \frac{1}{\varepsilon^2} F_{2,3}(n) + \zeta(2) \, F_{2,4}(n)]$$
$$+ \ldots,$$

where $x = m^2/q^2$ and

$$F_{N,k}(n) \sim \frac{S_{a,\ldots}}{n^b}, \qquad (a + b = M - N),$$

For many MI (there is some criteria) no mixing terms with different $a+b$. So, the transcendentality principe works!!!

Here

$$C_n \sim \frac{1}{n^c}, \quad \text{and} \quad C_n \sim \frac{(2n)!}{(n!)^2}\frac{1}{n^c}$$

with some c, for diagrams with one-massive-particle-cut and two-massive-particle-cut, respectively.

So, it is necessary to calculate some most singular part of MI (or some simpler diagram with "similar topology") and alter it is possible to predict the results for less singular part, where the corresponding $F_{N,k}$ are usually constans.

Using O.Tarasov and V.Smirnov investigations, Oleg Veretin and Misha Kalmykov prepared programs to calculate **analytically !!!** several (usually less then 100) terms in the above expansion. Moreover, every part is calculated independently.

Using the above ansatz and an information about the most singular part of MI (or some simpler diagram with "similar topology") it is possinble usually to reconsruct the complete MI result **without direct calculations**.

Now we will return to the anomalous dimensions.

Cheaks:

The NLO corrections have been proven by exact calculation (A.K., L.Lipatov, V.Velizhanin, 2003).

The NNLO corrections have been recalculated for $j < 70$ from Bethe ansatz (M.Staudacher et al., 2005)

The NNLO corrections have been recalculated for $j \to \infty$ (Z. Bern, L. Dixon, V. Smirnov, 2005)

E. The NNNLO universal anomalous dimension from Bethe Ansatz.

The long-range asymptotic Bethe equations for twist-two operators read (N.Beisert, M.Staudacher, 2005) **(in the section $M = j + 2$)**

$$\left(\frac{x_k^+}{x_k^-}\right)^2 = \prod_{m=1, m\neq k}^{M} \frac{x_k^- - x_m^+}{x_k^+ - x_m^-} \frac{(1 - g^2/x_k^+ x_m^-)}{(1 - g^2/x_k^- x_m^+)} \exp\left(2\, i\, \theta(u_k, u_j)\right),$$
$$\prod_{k=1}^{M} \frac{x_k^+}{x_k^-} = 1\,. \qquad (2)$$

These are M equations for $k = 1, \ldots, M$ Bethe roots u_k, with

$$x_k^\pm = x(u_k^\pm)\,, \qquad u^\pm = u \pm \frac{i}{2}\,, \qquad x(u) = \frac{u}{2}\left(1 + \sqrt{1 - 4\,\frac{g^2}{u^2}}\right), \qquad (3)$$

and where the dressing phase $\theta \sim \zeta(3)$ is a rather intricate function conjectured in (N.Beisert, B.Eden, M.Staudacher, 2006)

Once the M Bethe roots are determined from above equations for the state of interest, its asymptotic all-loop anomalous dimension is given by

$$\gamma^{ABA}(g) = 2\, g^2 \sum_{k=1}^{M} \left(\frac{i}{x_k^+} - \frac{i}{x_k^-} \right). \tag{4}$$

The above equations can be solved recursively order by order in g at arbitrary values of M once the one-loop solution for a given state is known.

The result for the four-loop asymptotic dimension: (A.K., L.Lipatov, A.Rey, M.Staudacher, V.Velizhanin, 2007)

$$
\begin{aligned}
& \frac{1}{256}\,\gamma_{uni}^{ABA}(j+2) \ = \\
& \boldsymbol{4\,S_{-7} + 6\,S_{7}} + 2\,(S_{-3,1,3} + S_{-3,2,2} + S_{-3,3,1} + S_{-2,4,1}) + 3\,(-S_{-2,5} \\
+ \;& S_{-2,3,-2}) + 4\,(S_{-2,1,4} + -S_{-2,-2,-2,1} - S_{-2,1,2,-2} - S_{-2,2,1,-2} - S_{1,-2,1,3} \\
- \;& S_{1,-2,2,2} - S_{1,-2,3,1}) + 5\,(-S_{-3,4} + S_{-2,-2,-3}) + 6\,(-S_{5,-2} \\
+ \;& S_{1,-2,4} - S_{-2,-2,1,-2} - S_{1,-2,-2,-2}) + 7\,(-S_{-2,-5} + S_{-3,-2,-2} \\
+ \;& S_{-2,-3,-2} + S_{-2,-2,3}) + 8\,(S_{-4,1,2} + S_{-4,2,1} - S_{-5,-2} - S_{-4,3} \\
- \;& S_{-2,1,-2,-2} + S_{1,-2,1,1,-2}) + 9\,S_{3,-2,-2} - 10\,S_{1,-2,2,-2} + 11\,S_{-3,2,-2} \\
+ \;& 12\,(-S_{-6,1} + S_{-2,2,-3} + S_{1,4,-2} + S_{4,-2,1} + S_{4,1,-2} - S_{-3,1,1,-2} - S_{-2,2,-2,1} \\
- \;& S_{1,1,2,3} - S_{1,1,3,-2} - S_{1,1,3,2} - S_{1,2,1,3} - S_{1,2,2,-2} - S_{1,2,2,2} - S_{1,2,3,1} - S_{1,3,1,-2} \\
- \;& S_{1,3,1,2} - S_{1,3,2,1} - S_{2,-2,1,2} - S_{2,-2,2,1} - S_{2,1,1,3} - S_{2,1,2,-2} - S_{2,1,2,2} \\
- \;& S_{2,1,3,1} - S_{2,2,1,-2} - S_{2,2,1,2} - S_{2,2,2,1} - S_{2,3,1,1} - S_{3,1,1,-2} - S_{3,1,1,2} - S_{3,1,2,1} \\
- \;& S_{3,2,1,1}) + 13\,S_{2,-2,3} - 14\,S_{2,-2,1,-2} + 15\,(S_{2,3,-2} + S_{3,2,-2}) \\
+ \;& 16\,(S_{-4,1,-2} + S_{-2,1,-4} - S_{-2,-2,1,2} - S_{-2,-2,2,1} - S_{-2,1,-2,2} - S_{-2,1,1,-3} \\
- \;& S_{1,-3,1,2} - S_{1,-3,2,1} - S_{1,-2,-2,2} - S_{2,-2,-2,1} + S_{-2,1,1,-2,1} + S_{1,1,-2,1,-2} \\
+ \;& S_{1,1,-2,1,2} + S_{1,1,-2,2,1}) - 17\,S_{-5,2} + 18\,(-S_{4,-3} - S_{6,1} + S_{1,-3,3}) \\
+ \;& 20\,(-S_{1,-6} - S_{1,6} - S_{4,3} + S_{-5,1,1} + S_{-4,-2,1} + S_{-3,-2,2} + S_{-2,-4,1} \\
+ \;& S_{-2,-3,2} + S_{1,3,3} + S_{3,1,3} + S_{3,3,1} - S_{1,1,-2,3} - S_{1,2,-2,-2} - S_{2,1,-2,-2}) \\
- \;& 21\,S_{3,4} + 22\,(S_{1,-2,-4} + S_{2,2,3} + S_{2,3,2} + S_{3,-2,2} + S_{3,2,2}) + 23\,(-S_{-3,-4} \\
- \;& S_{5,2} + S_{2,-2,-3}) + 24\,(-S_{-4,-3} + S_{1,-4,-2} - S_{1,-3,1,-2} - S_{1,1,1,4} - S_{1,1,4,1} \\
- \;& S_{1,3,-2,1} - S_{1,4,1,1} - S_{3,-2,1,1} - S_{3,1,-2,1} - S_{4,1,1,1} + S_{-2,-2,1,1,1} + S_{-2,1,-2,1,1} \\
+ \;& S_{1,-2,-2,1,1} + S_{1,-2,1,-2,1} + S_{1,1,-2,-2,1} + S_{1,1,1,-2,-2} + S_{1,1,2,-2,1} + S_{1,2,1,-2,1} \\
+ \;& S_{2,1,1,-2,1}) + 25\,S_{2,-3,-2} + 26\,(-S_{2,5} + S_{1,4,2} + S_{2,4,1} + S_{4,1,2} + S_{4,2,1}) \\
+ \;& 28\,(S_{1,2,4} + S_{2,1,4} - S_{-3,1,-2,1} - S_{-2,1,-3,1} - S_{1,-2,1,-3}) + 30\,S_{-3,1,-3} \\
+ \;& 32\,(S_{1,5,1} + S_{5,1,1} - S_{-3,-2,1,1} - S_{-2,-3,1,1} - S_{1,-3,-2,1} - S_{1,-2,-3,1}
\end{aligned}
$$

$$
\begin{aligned}
&- \; S_{2,2,-2,1} + S_{1,2,-2,1,1} + S_{2,1,-2,1,1} - S_{1,1,1,-2,1,1}) + 36\,(S_{1,1,5} + S_{1,3,-3} \\
&+ \; S_{3,1,-3} - S_{1,1,-3,-2} - S_{1,1,-2,-3} - S_{1,1,2,-3} - S_{1,2,-2,2} - S_{1,2,1,-3} - S_{2,1,-2,2} \\
&- \; S_{2,1,1,-3}) + 38\,S_{-3,-3,1} + 40\,(-S_{1,-4,1,1} - S_{2,-3,1,1} + S_{1,1,1,-2,2}) \\
&- \; 41\,S_{3,-4} + 42\,(-S_{2,-5} + S_{1,-4,2} + S_{1,-3,-3}) + 44\,(S_{1,-5,1} + S_{2,-3,2} + S_{3,-3,1}) \\
&+ \; 46\,S_{2,2,-3} + 48\,S_{1,1,-3,1,1} + 60\,(S_{1,1,-5} - S_{1,1,-3,2}) + 62\,S_{2,-4,1} + 64\,S_{1,1,1,-3,1} \\
&+ \; 68\,(S_{1,2,-4} + S_{2,1,-4} - S_{1,2,-3,1} - S_{2,1,-3,1}) - 72\,S_{1,1,1,-4} - 80\,S_{1,1,-4,1} \\
&- \; \boldsymbol{\zeta(3)\mathbf{S}_1(\mathbf{S}_3 - \mathbf{S}_{-3} + 2\,\mathbf{S}_{-2,1})}.
\end{aligned}
$$

BFKL and double-logarithmis constrains:

the above result is wrong.

Wrapping effects??

Including the wrapping effects:

(Z. Bajnok, R. Janik, T. Lukowski, Nov. 2008)

$$\gamma_{uni}(j+2) \;=\; \gamma_{uni}^{ABA}(j+2) + \gamma_{uni}^{wr}(j+2),$$

$$\frac{1}{256}\,\gamma_{uni}^{wr}(j+2) \;=$$

$$\frac{1}{2}\,\boldsymbol{S_1^2}\,[\boldsymbol{2\,S_{-5} + 2\,S_5}$$
$$+4\,\left(S_{4,1} - S_{3,-2} + S_{-2,-3} - 2\,S_{-2,-2,1}\right)$$
$$-4\,S_{-2}\zeta(3) - 5\,\zeta(5)]$$

F. The limit $j \to 1$.

The small-x behavior of parton distributions: there are new experimental data at small x produced by the H1 and ZEUS collaborations in HERA.

Using $j = 1 + \omega \to 1$

$$
\begin{aligned}
\gamma_{uni}^{(0)}(1+\omega) &= \frac{4}{\omega} + O(\omega^1), \\
\gamma_{uni}^{(1)}(1+\omega) &= -32\,\zeta_3 + O(\omega^1), \\
\gamma_{uni}^{(2)}(1+\omega) &= 32\zeta_3 \frac{1}{\omega^2} - 232\,\zeta_4 \frac{1}{\omega} + O(\omega^0)
\end{aligned}
$$

Full agreement with the predictions for $\gamma_{uni}^{(0)}(1+\omega)$, $\gamma_{uni}^{(1)}(1+\omega)$ and also for the first term of $\gamma_{uni}^{(2)}(1+\omega)$ coming from an investigation of BFKL equation at NLO accuracy.

G. $j \to \infty$ limit and the AdS/CFT correspondence

The great interest to this limit comes from the recent formulation of the AdS/CFT correspondence: the strong-coupling limit $\alpha_s N_c \to \infty$ is described by a classical supergravity in the anti-de Sitter space $AdS_5 \times S^5$:

$$\gamma(j) = \tilde{a}(z) \, \ln j + O(j^0) \,, \qquad z = \frac{\alpha N_c}{\pi} = 4\hat{a}$$

in the strong coupling regime : (S.Gupser, I.Klebanov, A.Polyakov, 2002; S. Frolov, Theytlin, 2002)

$$\lim_{z \to \infty} \tilde{a} = -z^{1/2} + \frac{3 \ln 2}{8\pi} + \mathcal{O}\left(z^{-1/2}\right) . \qquad (5)$$

In our studies, all anomalous dimensions $\gamma_i(j)$ and $\tilde{\gamma}_i(j)$ ($i = +, 0, -$) coincide at large j and our results for $\gamma_{uni}(j)$

$$\lim_{z \to 0} \tilde{a} = -z + \frac{\pi^2}{12} z^2 - \frac{11}{720} \pi^4 z^3 + \dots .$$

G1. Before calculation of the NNLO corrections, we suggested for $\tilde{a}$ (A.K., L.Lipatov, V.Velizhanin, 2003)

$$z = -\tilde{a} + \frac{\pi^2}{12}\,\tilde{a}^2 \qquad (6)$$

interpolating between its weak-coupling expansion up to NNLO

$$\tilde{a} = -z + \frac{\pi^2}{12}\,z^2 - \frac{1}{72}\pi^4 z^3 + O(z^4)$$

and strong-coupling asymptotics

$$\tilde{a} \approx -1.1026\; z^{1/2} + \mathcal{O}\left(1\right).$$

The prediction for NNLO based on the above simple equation is valid with the accuracy $\sim 10\%$. It means, that this extrapolation seems to be good for all values of z.

G2. Recently (N.Beisert, B.Eden, M.Staudacher, 2006):

- Integral equation for some function $f(x)$ ($f(0) = \tilde{a}$).
- Many coefficients in the expansion

$$f(0) \; = \sum_{m=0} c_m \; z^m$$

- Full agreement with our results for first foour terms and (for any terms) with transcendentality principe: $c_m \sim \zeta(2m)$ for $m > 0$ (or products of ζ-function with the sum of indices equal to $2m$), for all coefficients.
- Study of the $z \to \infty$ limit: can of Beisert-Eden-Staudacher equation reproduce Polyakov et al. asymptotics?

  Yes. It was shown numerically in (M.K.Benna, S.Benvenuti, I.R.Klebanov, A.Scardicchio, 2006) and analytically in (A.K., L.N.Lipatov, 2006).

- There were calculated several coefficients in the expansion

$$f(0) = \sum_{m=0} \tilde{c}_m \, z^{(1-m)/2}$$

c_0 in (S.Gupser, I.Klebanov, A.Polyakov, 2002)

c_1 in (S.Frolov, A.A.Theytlin, 2002)

c_1 in (R.Roiban, A.A.Theytlin, 2007), (A.Tirziu, A.A.Theytlin, 2008)

Very recently (B.Basso, G.P.Korchemsky, J.Kotanski, 2007) have found a method to evaluate the $\tilde{c}_m$ coefiicients from BES-equation and calculated several of them. The first three coefficients are in agreement with the results of above calculations. Moreover, the results are in agreement with transcendentality principe: $\tilde{c}_1 \sim \ln 2$ and $\tilde{c}_m \sim \zeta(m)$ for $m > 1$ (or products of ζ-function with the sum of indices equal to m), for all coefficients.

Conclusion

- I have demonstrated a way to find the universal AD $\gamma_{uni}(j)$ for the $\mathcal{N} = 4$ SUSY without a direct calculations (but based on calculations in QCD and on transcendentality principe).
- Bethe Ansatz and evaluation of the NNNLO universal anomalous dimension using transcendentality principe.
- There is a self-consistency in the Regge ($j \to 1$) and quasi-elastic ($j \to \infty$) regimes.

Next steps:

- Phenomenological applications ?
- 5-loop of $\gamma_{uni}(j)$. Calculations are finished by two groups (R.Janik et al.) and the results will be published soon.
  The partial case $(j = 4)$ (i.e. the 5-loop of the anomalous dimension of the so-called Konishi operator) has been publised very recently in (Z. Bajnok, A.Hegedus, R. Janik, T. Lukowski, Jun. 2009)
- All-loop of $\gamma_{uni}(j)$ at $j \to \infty$?