SUSY: From the Basics to Phenomenology

Sven Heinemeyer, IFCA (CSIC, Santander)

Dubna, 07/2009

1. SUSY Lagrangian and algebra

2. The MSSM and simplified versions

3. The Higgs sector of the MSSM

4. SUSY at the LHC and the ILC

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Literature:

1. Drees, Godbole, Roy: "Theory and Phenomenology of Sparticles"

2. Martin: "A Supersymmetry Primer"
   `arXiv.org/abs/hep-ph/9709356`

3. Signer: "ABC of SUSY"
   `arXiv.org/abs/0905.4630`

4. . . .

SUSY lectures (I): SUSY Lagrangian and algebra

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1. Motivation for SUSY

2. Symmetry and Algebra

3. Superfields and Superspace

4. Supersymmetric Lagrangians

1. Motivation for SUSY

The Standard Model (SM) cannot be the ultimate theory

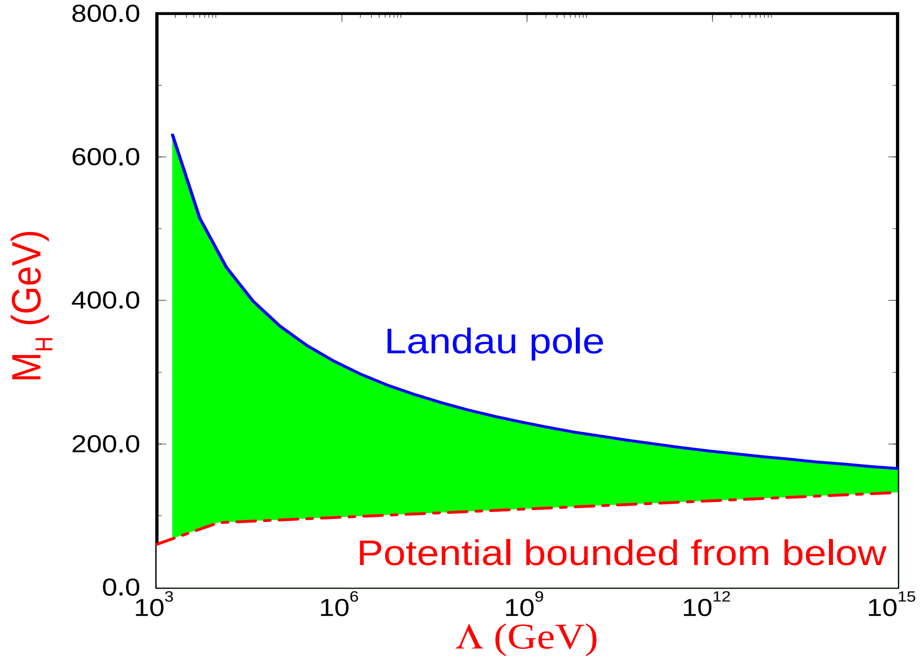
– The SM does not contain gravity

– Further problems: Hierarchy problem

Up to which energy scale Λ can the SM be valid?

– $\Lambda < M_{\mathrm{Pl}}$: inclusion of gravity effects necessary

– stability of Higgs potential: $\Rightarrow$

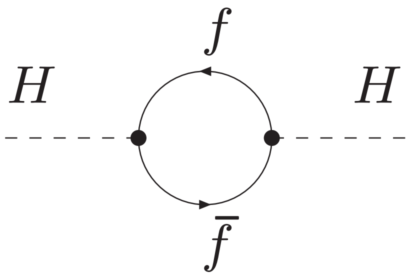


– Hierarchy problem :
Higgs mass unstable
w.r.t. quantum corrections
$\delta M_H^2 \sim \Lambda^2$
(but what does this mean?)

Mass is what determines the properties of the free propagation of a particle

Free propagation: 

inverse propagator: $i(p^2 - M_H^2)$

Loop corrections: 

inverse propagator: $i(p^2 - M_H^2 + \Sigma_H^f)$

QM: integration over all possible loop momenta k

dimensional analysis:

$$\Sigma_H^f \sim N_f \, \lambda_f^2 \int d^4k \left(\frac{1}{k^2 - m_f^2} + \frac{2m_f^2}{(k^2 - m_f^2)^2} \right)$$

$$\text{for } \Lambda \to \infty : \quad \Sigma_H^f \sim N_f \, \lambda_f^2 \left(\underbrace{\int \frac{d^4k}{k^2}}_{\sim \Lambda^2} + 2m_f^2 \underbrace{\int \frac{dk}{k}}_{\sim \ln \Lambda} \right)$$

$\Rightarrow$ quadratically divergent!

For $\Lambda = M_{\text{Pl}}$:

$$\Sigma_H^f \approx \delta M_H^2 \sim M_{\text{Pl}}^2 \quad \Rightarrow \quad \delta M_H^2 \approx 10^{30}\, M_H^2$$

(for $M_H \lesssim 1$ TeV)

– no additional symmetry for $M_H = 0$

– no protection against large corrections

$\Rightarrow$ Hierarchy problem is instability of small Higgs mass to large corrections in a theory with a large mass scale in addition to the weak scale

E.g.: Grand Unified Theory (GUT): $\delta M_H^2 \approx M_{\text{GUT}}^2$

Note however: there is another fine-tuning problem in nature, for which we have no clue so far – cosmological constant

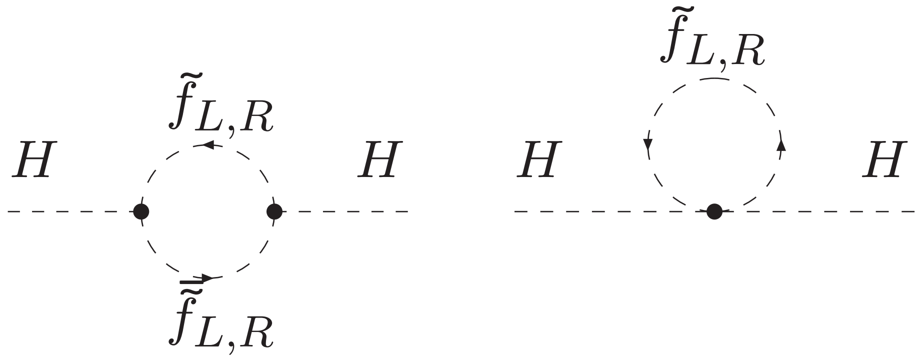
Supersymmetry:

Symmetry between fermions and bosons

$$Q|\text{boson}\rangle = |\text{fermion}\rangle$$
$$Q|\text{fermion}\rangle = |\text{boson}\rangle$$

Effectively: SM particles have SUSY partners (e.g. $f_{L,R} \to \tilde{f}_{L,R}$)

SUSY: additional contributions from scalar fields:



$$\Sigma_H^{\tilde{f}} \sim N_{\tilde{f}}\, \lambda_{\tilde{f}}^2 \int d^4k \left(\frac{1}{k^2 - m_{\tilde{f}_L}^2} + \frac{1}{k^2 - m_{\tilde{f}_R}^2} \right) + \text{ terms without quadratic div.}$$

for $\Lambda \to \infty$: $\Sigma_H^{\tilde{f}} \sim N_{\tilde{f}}\, \lambda_{\tilde{f}}^2\, \Lambda^2$

⇒ quadratic divergences cancel for

$$N_{\tilde{f}_L} = N_{\tilde{f}_R} = N_f$$
$$\lambda_{\tilde{f}}^2 = \lambda_f^2$$

complete correction vanishes if furthermore

$$m_{\tilde{f}} = m_f$$

Soft SUSY breaking: $m_{\tilde{f}}^2 = m_f^2 + \Delta^2, \quad \lambda_{\tilde{f}}^2 = \lambda_f^2$

$$\Rightarrow \Sigma_H^{f+\tilde{f}} \sim N_f \, \lambda_f^2 \, \Delta^2 + \ldots$$

⇒ correction stays acceptably small if mass splitting is of weak scale

⇒ realized if mass scale of SUSY partners

$$M_{\mathrm{SUSY}} \lesssim 1\,\mathrm{TeV}$$

⇒ SUSY at TeV scale provides attractive solution of hierarchy problem

2. Symmetry and Algebra

Symmetry: a group of transformations that leaves the Lagrangian invariant

Generators of the group fulfill certain algebra

Examples:

0. Angular rotation: $\Psi \to \Psi e^{i\theta^a L_a}$
   theory is invariant under rotation
   generators: L_a, algebra: $[L_a, L_b] = i\varepsilon_{abc}L^c$
   Quantum numbers: (max. spin)^2, spin $[l(l+1), m = +l \ldots -l]$

1. Internal symmetry: $SU(3) \times SU(2) \times U(1)$
   gauge symmetry for the description of the strong and electroweak force
   generators: T_a, algebra: $[T_a, T_b] = if_{abc}T^c$
   Quantum numbers: color, weak isospin, hyper charge

2. Poincaré symmetry (includes rotation)
   space–time symmetries:
   Lorentz transformations: $\Lambda^{\mu\nu}$, translations: P^ρ
   Quantum numbers: mass, spin

Lorentz group: Representations of Lorentz group are labeled by two 'spins', $j_1, j_2 = 0$, where $j_1, j_2 = 0, \frac{1}{2}, 1, \ldots$

Basic representations $M_\alpha{}^\beta$ act on:

$(\frac{1}{2}, 0)$: LEFT-handed 2-component Weyl spinor, ψ_α

$(0, \frac{1}{2})$: RIGHT-handed 2-component Weyl spinor, $\bar{\psi}^{\dot{\alpha}}$

The two component Weyl spinors ψ_α (left-handed) and $\bar{\psi}^{\dot{\alpha}}$ (right-handed) transform under Lorentz transformations as follows:

$$\begin{aligned} \psi'_\alpha &= M_\alpha{}^\beta \psi_\beta; & \bar{\psi}'_{\dot{\alpha}} &= (M^*)_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}} \\ \psi'^\alpha &= (M^{-1})_\beta{}^\alpha \psi^\beta; & \bar{\psi}'^{\dot{\alpha}} &= (M^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}} \bar{\psi}^{\dot{\beta}} \end{aligned}$$

where $M = \exp(i\frac{\vec{\sigma}}{2}(\vec{\vartheta} - i\vec{\varphi}))$ and $\vec{\vartheta}$ and $\vec{\varphi}$ are the three rotation angles and boost parameters, respectively

$\Rightarrow$ spinors with undotted indices (first two components of Dirac spinor) transform according to $(\frac{1}{2}, 0)$-representation of Lorentz group,
spinors with dotted indices (last two components of Dirac spinor) transform according to $(0, \frac{1}{2})$-representation

Our world (the SM) is described by:
- internal symmetry: T_a
- Poincaré symmetry: $\Lambda^{\mu\nu}$, P^ρ

internal symmetry is a trivial extension of the Poincaré symmetry:

$$[\Lambda^{\mu\nu}, T^a] = 0, \qquad [P^\rho, T^a] = 0$$

$\Rightarrow$ direct product: (Poincaré group) $\otimes$ (internal symmetry group)

Particle states characterized by maximal set of commuting observables:

$$|\underbrace{m, s; \vec{p}, s_3}_{\text{space–time}}; \underbrace{Q, I, I_3, Y, \ldots}_{\text{internal}}\rangle$$

quantum numbers

Wanted: extension of the SM

Theorem # 1: No-go theorem [*Coleman, Mandula '67*]

Any Lie-group containing Poincaré group P and
internal symmetry group $\tilde{G}$ must be direct product $P \otimes \tilde{G}$

$$|\underbrace{m, s; \vec{p}, s_3}_{\text{space–time}}; \underbrace{\tilde{g}, \ldots}_{\text{internal}}\rangle$$

quantum numbers

New group $\tilde{G}$ with generators Q^α and

$$[\Lambda^{\mu\nu}, Q^\alpha] \neq 0, \quad [P^\rho, Q^\alpha] \neq 0$$

impossible

Direct product $\Rightarrow$ no irreducible multiplets can contain particles with different mass or different spin

$\Rightarrow$ new symmetry must predict new particles with the same mass and spin as in the SM

$\Rightarrow$ experimentally excluded, no such symmetry possible :-(

Theorem # 2: How-To-Avoid-the-No-go theorem

[*Gol'fand, Likhtman '71*] [*Volkov, Akulov '72*] [*Wess, Zumino '73*]

No go theorem can be evaded if instead of Lie-group (generators fulfill commutator relations):

$$[\ldots,\ldots] \rightarrow \{\ldots,\ldots\}$$

Anticommutator: $\{A, B\} = AB + BA$

$\Rightarrow$ Generator Q^α is fermionic (i.e. it has spin $\frac{1}{2}$)

$\Rightarrow$ Particles with different spin in one multiplet possible

$$Q|\text{boson}\rangle = |\text{fermion}\rangle, \quad Q|\text{fermion}\rangle = |\text{boson}\rangle$$

Q changes spin (behavior under spatial rotations) by $\frac{1}{2}$

E.g.:

spin 2	$\rightarrow$	spin $\frac{3}{2}$	$\rightarrow$	spin 1
graviton		gravitino		photon

Simplest case: only one fermionic generator Q_α (and conjugate $\bar{Q}_{\dot\beta}$)

$\Rightarrow$ $N = 1$ SUSY algebra:

$$
\begin{aligned}
[Q_\alpha, P_\mu] = \left[\bar{Q}_{\dot\beta}, P_\mu\right] &= 0 \\
[Q_\alpha, M^{\mu\nu}] &= i(\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta \\
\left\{Q_\alpha, Q_\beta\right\} = \left\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\right\} &= 0 \\
\left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\} &= 2(\sigma^\mu)_{\alpha\dot\alpha} P_\mu
\end{aligned}
$$

Energy $= H = P_0$, $\Rightarrow$ $[Q_\alpha, H] = 0$ $\Rightarrow$ conserved charge

$\Rightarrow$ SUSY: symmetry that relates bosons to fermions

unique extension of Poincaré group of symmetries of $D = 4$ relativistic QFT

$\rightarrow$ A simple quantum mechanical SUSY example

A QM SUSY example

Harmonic oscillator ($\hbar = c = \omega = \ldots = 1$)

$[q, p] = i$

$a = \frac{1}{\sqrt{2}}(q + ip)$, $a^+ = \frac{1}{\sqrt{2}}(q - ip) \Rightarrow [a, a^+] = 1$

Eigenstates: $|n\rangle$: $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^+|n\rangle = \sqrt{n+1}|n+1\rangle$

Counting operator: $N_B = a^+ a$, $N_B|n\rangle = a^+ a|n\rangle = \sqrt{n} a^+|n-1\rangle = n|n\rangle$

Hamilton: $H_B = \frac{1}{2}(p^2 + q^2) = \ldots = N_B + \frac{1}{2} \Rightarrow H_B|n\rangle = (n + \frac{1}{2})|n\rangle$

Now add a two state system analogous to $|\vec{S}^2, S_z\rangle$ for spin $\frac{1}{2}$

states: $|\frac{1}{2}, +\frac{1}{2}\rangle =: |+\rangle$, $|\frac{1}{2}, -\frac{1}{2}\rangle =: |-\rangle$

operators: S_x, S_y, S_z : closed Lie algebra: $[S_i, S_y] = i\varepsilon_{ijk} S_k$

$S_\pm := S_x \pm i S_y$, $d^+ := S_+$, $d := S_-$

matrix representation: $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$S_x = \frac{1}{2}\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_y = \frac{1}{2}\sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $S_z = \frac{1}{2}\sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow (d^+)^2 = d^2 = 0, \; [d^+, d] = 2S_z$

$\Rightarrow$ commutator $[d^+, d]$ leaves the algebra of d, d^+

But what happens with the anticommutator?

$\{d^+, d\} = \ldots = 1,\ \{d, d\} = \{d^+, d^+\} = 0$
$\Rightarrow$ anticommutators form a closed algebra

Counting operator: $N_F = d^+ d$

Hamilton: $H_F = S_z = \ldots = N_F - \frac{1}{2}$

$d^+|-\rangle = \ldots = |+\rangle,\ d^+|+\rangle = \ldots = 0,$

$d|-\rangle = \ldots = 0,\ d|+\rangle = \ldots = |-\rangle$

$N_F|+\rangle = \ldots = |+\rangle,\ H_F|+\rangle = \frac{1}{2}|+\rangle$: fermion

$N_F|-\rangle = d^+ d|-\rangle = 0$: vacuum

Coupling of the two systems:

$$H := H_B + H_F = N_B + N_F = a^+ a + d^+ d$$

$$|n,+\rangle := |n\rangle \otimes |+\rangle, \; |n,-\rangle := |n\rangle \otimes |-\rangle$$

Spectrum: $H|n,+\rangle = (a^+ a + d^+ d)(|n\rangle \otimes |+\rangle) = (n+1)|n,+\rangle$

$$H|n,-\rangle = (n+0)|n,-\rangle$$

lowest state: $|0,-\rangle$ with $E = 0$, not degenerate

all other states are two-fold degenerate:

$$\begin{array}{l} E = 0 : |0,-\rangle \\ E = 1 : |1,-\rangle, |0,+\rangle \quad \leftarrow \text{multiplet} \\ E = 2 : |2,-\rangle, |1,+\rangle \\ E = 3 : |3,-\rangle, |2,+\rangle \\ \;\vdots \quad \vdots \quad \vdots \quad , \quad \vdots \end{array}$$

Operator that acts within one multiplet
(i.e. that leaves the energy unchanged?

$Q|n,+\rangle \to |n+1,-\rangle$, $Q^+|n+1,-\rangle \to |n,+\rangle$

$\Rightarrow Q = a^+ d \cdot c$, $Q^+ = a d^+ \cdot c^*$

Normalization: $\ldots c = c^* = \frac{1}{\sqrt{2}}$

Q, Q^+ leave energy unchanged $\Rightarrow [H,Q] = [H,Q^+] = 0$

$Q|\text{vac}\rangle = \ldots = 0$, $Q^+|\text{vac}\rangle = \ldots = 0$

$[N_F, Q] = \ldots = -Q$, $[N_F, Q^+] = \ldots = +Q^+$,
$[N_B, Q] = \ldots = +Q$, $[N_B, Q^+] = \ldots = -Q+$

$\{Q, Q^+\} = \ldots = \frac{1}{2}H$

$\Rightarrow$ energy expectation value:

$$\begin{aligned} \langle n, \pm | H | n, \pm \rangle \sim \langle n, \pm | \{Q, Q^+\} | n, \pm \rangle \\ = (\langle n, \pm | Q) \left(Q^+ | n, \pm \rangle \right) + \left(\langle n, \pm | Q^+ \right) (Q | n, \pm \rangle) \\ = (\ldots) + (\ldots)^+ \end{aligned}$$

$\Rightarrow$ positive definite

$\{Q, Q\} = 2Q^2 \sim d^2 = 0$

$\{Q^+, Q^+\} = 2(Q^+)^2 \sim (d^+)^2 = 0$

SUSY algebra of the HO:

$$\{Q, Q^+\} = \frac{1}{2}H$$
$$\{Q, Q\} = \{Q^+, Q^+\} = 0$$
$$[H, Q] = [H, Q^+] = 0$$

General structure:
$\{F, F\} = B,\ [B, B] = B,\ [B, F] = F$

⇒ Super-Lie / graded Lie algebra
contains commutators and anticommutators

⇒ Coleman-Mandula theorem does not apply :-)

→ end of the simple QM example

Can SUSY be an exact symmetry?

Consider fermionic state $|f\rangle$ with mass m
$\Rightarrow$ there is a bosonic state $|b\rangle = Q_\alpha |f\rangle$

$$P^2 |f\rangle = m^2 |f\rangle$$

$$\Rightarrow \quad P^2 |b\rangle = P^2 Q_\alpha |f\rangle = Q_\alpha P^2 |f\rangle = Q_\alpha m^2 |f\rangle = m^2 |b\rangle$$

$\Rightarrow$ for each fermionic state there is a bosonic state with the same mass

$\Rightarrow$ states are paired bosonic $\leftrightarrow$ fermionic

$\Rightarrow$ still experimentally excluded

$\Rightarrow$ SUSY must be broken

Further consequences of the SUSY algebra

$$\left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\} = 2\sigma^\mu_{\ \alpha\dot\beta} P_\mu$$

$$\Rightarrow \left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\} \bar\sigma_\nu^{\dot\beta\alpha} = 2 \underbrace{\sigma^\mu_{\ \alpha\dot\beta} \bar\sigma_\nu^{\dot\beta\alpha}}_{2g^\mu_{\ \nu}} P_\mu = 4P_\nu$$

$$\nu = 0 \Rightarrow H = P_0 = \frac{1}{4}\left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\} \bar\sigma_0^{\dot\beta\alpha} = \frac{1}{4}\left(\left\{Q_1, Q_1^\dagger\right\} + \left\{Q_2, Q_2^\dagger\right\}\right)$$

where $\bar{Q}_{\dot\alpha} = (Q_\alpha)^\dagger$

$\left\{Q_i, Q_i^\dagger\right\} = Q_i Q_i^\dagger + Q_i^\dagger Q_i$: hermitian operator, eigenvalues ≥ 0

$\Rightarrow$ for any state $|\alpha\rangle$: $\langle\alpha|H|\alpha\rangle \geq 0$

spectrum of H is bounded from below, ≥ 0

$\Rightarrow$ no negative eigenvalues

State with lowest energy: vacuum state $|0\rangle$

if vacuum state is symmetric, i.e. $Q|0\rangle = 0$, $Q^\dagger|0\rangle = 0$ for all Q

$\Rightarrow$ vacuum has zero energy, $\langle 0|H|0\rangle = E_{\text{vac}} = 0$

For spontaneous symmetry breaking: vacuum state is not invariant

$\Rightarrow$ If (global) SUSY is spontaneously broken, i.e. $Q_\alpha|0\rangle \neq 0$,
then $\langle 0|H|0\rangle = E_{\text{vac}} > 0$

$\Rightarrow$ non-vanishing vacuum energy

Further consequences for SUSY multiplets:

$\left\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\right\} = 0 \Rightarrow \bar{Q}_{\dot{\alpha}}^2 = 0$ (and $Q_{\alpha}^2 = 0$)

Consider multiplet, start with state of lowest helicity λ_0

application of $\bar{Q}_{\dot{\alpha}}$ $\Rightarrow$ one additional state with helicity $\lambda_0 + \frac{1}{2}$

further application of $\bar{Q}_{\dot{\alpha}}$ $\Rightarrow$ 0, no further state

$\Rightarrow$ one fermionic + one bosonic state

(N SUSY generators $\Rightarrow 2^{N-1}$ bosonic and 2^{N-1} fermionic states)

SUSY multiplet contains equal number of bosonic and fermionic state

Most relevant multiplets (possess also CPT conjugate 'mirrors'):

- chiral supermultiplet: $-\frac{1}{2}, 0$

  Weyl fermion (quark, lepton, ...) + complex scalar (squark, slepton)

- vector supermultiplet: $-1, -\frac{1}{2}$

  Gauge boson (massless vector) + Weyl fermion (gaugino)

- graviton supermultiplet: $-2, -\frac{3}{2}$

  graviton + gravitino

3. Superfields and Superspace

Translation transformation: P_μ , parameter: x^μ

SUSY transformation: $Q_\alpha, \bar{Q}_{\dot{\alpha}}$, parameter: $\theta, \bar{\theta}$ $\rightarrow$ anticommuting c-numbers

("Grassmann variables")

$\Rightarrow$ Extension of 4-dim. space–time by coordinates θ^α, $\bar{\theta}^{\dot{\alpha}}$: superspace

Point in superspace: $X = (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, Superfield: $\phi(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$

Grassmann variables:

SUSY transformation: $Q_\alpha, \bar{Q}_{\dot\alpha}$, parameter: $\theta, \bar\theta \to$ anticommuting c-numbers ("Grassmann variables")

Without spinor index:

$$\{\theta, \theta\} = 0 \;\Rightarrow\; \theta\theta = 0$$

With two-component spinor index (as it is our case):

$$\theta\theta \equiv \theta^\alpha \theta_\alpha = \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta \;\Rightarrow\; \theta\theta \neq 0, \; (\theta^1\theta_2 \neq 0)$$

Taylor expansion in Grassmann variable: $\theta^\alpha\theta^\beta\theta^\gamma = 0$ $(\alpha, \beta, \gamma = 1, 2)$

$\Rightarrow$ Taylor expansion terminates after second term, i.e. $\phi(\theta) = a + \theta\psi + \theta\theta\, f$

Integration: $\int d\theta = 0$, $\int d\theta\, \theta = 1$

$d^2\theta = -\frac{1}{4}\epsilon_{\alpha\beta} d\theta^\alpha d\theta^\beta$

$$\Rightarrow \quad \int d^2\theta\, \phi(\theta) = \int d^2\theta (a + \theta\psi + \theta\theta\, f) = f$$

SUSY transformations: (Lagrangian should be invariant!)

Group element of finite SUSY transformation:

$$S(y,\xi,\bar\xi) = \exp i\left(\xi Q + \bar\xi\bar Q - y^\mu P_\mu\right)$$

in analogy to group elements for Lie-groups

ξ, $\bar\xi$ are independent of y^μ: global SUSY transformation

Transformation of superfield: $S(y,\xi,\bar\xi)\phi(x,\theta,\bar\theta)$

$$\Rightarrow \quad S(y,\xi,\bar\xi)\phi(x,\theta,\bar\theta) = \phi(x^\mu + y^\mu - i\xi\sigma^\mu\bar\theta + i\theta\sigma^\mu\bar\xi, \xi+\theta, \bar\xi+\bar\theta)$$

representations of generators obtained from infinitesimal transformation of superfield

$$\Rightarrow \quad P_\mu = i\partial_\mu, \quad Q_\alpha = -i\,\partial_\alpha + (\sigma^\mu\bar\theta)_\alpha\partial_\mu, \quad \bar Q_{\dot\alpha} = i\,\partial_{\dot\alpha} - (\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$$

with $\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}$, $\bar\partial_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}$

Definition of covariant derivatives:

$$D_\alpha = -i\,\partial_\alpha - (\sigma^\mu\bar\theta)_\alpha\partial_\mu, \quad \bar D_{\dot\alpha} = i\,\bar\partial_{\dot\alpha} + (\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$$

General superfield in component form

Most general form of field depending on x, θ, $\bar{\theta}$:

$$\Phi(x,\theta,\bar{\theta}) = \varphi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta F(x) + \bar{\theta}\bar{\theta}H(x) + \theta\sigma^{\mu}\bar{\theta}A_{\mu}(x) + (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\xi(x) + (\theta\theta)(\bar{\theta}\bar{\theta})D(x)$$

Further terms vanish because of $\theta\theta\theta = \bar{\theta}\bar{\theta}\bar{\theta} = 0$

Components (can be complex):

φ, F, H, D: scalar fields

A_{μ}: vector field

ψ, $\bar{\chi}$, $\bar{\lambda}$, ξ: Weyl-spinor fields

$\Rightarrow$ Too many components in 4-dim. for irreducible representation of SUSY with spin ≤ 1 (chiral or vector multiplet)

$\Rightarrow$ representation is reducible
(not all component fields mix with each other under SUSY transf.)

⇒ Irreducible superfields (smallest building blocks) from imposing conditions on general superfield
conditions need to be invariant under SUSY transformations:

$\bar{D}_{\dot{\alpha}}\Phi = 0$: left-handed chiral superfield (LHχSF)

$D_{\alpha}\Phi = 0$: right-handed chiral superfield (RHχSF)

$\Phi = \Phi^{\dagger}$: vector superfield

⇒ chiral superfields describe left- or right-handed component of SM fermion
+ scalar partner

simplified LHχSF in components:

$$\phi(x,\theta) = \varphi(x) + \sqrt{2}\theta\,\psi(x) - (\theta\theta)F(x)$$

φ, F: scalar fields , ψ: Weyl-spinor field

→ Explicit evaluation for the simplified case

Example: simplified left-handed chiral superfield

$$\phi_L(x,\theta) \;=\; \varphi(x) + \sqrt{2}\theta\,\psi(x) - (\theta\theta)F(x)$$

mass dimensions: $[\varphi] = 1$, $[\psi] = \frac{3}{2}$, $[F] = 2$

$\theta\theta = \varepsilon^{\alpha\beta}\theta_\alpha\theta_\beta$, $\quad \theta^\alpha\theta_\alpha = -\theta_1\theta_2 + \theta_2\theta_1$

Infinitesimal SUSY transformations: (ε infinitesimal)

$$\theta^\alpha \to \theta^\alpha + \varepsilon^\alpha$$
$$x^\mu \to x^\mu + 2i\theta\sigma^\mu\bar{\varepsilon}$$

$$\Rightarrow \delta\phi_L = \left(\varepsilon\frac{\partial}{\partial\theta} + \bar{\varepsilon}\frac{\partial}{\partial\bar{\theta}} + 2i\theta\sigma^\mu\bar{\varepsilon}\partial_\mu\right)\phi_L$$

NR: $\varepsilon^\alpha\frac{\partial}{\partial\theta^\alpha}\theta^\beta\varepsilon_{\beta\gamma}\theta^\gamma = \ldots = 2\varepsilon^\alpha\theta_\alpha$ (*)

$$
\begin{aligned}
\delta\phi_L &= 2i\theta\sigma^\mu\bar{\varepsilon}\partial_\mu\varphi \\
&\quad + \sqrt{2}\varepsilon^\alpha\psi_\alpha + \sqrt{2}2i\theta\sigma^\mu\bar{\varepsilon}\partial_\mu\theta^\alpha\psi_\alpha \\
&\quad +^{(*)} 2\varepsilon^\alpha\theta_\alpha F + \mathcal{O}\left(\theta^3\right)
\end{aligned}
$$

NR: $\theta^\beta(\sigma^\mu)_{\beta\dot{\beta}}\bar{\varepsilon}^{\dot{\beta}}\theta^\alpha = \ldots = -\frac{1}{2}\theta\theta(\sigma^\mu)^\alpha_{\dot{\beta}}\bar{\varepsilon}^{\dot{\beta}}$ (**)

$$
\begin{aligned}
\delta\phi_L &= 2i\theta\sigma^\mu\bar{\varepsilon}\partial_\mu\varphi \\
&\quad + \sqrt{2}\varepsilon^\alpha\psi_\alpha -^{(**)} \sqrt{2}i(\theta\theta)(\sigma^\mu)^\alpha_{\dot{\beta}}\bar{\varepsilon}^{\dot{\beta}}\partial_\mu\psi_\alpha \\
&\quad + 2\varepsilon^\alpha\theta_\alpha F + \mathcal{O}\left(\theta^3\right) \\
&\overset{!}{=} \delta\varphi + \sqrt{2}\theta^\alpha\delta\psi_\alpha + (\theta\theta)\delta F
\end{aligned}
$$

SUSY transformation of a LHχSF should yield a LHχSF!

Comparison of θ components:

$$\theta^0 : \delta\varphi = \sqrt{2}\varepsilon\psi \qquad \text{boson} \to \text{fermion}$$

$$\theta^1 : \delta\psi_\alpha = \sqrt{2}\varepsilon_\alpha F + i\sqrt{2}(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\varepsilon}^{\dot{\alpha}}\partial\varphi \qquad \text{fermion} \to \text{boson}$$

$$\theta^2 : \delta F = -i\sqrt{2}\partial\left((\sigma^\mu)^\alpha_{\dot{\beta}}\,\bar{\varepsilon}^{\dot{\beta}}\psi_\alpha\right) \qquad \text{total derivative!}$$

F transforms as total derivative

F is the component with the highest power in θ

$\Rightarrow$ Construction of $\mathcal{L}$ (invariant under SUSY transformations)
with highest component

$\to$ end of example

$\Rightarrow$ Irreducible superfields (smallest building blocks) from imposing conditions on general superfield
conditions need to be invariant under SUSY transformations:

$\bar{D}_{\dot{\alpha}}\Phi = 0$: left-handed chiral superfield (LHχSF)

$D_{\alpha}\Phi = 0$: right-handed chiral superfield (RHχSF)

$\Phi = \Phi^{\dagger}$: vector superfield

$\Rightarrow$ chiral superfields describe left- or right-handed component of SM fermion
+ scalar partner

LHχSF in components:

$$\begin{aligned}\phi(x,\theta,\bar{\theta}) \; = \; & \varphi(x) + \sqrt{2}\theta\psi(x) - i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\varphi(x) + \frac{i}{\sqrt{2}}(\theta\theta)(\partial_{\mu}\psi(x)\sigma^{\mu}\bar{\theta}) \\ & - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial^{\mu}\partial_{\mu}\varphi(x) - (\theta\theta)F(x)\end{aligned}$$

φ, F: scalar fields , ψ: Weyl-spinor field

LHχSF: Transf. of component fields with infinitesimal SUSY param. $\xi, \bar{\xi}$:

$$\delta\phi(x,\theta,\bar{\theta}) = i(\xi Q + \bar{\xi}\bar{Q})\phi(x,\theta,\bar{\theta})$$

Comparison with

$$\begin{aligned}\delta\phi(x,\theta,\bar{\theta}) &= \delta\varphi(x) + \sqrt{2}\theta\delta\psi(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\delta\varphi(x) + \frac{i}{\sqrt{2}}(\theta\theta)(\partial_\mu\delta\psi(x)\sigma^\mu\bar{\theta}) \\ &\quad - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial^\mu\partial_\mu\delta\varphi(x) - (\theta\theta)\delta F(x)\end{aligned}$$

$\Rightarrow$ determination of $\delta\varphi$, $\delta\psi$, δF:

$$\delta\varphi = \sqrt{2}\xi\psi \qquad \text{boson} \rightarrow \text{fermion}$$

$$\delta\psi_\alpha = -\sqrt{2}F\xi_\alpha - i\sqrt{2}(\sigma^\mu\bar{\xi})_\alpha\partial_\mu\varphi \qquad \text{fermion} \rightarrow \text{boson}$$

$$\delta F = \partial_\mu(-i\sqrt{2}\psi\sigma^\mu\bar{\xi}\,) \qquad F \rightarrow \text{total derivative}$$

RHχSF: analogously

Vector superfield in components:

$$\begin{aligned} V(x,\theta,\bar\theta) &= c(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) + \theta\sigma^\mu\bar\theta v_\mu(x) \\ &+ \frac{i}{2}(\theta\theta)(M(x)+iN(x)) - \frac{i}{2}(\bar\theta\bar\theta)(M(x)-iN(x)) \\ &+ i(\theta\theta)\bar\theta\left(\bar\lambda(x) + \frac{i}{2}\partial_\mu\chi(x)\sigma^\mu\right) - i(\bar\theta\bar\theta)\theta\left(\lambda(x) - \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right) \\ &+ \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)\left(D(x) - \frac{1}{2}\partial^\mu\partial_\mu c(x)\right) \end{aligned}$$

Number of components can be reduced by SUSY gauge transformation:

Wess-Zumino gauge: $\chi(x) = c(x) = M(x) = N(x) \equiv 0$

Vector SF: $V(x,\theta,\bar\theta) = \ldots + i(\theta\theta)\bar\theta\bar\lambda(x) - i(\bar\theta\bar\theta)\theta\lambda(x) + \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)D(x) + \ldots$

$$\delta D = -\xi\sigma^\mu\partial_\mu\bar\lambda(x) - \partial_\mu\lambda(x)\sigma^\mu\bar\xi \qquad D \to \text{total derivative}$$

4. Supersymmetric Lagrangians

Aim: construct an action that is invariant under SUSY transformations:

$$\delta \int d^4x \mathcal{L}(x) = 0$$

Satisfied if $\mathcal{L} \longrightarrow \mathcal{L} +$ total derivative

F and D terms (the terms with the largest number of θ and $\bar{\theta}$ factors) of chiral and vector superfields transform into a total derivative under SUSY transformations

$\Rightarrow$ Use F-terms (LHχSF, RHχSF) and D-terms (Vector SF) to construct an invariant action:

$$S = \int d^4x \left(\int d^2\theta \mathcal{L}_F + \int d^2\theta d^2\bar{\theta} \mathcal{L}_D \right)$$

If Φ is a LHχSF $\Rightarrow$ Φ^n is also a LHχSF (since $\bar{D}_{\dot{\alpha}}\Phi^n = 0$ for $\bar{D}_{\dot{\alpha}}\Phi = 0$)

$\Rightarrow$ products of chiral superfields are chiral superfields, products of vector superfields are vector superfields

F-term Lagrangian:

$$\mathcal{L}_F = \int d^2\theta \sum_{ijk} \left(a_i \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} \lambda_{ijk} \Phi_i \Phi_j \Phi_k \right) + \text{ h.c.}$$

Terms of higher order in Φ_i lead to non-renormalizable Lagrangians

$\Rightarrow$ F-term Lagrangian contains mass terms, scalar–fermion interactions ($\rightarrow$ superpotential), but no kinetic terms

D-term Lagrangian:

$$\mathcal{L}_D = \int d^2\theta d^2\bar{\theta} V$$

$\Rightarrow$ D-term Lagrangian contains kinetic terms

Example: the Wess–Zumino Lagrangian

Construction of Lagrangian from chiral superfields Φ_i

$\Rightarrow$ Φ_i, $\Phi_i\Phi_j$, $\Phi_i\Phi_j\Phi_k$

$\Phi_i^\dagger\Phi_i$: vector superfield, $(\Phi_i^\dagger\Phi_i)^\dagger = \Phi_i^\dagger\Phi_i$

$$\left[\Phi_i^\dagger\Phi_i\right]_{\theta\theta\bar\theta\bar\theta} = F^\dagger F + (\partial_\mu\varphi^*)(\partial^\mu\varphi) + \frac{i}{2}(\psi\sigma^\mu\partial_\mu\bar\psi - \partial_\mu\psi\sigma^\mu\bar\psi) + \partial_\mu(\ldots)$$

Auxiliary field F can be eliminated via equations of motion

$$\begin{aligned}\Rightarrow \quad \mathcal{L}_D \; = \; & \frac{i}{2}(\psi_i\sigma^\mu\partial_\mu\bar\psi_i - (\partial_\mu\psi_i)\sigma^\mu\bar\psi_i) - \frac{1}{2}m_{ij}(\psi_i\psi_j + \bar\psi_i\bar\psi_j) \\ & + (\partial_\mu\varphi_i^*)(\partial^\mu\varphi_i) - \sum_i \left| a_i + \frac{1}{2}m_{ij}\varphi_j + \frac{1}{3}\lambda_{ijk}\varphi_j\varphi_k \right|^2 \\ & - \lambda_{ijk}\varphi_i\psi_j\psi_k - \lambda^\dagger_{ijk}\varphi_i^*\bar\psi_j\bar\psi_k\end{aligned}$$

Auxiliary fields are eliminated via equations of motions:

$$\begin{aligned} \text{abelian :} \quad F &= m\varphi^* + g\varphi^{*2} \\ \text{non-abelian, gauge group } G \text{ :} \quad D^G &= \ldots \sum_a g_G \left(\varphi_i^\dagger (T_G)^a \varphi_i \right) \end{aligned}$$

(internal indices of T_G, φ_i suppressed)

$$\Rightarrow \quad \mathcal{L}_D = F\, F^* + \frac{1}{2} \sum_G D^G (D^G)^\dagger + \ldots$$

Lagrangian for scalar fields φ_i and spinor fields ψ_i with the same mass m_{ii}

contains couplings of type $h f \bar{f}$ and $\tilde{h} \tilde{f} \bar{f}$ with the same strength

$\Rightarrow$ SUSY implies relations between masses and couplings

$\mathcal{L}$ can be rewritten as kinetic part + contribution of superpotential $\mathcal{V}$:

$$\mathcal{V}(\varphi_i) = a_i\varphi_i + \frac{1}{2}m_{ij}\varphi_i\varphi_j + \frac{1}{3}\lambda_{ijk}\varphi_i\varphi_j\varphi_k$$

$$\Rightarrow \quad \mathcal{L} \;=\; \frac{i}{2}(\psi_i\sigma^\mu\partial_\mu\bar{\psi}_i - (\partial_\mu\psi_i)\sigma^\mu\bar{\psi}_i) + (\partial_\mu\varphi_i^*)(\partial^\mu\varphi_i)$$
$$- \sum_i \left|\frac{\partial\mathcal{V}}{\partial\varphi_i}\right|^2 - \frac{1}{2}\frac{\partial^2\mathcal{V}}{\partial\varphi_i\partial\varphi_j}\psi_i\psi_j - \frac{1}{2}\frac{\partial^2\mathcal{V}^*}{\partial\varphi_i^*\partial\varphi_j^*}\bar{\psi}_i\bar{\psi}_j$$

$\mathcal{V}$ determines all interactions and mass terms

Without proof:

characteristics of $\mathcal{V}$ = characteristics of $\mathcal{L}$

Special case $a_i = 0$: Wess–Zumino model